RG folklore

Invariance with respect to change of the reference scale μ

$$\frac{dF}{d\mu} = 0 \; . \tag{1}$$

can be detailed as a linear partial DE

$$\left[x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g} \right] \; F(x,g) = 0 \, ; \quad x = q^2/\mu^2 , \quad g = g_\mu \; . \tag{2}$$

$$\beta(g_\mu) = z\frac{\partial \bar{g}(z)}{\partial z} \quad \text{at} \quad z = \mu^2 \; . \tag{3}$$

Running coupling $\bar{g}$ is a function of 2 arguments : $q^2/\mu^2 = x$ and g_μ with property $\bar{g}(1,g) = g$. The $\bar{g}$ satisfies eqs. (1),(2). Due to this it is *invariant coupling* function.

RG folklore; cont.

Besides,

$$x\frac{\partial \bar{g}(x,g)}{\partial x} = \beta(\bar{g}(x,g)) \ . \qquad (4)$$

Also of interest are covariant objects $s(x,g)$ with

$$\left[x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g} + \gamma_s(g)\right] \ s(x,g) = 0 \quad , \qquad (5)$$

$\gamma_s(g)$ being anomalous dimension of s .

Mathematical Grounds

Functional and Diff. Equations

The central is Funct. Eq (FE) for invariant coupling

$$\bar{g}(x,g) = \bar{g}\left(\frac{x}{t}\,,\bar{g}(t,g)\right)\quad . \qquad (6)$$

Non-linear DEq (4) is obtained from it by differentiating over x with $t = x$. In parallel, by diff-ing over t at $t = 1$ one gets (partial) PDEq (2) with *the Lie operator* $L(x,g)$

$$L(x,g)\bar{g}(x,g) = 0;\;\; L(x,g) = \left[x\frac{\partial}{\partial x} - \beta(g)\frac{\partial}{\partial g}\right]. \qquad (7)$$

Functional Group Eqs

Due to this, Funct. eqs (6) and

$$\bar{s}(x,g) = \bar{s}(t,g)\,\bar{s}\left(\frac{x}{t}\,,\bar{g}(t,g)\right) \qquad (8)$$

presents *most general form of RG symmetry in QFT.*
From (6), (8) stem (4) and

$$x\frac{\partial s(x,g)}{\partial x} = s(x,g)\,\gamma_s\left(\bar{g}(x,g)\right)\ , \qquad (9)$$

Meanwhile, these Funct. Eqs.(8) and (6)

$$\bar{g}(x,g) = \bar{g}\left(\frac{x}{t},\bar{g}(t,g)\right)\ . \qquad (6)$$

just contain the group composition law and
have no physical contents !!

RG transformation

Consider change $[\mu_i \longrightarrow \mu_k\ ,\ g_i \longrightarrow g_k]$, as *operation* with continuous positive parameter t , acting on *group element* $\mathcal{G}_i(\mu_i, g_i)$, specified by 2 coordinates.

Operation R_t

$$R_t \cdot \mathcal{G}_i = \mathcal{G}_k \sim R_t \left\{ \mu_i^2 \longrightarrow \mu_k^2 = t\mu_i^2 ,\ g_i \longrightarrow g_k = \bar{g}(t, g_i) \right\} \tag{10}$$

contains dilatation of μ , and funct'l transf-n of g_μ .

The R_t group structure is provided just by eq.(6).

