Lecture 4: Causality and Analyticity

- Microscopical Causality in local QFT
- Analyticity from Causality
- Dispersion Relation for forward scattering amplitude
- Källen – Lehmann representation for propagator
- Jost-Lehmann-Dyson repres'n; virtual scattering
- Källen–Lehmann representation, invariant coupling

Microscopical Causality in local QFT

The Fourier image

$$F(E) = \int_{-\infty}^{\infty} e^{itE} A(t)\, dt \qquad (1)$$

of forward scattering amplitude subdue to non-relativistic "causality condition"

$$\boxed{A(t) = 0 \quad \text{at} \quad t < 0}$$

can be analytically continued from real E values to upper half plane $Q \to z = E + i\xi;\ \xi = \Im m z > 0$ as in integrand

$$F(z) = \int_{0}^{\infty} e^{itE - t\xi} A(t)\, dt \qquad (2)$$

factor $e^{-t\xi}$ provides convergence.

Microscopical Causality in local QFT

Due to Analyticity, one can use Cauchy theorem for

$$F(E) = \int_{-\infty}^{\infty} e^{itE} A(t)\, dt \qquad (3)$$

with integration contour Γ in the upper half-plane

$$\oint_{\Gamma} \frac{f(z')}{z' - z} dz' = 0$$

and get Dispersion Relation

$$\Re e f(E) = \frac{\mathcal{P}}{\pi} \int_{-\infty}^{\infty} \frac{\Im m f(E')}{E - E'}\, dE' = \frac{\mathcal{P}}{\pi} \int_{m}^{\infty} \frac{k\sigma(E')}{E - E'}\, dE'$$

connecting two observable functions.

Causality and Dispersion Relation

In obtaining this Dispersion Relation for forward scattering amplitude

$$\Re e f(E) = \frac{\mathcal{P}}{\pi} \int_m^\infty \frac{k\,\sigma(E')}{E - E'}\, dE' \,, \qquad (4)$$

we assumed “good" asymptotic behavior and used Optical Theorem

$$f(z) \lesssim C/z \quad \text{as} \quad |z| \to \infty \quad \text{and} \quad \Im m f(E) = k\,\sigma(E)\,.$$

In a more realistic case, one uses relativistic causality and symmetry crossing property of forward scattering.

Causality in QFT

Causality in local QFT states that signal velocity is limited by c and formulated as Local Commutativity for Lagrangian

$$[\mathcal{L}(x), \mathcal{L}(y)] \equiv \mathcal{L}(x)\,\mathcal{L}(y) - \mathcal{L}(y)\,\mathcal{L}(x) = 0, \qquad (5)$$

for space-like $(x-y)^2 = (x_0 - y_0)^2 - (\mathbf{x} - \mathbf{y})^2 < 0$ intervals.

Eq.(5) is provided by Loc. Comm. conditions for field operators

$$[\phi(x), \phi(y)] = 0, \quad (x-y)^2 < 0\,. \qquad (6)$$

Stueckelberg-Feynman propagator

Along with Pauli-Willars commutator

$$D(x-y) = \frac{1}{i}\,\langle 0\,[\,(\phi(x)\,,\,\phi(y))|\,0\rangle\ , \tag{7}$$

vanishing outside light cone

$$D(x-y)=0, \quad \text{at} \quad (x-y)^2=(x_0-y_0)^2-(\mathbf{x}-\mathbf{y})^2<0\,. \tag{8}$$

In calculation, we use Stueckelberg–Feynman propagator

$$D_c(x-y) = D_F(x-y) = \frac{1}{i}\,\langle 0\,|T\,[\phi(x)\,\phi(y)]|\,0\rangle\ , \tag{9}$$

the vacuum average of time-ordered product $T\,[\phi(x)\,\phi(y)]$

Källen – Lehmann eq. for propagator, 2

For the causal Stueckelberg-Feynman propagator the Källen–Lehmann (KL) spectral representation is

$$D_c(q^2) = \frac{1}{\pi}\int_0^\infty \frac{d\sigma}{\sigma - q^2 - \mathrm{i}\epsilon}\,. \qquad (10)$$

Its “dressed” counterpart looks like

$$D_c(q^2, \alpha_s) = \frac{1}{\pi}\int_0^\infty d\sigma \frac{\rho(\sigma, \alpha_s)}{\sigma - q^2 - \mathrm{i}\epsilon} \qquad (11)$$

with $\rho(\sigma, \alpha_s)$ behaving as $1/\ln^2\sigma$, that allows one to use it in the non-subtracted form.

