Física I, Engenharia Química (2011-2012)

1. Considere uma gota esférica de uma certa substância que se encontra no fundo de um poço de água. A densidade da substância é igual a 0.85 g/cm^3 e o raio da gota igual a $R = 90$ micrómetros. Na sua trajectória vertical a caminho da superfície da água, a gota sofre uma força de resistência da água dada por

$$F_{\text{res}} = -6\pi\mu R v$$

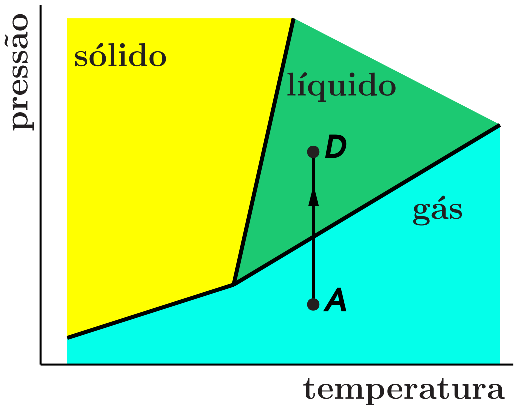
onde v representa a velocidade instantânea da gota e $\mu = 1.00$ g/ms, a viscosidade da água.

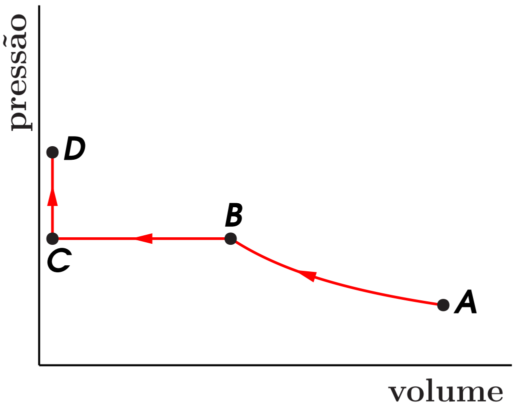
a Quais as outras forças que determinam a aceleração da gota?

b Determine a velocidade terminal da gota, considerando $g = 9.8$ m/s^2.

c Caso a gota se mova com a velocidade terminal, determine o valor, em Newtons, de cada uma das forças da resposta à alínea **a**.

2. Considere uma certa substância dentro de um cilindro fechado por um pistão. Inicialmente a substância encontra-se no estado de gás (gás ideal), indicado por A nos diagramas de fase e de *volume-pressão* abaixo. Pelo aumento da pressão sobre o pistão, a substância transforma-se num líquido, por um processo isotérmico, até ao estado indicado por D nos diagramas de fase e de *volume-pressão*.
(O vídeo http://www.youtube.com/watch?v=7MWAy9Jbsoc mostra o início do processo ao contrário $D \to A$)





A curva no diagrama de *volume-pressão* corresponde à curva $A \to D$ indicada no diagrama de fase.

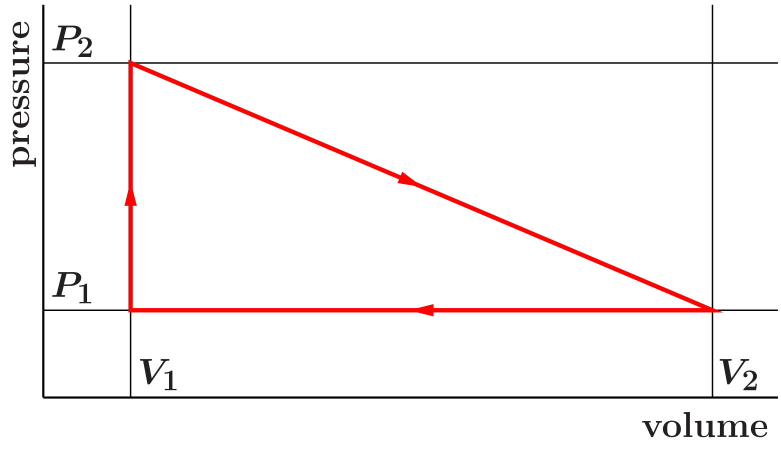
a Explique a forma da curva $A \to B$ no diagrama de *volume-pressão*.

b Explique a forma da curva $B \to C$ no diagrama de *volume-pressão*.

c Copie o diagrama de fase para a sua folha e indique na curva $A \to D$ a posição dos estados B e C da substância.

d Explique a forma da curva $C \to D$ no diagrama de *volume-pressão*.