Indeed, if we put $x = \tau\, t$, then its l.h.s. describes the $R_{\tau t}$ acting on g , while r.h.s one – $R_\tau \otimes R_t\, g$

$$R_{\tau\, t}\, g = \bar{g}(\tau\, t, g)\,; \qquad R_\tau \otimes R_t\, g = R_\tau\, \bar{g}(t, g) = \bar{g}\left(\tau, \bar{g}(t, g)\right)$$

Lie Group of Transformations

Combination of

$$R_t \cdot \mathcal{G}_i = \mathcal{G}_k \sim R_t \{\mu_i^2 \to \mu_k^2 = t\mu_i^2,\ g_i \to g_k = \bar{g}(t, g_i)\} \qquad (10)$$

and

$$R_{\tau\, t}\, g = \bar{g}(\tau\, t, g)\,; \qquad R_\tau \otimes R_t\, g = R_\tau\, \bar{g}(t, g) = \bar{g}\,(\tau, \bar{g}(t, g))$$

results in

$$\bar{g}(x, g) = \bar{g}\left(\frac{x}{t}, \bar{g}(t, g)\right)\ . \qquad (6)$$

Hence, the eq.(6) provides the group composition law $R_{\tau t} = R_\tau \otimes\ R_t$, that is operations R_t (10)) form continuous Sophus Lie(1880) group of transformations

Abstract formulation of composition law

Let $T(l)$ be a transf-tion of an abstract set $\mathcal{M}$ of elements M_i to itself, depending on continuous real parameter l, varying in $(-\infty < l < \infty)$, That is, for each M one can write

$$T(l)\,M = M' \quad (M, M' \subset \mathcal{M}) \; .$$

Assume, set $\mathcal{M}$ can be projected on numerical axis, i.e., to each M_i there correspond a number g_i.
Then

$$T(l)g = g' = G(l, g) \; ,$$

with G – continuous function of 2 arguments.

Abstract form-n of composition law, cont'd

$$T(l)g = g' = G(l,g) \ ,$$

with G – continuous function with property

$$G(0,g) = g \ , \quad \text{that relates to unity trans-n} \quad T(0) = \mathbf{E} \, .$$

Trans-s $T(l)$ form a group provided the composition law $T(\lambda) \oplus T(l) = T(\lambda + l)$, and funct'l eq for G

$$G\{\lambda, G(l,g)\} = G(\lambda + l, g) \tag{11}$$

holds.

Diff. Group Equations

According to Lie group theory, it's sufficient to consider infinitesimal (at $\lambda \ll 1$) version of (11) – the Diff. eq.

$$\frac{\partial G(l,g)}{\partial l} = \beta\{G(l,g)\}\ . \qquad (12)$$

with generator defined via derivative

$$\beta(g) = \frac{\partial G(\epsilon, g)}{\partial \epsilon}, \quad \text{at} \quad \epsilon = 0.$$

After logarithmic change of variables

$$l = \ln x\,, \quad \lambda = \ln t\,, \quad G(l,g) = \bar{g}(x,g)\,, \quad T(\ln t) = R_t\,, \qquad (13)$$

we get multiplicative (6), (4) instead of additive (11), (12).

Transformation of reparameterisation

A particular solution $f(x)$ of some boundary problem is specified by boundary condition $f(x_0) = f_0$. It can be given as $F(x/x_0, f_0)$ with property $F(1, \gamma) = \gamma$. Now equation

$$F(x/x_0, f_0) = F\left(x/x_1, f_1\right)$$

expresses *the reparameterization invariance* as in the explicit case $F(x, \gamma) = \Phi(\ln x + \gamma)$). Using relations

$$f_1 = F(x_1/x_0, f_0); \quad \xi = x/x_0, \quad t = x_1/x_0\,,$$

we come to the funct'l eq.

$$F(\xi, f_0) = F\left(\xi/t, F(t, f_0)\right) \qquad (6 - bis),$$

Transf-n of reparameterisation; cont'd

$$F(\xi, f_0) = F\left(\xi/t, F(t, f_0)\right) \qquad (6 - bis),$$

is equivalent to (6). The involved operation can presented as

$$G_t \ : \ \{ \ \xi \to \xi/t \ , \ f_0 \to f_1 = F(t, f_0) \ \} \ . \qquad (14)$$

The additive version of these eqs is

$$R(l) \ : \ \{ \ q \to q' = q - l \ , \ g \to g' = G(l, g) \ \} \ , \qquad (15)$$

and (11).