Jost–Lehmann–Dyson representat'n for virtual scattering

For Deep-Inelastic Scattering (DIS) probability, one uses hadronic tensor

$$W_{\mu\nu}(q,P) \sim \int dz \exp^{iq\cdot z} \left\langle P,\sigma \left| \left[J_\mu(\frac{z}{2}), J_\nu(-\frac{z}{2}) \right] \right| P,\sigma \right\rangle \qquad (12)$$

defined via current commutator. For structure functions W_n , a more involved Jost–Lehmann–Dyson representation holds

$W(\nu,Q^2) = \qquad (\nu = P\cdot q > 0;\ Q^2 = -q^2 > 0)$

$$= \int\limits_0^1 d\rho\rho^2 \int\limits_{\lambda^2_{\rm min}}^{\infty} d\lambda^2 \int\limits_{-1}^{1} dz\delta\left(Q^2 + M^2\rho^2 + \lambda^2 - 2z\rho\sqrt{\nu^2 + M^2Q^2}\right)\psi(\rho,\lambda^2)\,.$$

$$\lambda^2_{\rm min} = M^2\left(1 - \sqrt{1-\rho^2}\right)^2 .$$

It turns to be useful for formulating analyticity of the structure functions moments.

Källen – Lehmann representation for invariant coupling

In QED, an invariant coupling = product of coupling constant and transverse photon propagator amplitude $\bar{\alpha}(Q^2, \alpha) = \alpha\, d_{tr}(Q^2 = -q^2, \alpha)$, satisfies KL eq.(10) by construction

$$\bar{\alpha}(Q^2, \alpha) = \frac{1}{\pi} \int_0^\infty d\sigma \frac{\rho(\sigma, \alpha)}{\sigma + Q^2 - \mathrm{i}\epsilon} \,. \qquad (13)$$

As it can be shown, the QCD invariant coupling $\bar{\alpha}_s(Q^2, \alpha_s)$ satisfies the KL representation as well

$$\bar{\alpha}_s(Q^2, \alpha_s) = \frac{1}{\pi} \int_0^\infty d\sigma \frac{\rho(\sigma, \alpha_s)}{\sigma + Q^2 - \mathrm{i}\epsilon} \,. \qquad (14)$$

RENORMALIZATION GROUP METHOD

Introductive Illustration For this, consider effective coupling $\bar{g}$ in the UV region with 1-loop log contribution

$$\bar{g}^{[1]}_{PT}(x,g) = g + g^2\beta\ln x. \qquad (15)$$

By simple arithmetics within RG FEq.

$$\bar{g}(x,g) = \bar{g}\left(\frac{x}{t}, \bar{g}(t,g)\right). \qquad (1.6)$$

one gets

$$Disc[\bar{g}^{[1]}_{PT}] = \bar{g}^{[1]}_{PT}(x,g) - \bar{g}^{[1]}_{PT}\left(\frac{x}{t}, \bar{g}^{[1]}_{PT}(t,g)\right) =$$

$$= [g + g^2\beta\ln x] - [g + g^2\beta\ln x + 2g^3\beta^2\ln t\ln(x/t)] \neq 0$$

- error of g^3 order.

RENORM-GROUP METHOD, 2

This error of g^3 order is liquidated by adding next order term $g^3\beta^2\ln^2 x$ into the r.h.s. of (15):

$$\bar{g}_{PT}^{[2]} = g + g^2\beta\ln x + g^3\beta^2\ln^2 x \;\rightarrow\; Disc[\bar{g}_{PT}^{[2]}] \sim g^4\ln^4 x.$$

This "improved" expression yields g^4 error and can be killed by adding $g^4\ln^3$ term into (1) and so on. Thus, on the one hand, finite polynomials cannot satisfy the condition of RG invariance. On the other, we conclude that FEq (1.6) is a tool for iterative restoring of RG-invariant expression in form of infinite series.

RENORM-GROUP METHOD,3

This example illustrates a general situation. As a rule, approximate solutions do not satisfy RG symmetry. In our case, this is happened in UV limit at $\ln x \to \infty$ where the observed discrepancy becomes quantitatively important. Note, that sum of mentioned iterative series is rather simple

$$\bar{g}^{[n]}_{PT} = g \sum_{k=0}^{n} (g\,\beta\,\ln x)^k; \quad \lim_{n\to\infty} \bar{g}^{[n]}_{PT} = \frac{g}{1 - g\,\beta\,\ln x}\,.$$

RENORM-GROUP METHOD,3

This is famous 1-loop approximation for the effective coupling in QFT

$$\bar{g}^{(1)}(x,g) = \frac{g}{1 - g\,\beta\,\ln x}\,. \tag{16}$$

It is instructive exercise, to check that it exactly satisfies the FEq (1.6). At the same time, expression (16) gives birth to grave issue - the problem of unphysical pole (“Landau ghost”) at $x = x* = e^{1/g\,\beta}$