3. Considere um cilindro de um motor de combustão cujo funcionamento é representado pelo diagrama P-V da figura seguinte.



Considere também as seguintes relações: $V_2 = 2.5V_1$ e $P_2 = 3P_1$.

a Se a temperatura no ponto (V_1,P_1) é igual a 300° K, qual a temperatura nos pontos (V_1,P_2) e (V_2,P_1)?

b Determine a quantidade de calor, em unidades P_1V_1, fornecida ao/pelo combustível na transformação $(V_1, P_2) \rightarrow (V_2, P_1)$.

c Considere a eficácia de um motor definida como o rácio entre o trabalho efectuado e o calor fornecido. Qual é a eficácia do motor representado pelo diagrama P-V em causa?

4. Duzentos gramas de água a 18^o C, postos dentro do congelador de um frigorífico para fazer cubos de gelo, congelam até a uma temperatura de -20^o C. O calor específico de água é igual a 4.18×10^3 J/kg K, de gelo a 2.11×10^3 J/kg K e o calor latente de solidificação de água é igual a 3.34×10^5 J/kg.
Determine a quantidade de calor retirada à água.

5. Uma mangueira para regar o quintal está ligada a um tanque. O nível de água dentro do tanque está 7.00 m acima do chão. A última parte da mangueira até à saída faz um ângulo de 30^o com a vertical, estando a saída da mangueira a uma altura de 70 cm acima do chão. O interior da mangueira tem uma secção de 2.3 cm^2.

a Determine a quantidade de água, em litros por minuto, que sai da mangueira.

b Determine a distância horizontal, medida a partir da saída da mangueira, onde o jacto de água toca no chão.

6. Quando se prende uma massa pontual de 0.42 kg a uma mola que está pendurada, a mola estica 9.8 cm. Determine a expressão $u(t)$ do deslocamento da massa em função do tempo, após uma perturbação temporária em que a massa sofreu um deslocamento de 2 cm para baixo.

7. Um feixe de luz monocromático passa por duas fendas estreitas e paralelas, que estão a uma distância uma da outra igual a 20 μm.
Determine a distância entre os máximos mais brilhantes observados num ecrã a 5.0 metros de distância das fendas, para a luz vermelha ($\lambda = 0.650$ μm) e para a luz azul ($\lambda = 0.475$ μm).

8. Uma caixa cilíndrica, com uma altura de 1.0 m e um fundo com 30 cm de raio, está em repouso no convés de um navio. O centro de massa da caixa encontra-se no meio da caixa e o coeficiente de atrito entre a caixa e o convés é igual a 0.75. As ondas do mar causam uma tal inclinação do convés que a caixa comece a mexer-se.
Determine se a caixa desliza ou se tomba.

9. Uma roda de raio R e massa m rola uma descida abaixo sem deslizar. A descida faz um ângulo θ com a horizontal e o momento de inércia da roda é dado por mR^2.
Determine a aceleração da roda.

10. Considere um sistema de duas massas pontuais, m_1 e m_2, ligadas por uma mola sem massa (constante de mola C_{el}) e cujos movimentos estão restritos a uma dimensão (x). As duas masses oscilam em torno da sua posição de equilíbrio de tal forma que o centro de massa (CM) do sistema não se mexe.
Sejam as distâncias entre as posições de equilíbrio das massas m_1 e m_2 e o CM dadas por r_1 e r_2, respectivamente.
Conclui-se logo que $m_1 r_1 = m_2 r_2$.
Um deslocamento x_1 da massa m_1 tem que ser compensado por um deslocamento x_2 da massa m_2 de tal forma que $m_1 (r_1 + x_1) = m_2 (r_2 + x_2)$, para que o CM não se mexa.
As oscilações das duas massas são descritas pelas seguintes expressões:

$$x_1(t) = A_1 \sin(\omega t) \quad \text{e} \quad x_2(t) = A_2 \sin(\omega t) \ ,$$

com $A_1 \ll r_1$ e $A_2 \ll r_2$.

a Demonstre que $m_1 A_1 = m_2 A_2$.