The additive version

$$R(l) \quad : \quad \{\, q \to q' = q - l \;, \quad g \to g' = G(l, g) \,\} \;, \qquad (16)$$

By change of variables $q \to x = e^q, \;\; l \to t = e^l$ and of function (13) one gets (4), (6) and transf-n

$$R_t \quad : \quad \{\, x' = x/t \;, \quad g' = \bar{g}(t, g) \,\} \qquad (17)$$

instead of eqs.(11), (12),(16).

One can treat eqs.(4),(6), (17) as *multiplicative* version of RG eqs. for effective coupling in massless QFT with 1 coupling g .

Here, $x = Q^2/\mu^2$. For propagator amplitude one has

$$\phi(q, g) \;\to\; R(l)\phi = z(l, g)\phi(q', g') \;, \qquad (18)$$

that corresponds to (8).

Simple Generalizations

”Massive” Case. For example in QFT, if we do not neglect the particle mass m, we should insert one more argument into the effective coupling $\bar{g}$ which now has to be considered as a function of 3 variables $x = Q^2/\mu^2,\ y = m^2/\mu^2,\ g$. The presence of a "mass" argument y modifies group transf-n

$$R_t \ : \ \{ \, x' = x/t \, , \ \ y' = y/t \, , \ \ g' = \bar{g}(t, y; g) \, \} \tag{19}$$

and the functional equation

$$\bar{g}(x, y; g) = \bar{g}\left(\frac{x}{t} \, , \ \frac{y}{t} \, ; \ \bar{g}(t, y; g) \right) \ . \tag{20}$$

Simple Generalization, 1

$$\bar{g}(x,y;g) = \bar{g}\left(\frac{x}{t} \, , \, \frac{y}{t} \, ; \, \bar{g}(t,y;g) \right) \; .$$

New parameter y enters also into the transformation law of g .

Let QFT model has several masses (like, QCD). Then there will be several mass arguments $y \longrightarrow \{y\} = y_1, y_2, \ldots y_n$.

Multi-coupling case

Another generalization relates to several coupling constants case: $g \to \{g\} = g_1, \ldots g_k$. Here arises "family" of effective couplings

$$\bar{g} \to \{\bar{g}\} \,, \quad \bar{g}_i = \bar{g}_i(x, y; \{g\}) \,, \qquad i = 1, 2, \ldots k \,, \tag{21}$$

satisfying the system of coupled funct'l eqs

$$\bar{g}_i(x, y; \{g\}) = \bar{g}_i\left(\frac{x}{t}, \frac{y}{t} \,; \ \ldots \ \bar{g}_j(t, y; \{g\}) \ \ldots\right). \tag{22}$$

Multi-coupling case; cont'd

This system is a generalization of (5) and (20) to the case when every element M_i of $\mathcal{M}$ can be described by k parameters, i.e., by the point $\{g\}$ in a k-dimensional real parameter space.

The RG transformation looks like

$$R_t : \left\{ x \to \frac{x}{t},\ y \to \frac{x}{t},\ \{g\} \to \{\bar{g}(t)\} \right\};$$

$$\bar{g}_i(t) = \bar{g}_i(t, y; \{g\}). \qquad (23)$$

1st Illustration: Elastic Rod

The symmetry of the FSS group transf'ns can be 'discovered' in many problems taken from diverse fields of physics.