b Demonstre que para as acelerações $a_1(t) = d^2 x_1(t)/dt^2$ e $a_2(t) = d^2 x_2(t)/dt^2$ se verifica a seguinte relação:

$$m_1 a_1(t) = m_2 a_2(t) = C_{\mathrm{el}} \left(x_1(t) + x_2(t) \right) \ .$$

c Demonstre que a frequência angular é dada por

$$\omega^2 = \frac{C_{\mathrm{el}}}{\mu} \ ,$$

onde a *massa reduzida* μ é definida por

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \ .$$

d Utilizando uma amplitude A definida por $\mu A = m_1 A_1 = m_2 A_2$, demonstre que a energia cinética total das duas massas é dada por

$$E_{\mathrm{cin}} = \frac{1}{2} \mu \, \omega^2 A^2 \cos^2(\omega t) \ .$$

e Demonstre que a energia potencial total da mola é dada por

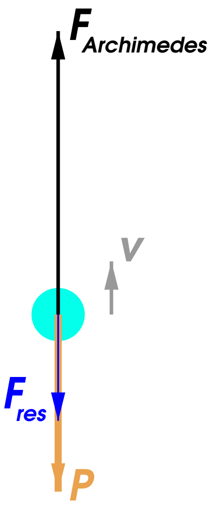
$$E_{\mathrm{el}} = \frac{1}{2} \mu \, \omega^2 A^2 \sin^2(\omega t) \ .$$

f Determine a energia total do sistema massas-mola.

Solutions

Exercício 1

a: For as far as we are concerned, there are in total three forces acting on the droplet: the gravitational force $P = -mg$, the Archimedes force (impulsão) A equal to the weight of the displaced water $A = V\rho_{\text{water}}g$ (V volume of the object $V = \frac{4}{3}\pi R^3$) and the resistance force of water against movement (arrasto) $-6\pi\mu Rv$. The latter force is downward, because the velocity of the droplet is upward.



b: When the droplet moves with a constant terminal velocity, the acceleration of the droplet equals zero. Consequently

$$P + F_{\text{res}} + F_{\text{Archimedes}} = F_{\text{total}} = ma = 0$$

We obtain the equation

$$-mg - 6\pi\mu R\, v_{\text{term}} + V_{\text{droplet}}\, \rho_{\text{water}}\, g = 0$$

From which we have

$$\begin{aligned}
v_{\text{term}} &= \frac{V_{\text{droplet}}\, \rho_{\text{water}}\, g - mg}{6\pi\mu R} = \frac{V_{\text{droplet}}\, \rho_{\text{water}}\, g - V_{\text{droplet}}\, \rho_{\text{substance}}\, g}{6\pi\mu R} \\
&= \frac{V_{\text{droplet}}\, g\, \left(\rho_{\text{water}} - \rho_{\text{substance}}\right)}{6\pi\mu R} = \frac{\frac{4}{3}\pi R^3 g \left(\rho_{\text{water}} - \rho_{\text{substance}}\right)}{6\pi\mu R} \\
&= \frac{2R^2 g \left(\rho_{\text{water}} - \rho_{\text{substance}}\right)}{9\mu} \\
&= \frac{2\left(90 \times 10^{-6}\ \text{m}\right)^2 \left(9.8\ \text{m/s}^2\right) \left(\left(1.00 \times 10^3\ \text{kg/m}^3\right) - \left(0.85 \times 10^3\ \text{kg/m}^3\right)\right)}{9\left(1.00 \times 10^{-3}\ \text{kg/ms}\right)}
\end{aligned}$$

$$= \frac{2\left(0.81\times 10^{-8}\ \mathrm{m}^2\right)\left(9.8\ \mathrm{m/s}^2\right)\left(0.15\times 10^{3}\ \mathrm{kg/m}^3\right)}{\left(9.00\times 10^{-3}\ \mathrm{kg/ms}\right)} = 0.265\times 10^{-2}\ \mathrm{m/s} = 2.65\ \mathrm{mm/s}$$

c:

$$V_{\text{droplet}} = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left(90\times 10^{-6}\ \mathrm{m}\right)^3 = 3.054\times 10^{-12}\ \mathrm{m}^3$$

$$\begin{aligned} |P| &= mg = V_{\text{droplet}}\,\rho_{\text{substance}}\,g \\ &= \left(3.054\times 10^{-12}\ \mathrm{m}^3\right)\left(0.85\times 10^{3}\ \mathrm{kg/m}^3\right)\left(9.8\ \mathrm{m/s}^2\right) = 2.54\times 10^{-8}\ \mathrm{N} \end{aligned}$$

$$\begin{aligned} |F_{\text{Archimedes}}| &= V_{\text{droplet}}\,\rho_{\text{water}}\,g \\ &= \left(3.054\times 10^{-12}\ \mathrm{m}^3\right)\left(1.00\times 10^{3}\ \mathrm{kg/m}^3\right)\left(9.8\ \mathrm{m/s}^2\right) = 2.99\times 10^{-8}\ \mathrm{N} \end{aligned}$$

$$\begin{aligned} |F_{\text{res}}| &= 6\pi\mu R\,v_{\text{term}} \\ &= 6\pi\left(1.00\times 10^{-3}\ \mathrm{kg/ms}\right)\left(90\times 10^{-6}\ \mathrm{m}\right)\left(0.265\times 10^{-2}\ \mathrm{m/s}\right) = 0.45\times 10^{-8}\ \mathrm{N} \end{aligned}$$

Exercício 2

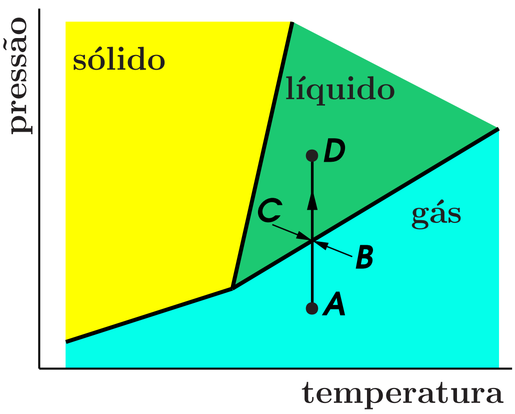
a: Since the process is isothermic one has the ideal-gas relation (Boyle):

$$PV = nRT = \text{constant} = X \quad\Longleftrightarrow\quad P(V) = \frac{X}{V}$$

The pressure P is inversely proportional to V.

b: When the substance condensates the pressure remains constant until no more gas or vapor is left. The volume reduces since gas takes more space than liquid.

c: In the phase diagram B and C represent the same state, because the pressure is the same in B as in C.



Nevertheless, in B one has only gas, in C only liquid.

d: One may increase the pressure, but it does only marginally effect the volume of a liquid. Hence, the volume remains constant in the process $C \to D$.

Exercício 3

a: The temperatures can be obtained from the ideal-gas law:

$$PV = nRT \quad \Longleftrightarrow \quad T_{11} = \frac{V_1 P_1}{nR}$$

with $T_{11} = 300$ K. The number of moles n and the gas constant R are constant during the process. We find then

$$T_{12} = \frac{V_1 P_2}{nR} = \frac{V_1 3P_1}{nR} = 3\frac{V_1 P_1}{nR} = 3T_{11} = 900 \text{ K}$$

and

$$T_{21} = \frac{V_2 P_1}{nR} = \frac{2.5 V_1 P_1}{nR} = 2.5\frac{V_1 P_1}{nR} = 2.5T_{11} = 750 \text{ K}$$

b: Here, the difficulty is to evaluate the work done by the engine. But, we know that the total effective work $W_{\text{effective}}$ performed by the engine in the complete cycle, is equal to the enclosed area. Consequently,

$$W_{\text{effective}} = \frac{1}{2}(V_2 - V_1)(P_2 - P_1) = \frac{1}{2}(1.5V_1)(2P_1) = 1.5P_1V_1$$