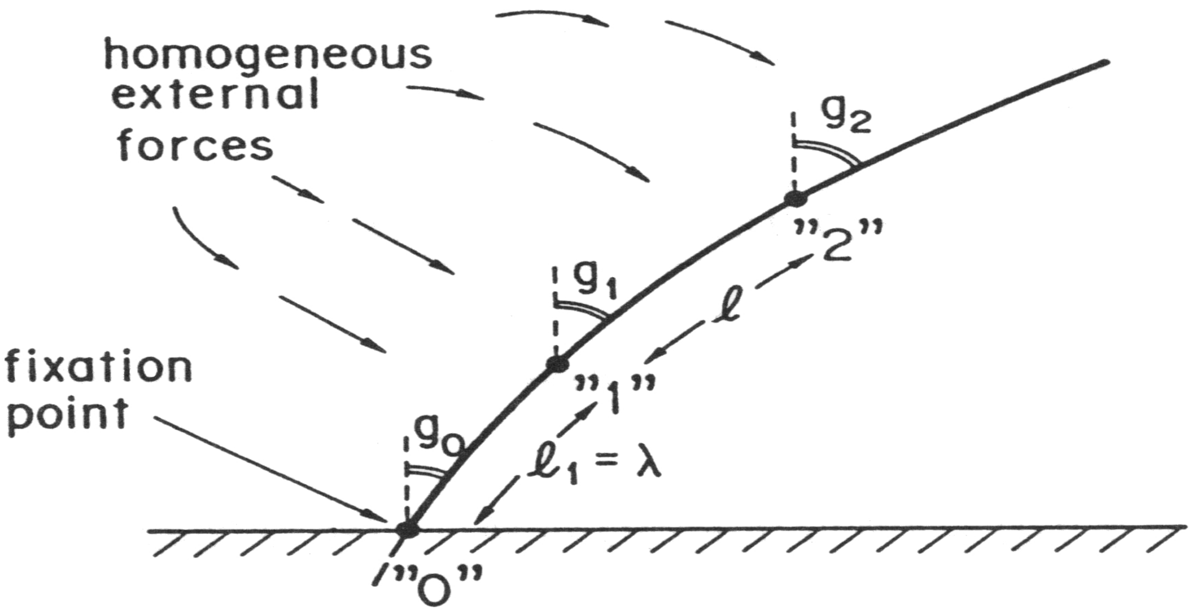

Figure 1: "Elastic rod" model

Imagine an elastic rod with a fixed point (point "0" in Fig. 1) bent by some external force, e.g., gravity or pressure of a moving gas or liquid.

Elastic Rod, 2

The form of rod can be described by angle g between tangent to the rod and vertical direction considered as function of distance l along rod from the fixation point, that is by function $g(l)$. If the properties of the rod material and of external forces are homogeneous along its length (i.e.independent of l), then $g(l)$ can be expressed as function $G(l, g_0)$ depending also on g_0 , deviation angle at fixation point from which distance l is measured.

Naturally, G should depend on other arguments, like extra forces and rod material parameters, as well but in this context they are irrelevant.

Elastic Rod, 3

Take two arbitrary points on the rod, "1" and "2" (see Fig.1 with $l_1 = ambda$ and $l_2 = \lambda + l$. The angles g_i at points "0", "1" and "2" are related via G function :

$$g_1 = G(\lambda, g_0), \quad g_2 = G(\lambda + l, g_0) = G(l, g_1). \tag{24}$$

To get the very r.h.s. of 2nd eq., one has to imagine that fixation point now is "1" as in Fig. 2.