Furthermore, in the isochoric process $(V_1, P_1) \to (V_1, P_2)$ no work is performed, since the volume is constant, whereas, in the isobaric process $(V_2, P_1) \to (V_1, P_1)$ the engine recieved work $W_{21\to 11}$ from the exterior, given by

$$W_{21\to 11} = P_1(V_1 - V_2) = P_1(-1.5V_1) = -1.5P_1V_1$$

So, the work $W_{12\to 21}$ performed by the engine in the process $(V_1, P_2) \to (V_2, P_1)$ is given by

$$W_{12\to 21} = W_{\text{effective}} - W_{21\to 11} = 3P_1V_1$$

The kinetic energy decreases in the process $(V_1, P_2) \to (V_2, P_1)$ because the initial temperature (900 K) is larger than the final temperature (750 K). The difference in kinetic energy K equals

$$K_{12\to 21} = K_{21} - K_{12} = \frac{3}{2}V_2P_1 - \frac{3}{2}V_1P_2 = \frac{3}{2}2.5V_1P_1 - \frac{3}{2}V_1 3P_1 = -0.75P_1V_1$$

The heat $Q_{12\to 21}$ which is consumed by the engine in the process $(V_1, P_2) \to (V_2, P_1)$, is thus given by

$$Q_{12\to 21} = W_{12\to 21} + K_{12\to 21} = 3P_1V_1 - 0.75P_1V_1 = 2.25P_1V_1$$

c: In the isochoric process $(V_1, P_1) \to (V_1, P_2)$ the engine also consumes heat, given by

$$Q_{11\to 12} = K_{11\to 12} = K_{12} - K_{11} = \frac{3}{2}V_1P_2 - \frac{3}{2}V_1P_1 = 3P_1V_1$$

Hence, the total heat consumption by the engine equals

$$Q_{\text{total}} = Q_{11\to 12} + Q_{12\to 21} = 5.25P_1V_1$$

The efficiency is thus easily determined

$$\text{efficiency} = \frac{W_{\text{effective}}}{Q_{\text{total}}} = \frac{1.5P_1V_1}{5.25P_1V_1} = 0.29$$

Some extra information (not necessary for the exam):

The result $W_{12\to21} = 3P_1V_1$ can be obtained directly from the relation of the pressure P in function of the volume V for the process $(V_1, P_2) \to (V_2, P_1)$. That relation is here given by

$$P(V) = \frac{P_1}{3}\left(13 - 4\frac{V}{V_1}\right)$$

You can easily verify the above relation by checking $P(V_1) = P_1(13-4)/3 = 3P_1 = P_2$ and $P(V_2) = P_1(13-4V_2/V_1)/3 = P_1(13-4\times2.5)/3 = P_1$, which are indeed the values of the pressure at respectively $V = V_1$ and $V = V_2$ along the line from (V_1, P_2) to (V_2, P_1). Now, the work along the line follows from

$$W_{12\to21} = \int_{V_1}^{V_2} dV\ P(V) = \int_{V_1}^{2.5V_1} dV\ \frac{P_1}{3}\left(13 - 4\frac{V}{V_1}\right)$$

The integral is not difficult to be evaluated. One finds

$$\begin{aligned}
W_{12\to21} &= \int_{V_1}^{2.5V_1} dV\ \frac{P_1}{3}\left(13 - 4\frac{V}{V_1}\right) = \frac{P_1}{3}\left[13V - 2\frac{V^2}{V_1}\right]_{V_1}^{2.5V_1} \\
&= \frac{P_1}{3}\left\{\left(13\times2.5V_1 - 2\frac{(2.5V_1)^2}{V_1}\right) - \left(13\times V_1 - 2\frac{(V_1)^2}{V_1}\right)\right\} \\
&= \frac{P_1}{3}\{32.5 - 12.5 - 13 + 2\}\ V_1 = 3P_1V_1
\end{aligned}$$

Exercício 4

process	formula	value 10^3 J
18^oC $\to$ 0^oC	(18 K) (0.2 kg) (4.18×10^3 J/kg K)	15.05×10^3 J
water $\to$ ice	(0.2 kg) (3.34×10^5 J/kg)	66.80×10^3 J
0^oC $\to$ -20^oC	(20 K) (0.2 kg) (2.11×10^3 J/kg K)	8.44×10^3 J
total		90.3×10^3 J

Exercício 5

a: From Bernoulli's principle we obtain (H represents the height of the water level in the reservoir, h the height of the water hose.)