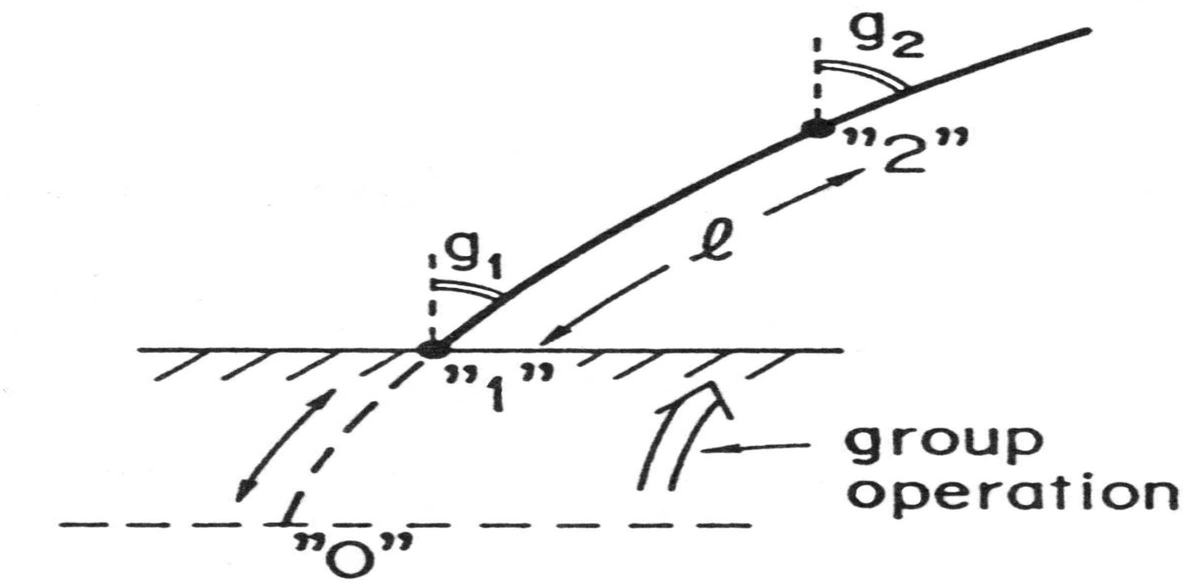

Figure 2: Group operation for the "Rod".

Elastic Rod, 4

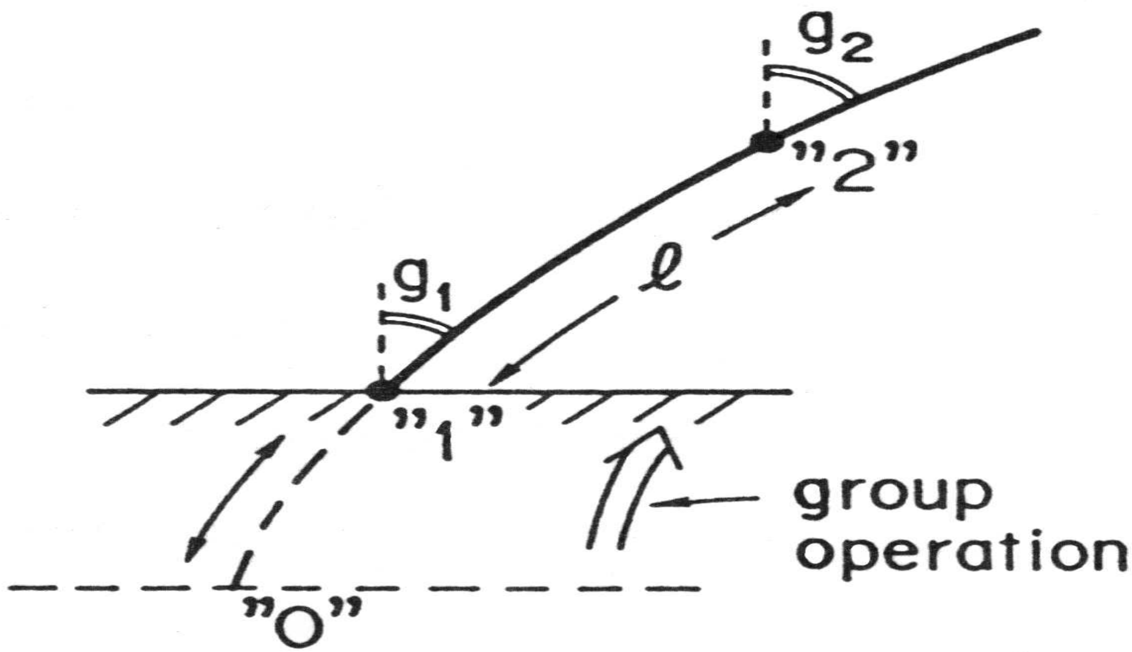

Figure 2: Group operation for the "Rod".

Combining all Eqs.

$$g_1 = G(\lambda, g_0),\ g_2 = G(\lambda + l, g_0) = G(l, g_1)\,, \qquad (23)$$

we get group composition law (7) for the function $G(l, g)$.

Elastic Rod, 5

In course of deriving the 2nd of Eqs. (24) we have tacitly assumed that rod is of infinite length. If we introduce a finite length L - Fig. 3,

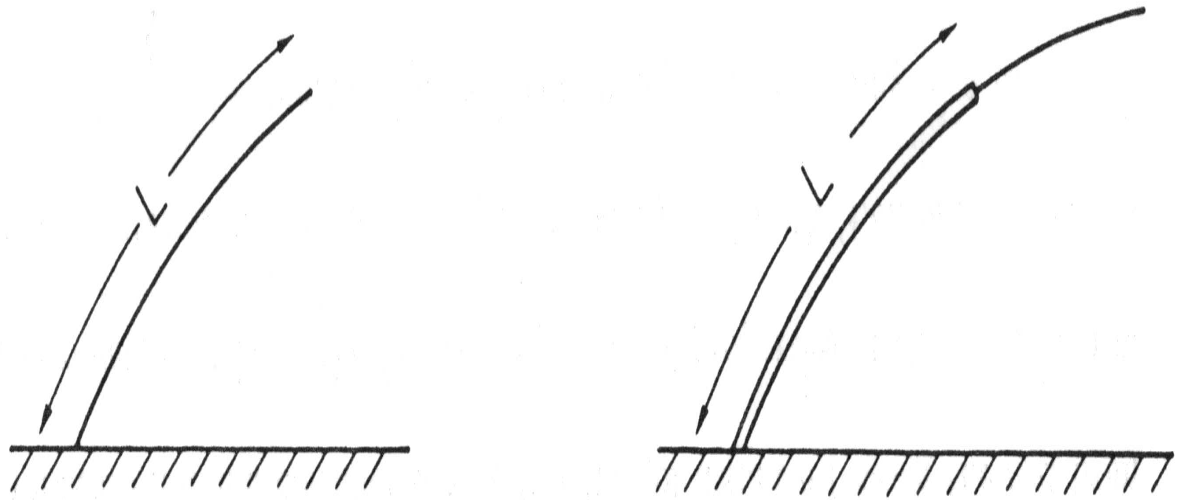

Figure 3: Rod with discrete inhomogeneity.

then $G(l, g)$ must be replaced by function $G(l, L, g)$ of 3 essential arguments, where the 2nd one is distance between the fixation point and the free end.

Combining now

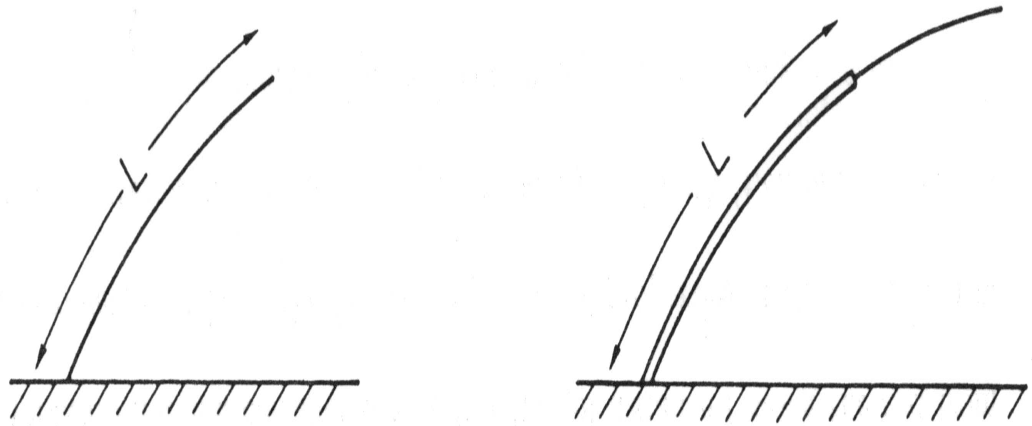

Figure 3: Rod with discrete inhomogeneity.