$$\frac{1}{2}\rho\, v_0^2 = P = \rho\, g(H-h) \quad\Longleftrightarrow\quad v_0^2 = 2g(H-h)$$

$$v_0^2 = 2\left(9.8\ \text{m/s}^2\right)(7.00\ \text{m} - 70\ \text{cm}) = 123.48\ (\text{m/s})^2 \quad \Longleftrightarrow \quad v_0 = 11.11\ \text{m/s}$$

$$\text{flow} = v_0 \times \text{section} = (11.11\ \text{m/s})\left(2.3\ \text{cm}^2\right) = 25.6 \times 10^{-4}\ \text{m}^3/\text{s} = 2.56\ \text{litres/s} = 153\ \text{litres/min}$$

b:

$$v_{0x} = v_0 \cos(60^o) \quad \text{and} \quad v_{0y} = v_0 \sin(60^o)$$

$$y(t) = h + v_{0y}t - \frac{1}{2}gt^2$$

If the water takes a time $t = \tau$ to reach the ground, then

$$h + v_{0y}\tau - \frac{1}{2}g\tau^2 = y(\tau) = 0 \quad \Longleftrightarrow \quad \tau = \frac{1}{g}\left(v_{0y} + \sqrt{v_{0y}^2 + 2gh}\right)$$

In the horizontal direction one has

$$x(t) = v_{0x}t$$

Hence the distance where the water reaches the ground is given by

$$x(\tau) = v_{0x}\tau = \frac{v_{0x}}{g}\left(v_{0y} + \sqrt{v_{0y}^2 + 2gh}\right)$$

Substitution of all the previous information gives

$$x(\tau) = v_{0x}\tau = 11.3\ \text{m}$$

Exercício 6

We search for the expression

$$u(t) = A\sin(\omega t) \quad \text{with} \quad \omega^2 = \frac{C_{\text{elastic}}}{m}$$

The amplitude is probably given by $A = 2$ cm, whereas the mass equals 0.42 kg. We need to know the elasticity constant C_{elastic}, which can be obtained from the equilibrium position of the spring-mass system, since it is given how much the spring extended to reach that position

$$C_{\text{elastic}} \times \text{extension} = mg \quad \Longleftrightarrow \quad \omega^2 = \frac{g}{\text{extension}} = \frac{9.8\ \text{m/s}^2}{9.8\ \text{cm}} = 100/\text{s}^2$$

We find then

$$u(t) = 2.0\sin(10\,t)\ \text{cm}$$

Exercício 7

In the lectures we derived that the maxima and minima can be found from (L distance slits-screen, x distance from center on the screen, d distance slits, λ wavelength light.)

$$\cos\left(\pi\frac{xd}{\lambda L}\right)$$

which gives maxima for

$$\frac{xd}{\lambda L} = 0,\ \pm 1,\ \pm 2, \ldots$$

Hence, the distance between maxima is given by

$$\frac{xd}{\lambda L} = 1 \quad \Longleftrightarrow \quad x = \frac{\lambda}{d} L$$

For blue light

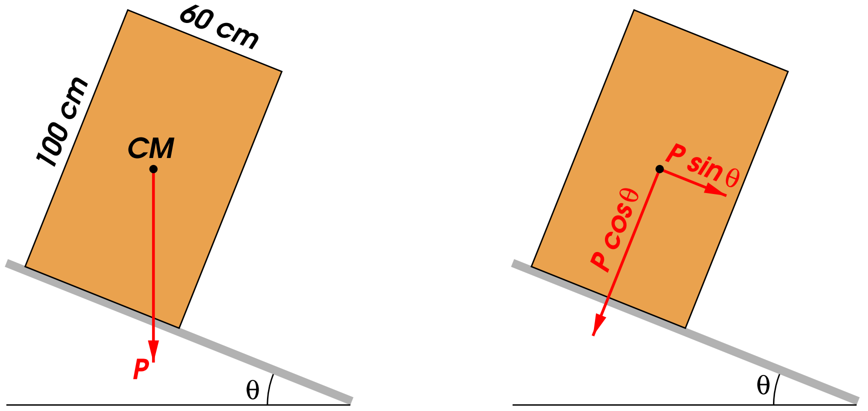
$$\lambda = 0.475\ \mu\text{m} \quad \Longleftrightarrow \quad x = \frac{0.475\ \mu\text{m}}{20\ \mu\text{m}} (5.0\ \text{m}) = 0.12\ \text{m}$$