$$g_1 = G(\lambda, L, g_0), \quad g_2 = G(\lambda + l, L, g_0) = G(l, L - \lambda, g_1) \quad (25)$$

we come to the functional equation

$$G(l + \lambda, L, g) = G(l, L - \lambda, G(\lambda, L, g)) \, , \quad (26)$$

which is just an "additive" version of the massive QFT one

$$\bar{g}(x, y; g) = \bar{g}\left(\frac{x}{t}, \frac{y}{t} ; \bar{g}(t, y; g)\right) . \quad (19)$$

and can be transformed to it by log change of variables used to get (7) from (5).

Breaking of Homogeneity

In (25) and (26) the 2nd argument L is not necessarily the rod length. It can be treated as a distance from the fixation point to a place where the rod properties undergo a discrete change (say, in thickness or in material).

Generally, the additional argument L describes the *discrete breaking of homogeneity* property of the system. It can take place at several points. Their coordinates must be introduced as G additional arguments: $L \to \{L\}$. In the QFT case this corresponds to the introduction of particle masses.

RG symmetry as Functional Self-Similarity

RG symmetry and RG transf-n are close to the notion of Self-Similarity well known in math. physics since the end of XIX. The Self-Similarity transf-n is a simultaneous power scaling of arguments $z = \{x, t, \ldots\}$ and functions $V_i(x, t, \ldots)$

$$S_\lambda : \quad \{ \qquad x \to x\lambda \,, \quad t \to t\lambda^a \quad \} \ ,$$

$$\{V_i(z) \to V_i^{`}(z') = \lambda^{\nu_i} V_i(z') \ \} \, .$$

We call it *Power Self-Similarity*=PSS transformation.

RG vs Power Self-Simil 2

According to Zeldovich and Barenblatt, PSS is of 2 kinds:

a/ The PSS of the 1st kind, with all the powers $a, \nu, \ldots$ being rational numbers defined from dimensions) = *(rational PSS)*.

b/ The PSS of the 2bd kind, with some of powers being irrational and defined from dynamics (fractal PSS).

To relate RG with PSS, turn to solution of basic RG FEq

$$\bar{g}(xt, g) = \bar{g}(x, \bar{g}(t, g)) \; . \qquad (1.6')$$

RG vs Power Self-similarity

The general solution of

$$\bar{g}(xt, g) = \bar{g}(x, \bar{g}(t, g)) \; . \qquad (1.6')$$

depends on arbitrary 1-argument function - see below.

Here, we look for partial solution, linear in 2nd argument $\bar{g}(x, g) = g f(x)$.

Function $f(x)$ satisfies eq. $f(xt) = f(x) f(t)$, with solution: $f(x) = x^{\nu}$ and $\bar{g}(x, t) = g x^{\nu}$. In our case the RG tran-n is reduced to PSS one,

$$R_t \to \{x \to x t^{-1}, \; g \to g t^{\nu}\} = S_t \; .$$

RG vs PSS, cont'd

Thus, the PSS transf-n is a special case of RG one,

$$R_t \to S_t = \{x \to xt^{-1},\ g \to gt^{\nu}\} \,. \tag{27}$$

Generally, in RG, instead of a power law, one has arbitrary functional dependence. Hence, one can consider all the RG transf-s as *functional* generalizations of PSS transf-n.

It is natural, to refer to them as to transf-s of funct'l scaling or **F**unctional **s**elf-**s**imilarity(FSS) transf-n.

In short

$$\mathrm{RG} \equiv \mathrm{FSS}$$