For red light

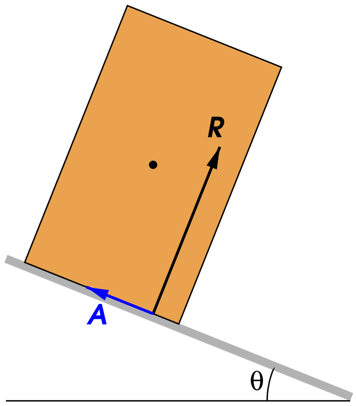
$$\lambda = 0.650\ \mu\text{m} \quad \Longleftrightarrow \quad x = \frac{0.650\ \mu\text{m}}{20\ \mu\text{m}} (5.0\ \text{m}) = 0.16\ \text{m}$$

Exercício 8

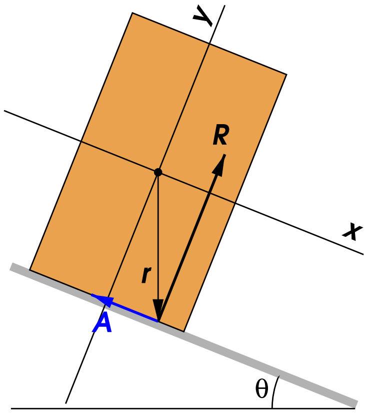
Let us first study the forces and torques which work at the cylindrical box (barrel) when the ship makes an angle θ with the horizontal.



The weight P acts in the center of mass (CM) of the barrel. In the righthand-side figure, we show the components of P in the directions parallel and perpendicular to the deck of the ship. The parallel component has magnitude $P\sin(\theta)$, whereas the perpendicular component has magnitude $P\cos(\theta)$.



The reaction force of the deck of the ship is perpendicular to the surface of the deck and acts vertically below the center of mass. Its magnitude equals the magnitude of the perpendicular component of P. Hence, $|R| = P\cos(\theta)$. The friction force A is parallel to the surface. Its magnitude equals the magnitude of the parallel component of P. Hence, $|R| = P\sin(\theta)$.



The perpendicular component of P and the reaction force R result in a torque which would rotate the barrel counterclockwise, whereas the parallel component of P and the friction force give a torque which would rotate the barrel clockwise.
When we take the coordinate system as indicated in the above figure, we calculate for the respective torques:

$$\vec{r} \times \vec{R} = \begin{pmatrix} (50\text{ cm})\tan(\theta) \\ 50\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ P\cos(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (50\text{ cm})P\sin(\theta) \end{pmatrix}$$

and

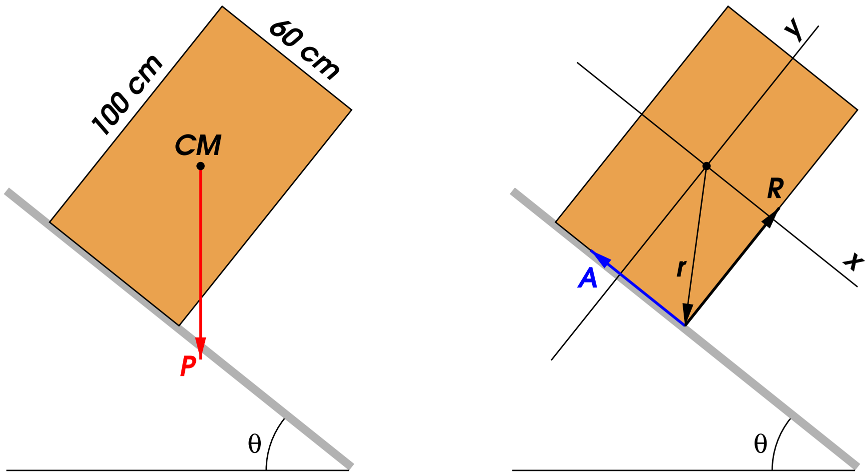
$$\vec{r} \times \vec{A} = \begin{pmatrix} (50\text{ cm})\tan(\theta) \\ 50\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} P\sin(\theta) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(50\text{ cm})P\sin(\theta) \end{pmatrix}$$

The torque $\vec{r} \times \vec{R}$ is in the positive z direction because the rotation direction is counterclockwise.
The torque $\vec{r} \times \vec{A}$ is in the negative z direction because the rotation direction is clockwise.
The sum of the torques $\vec{r} \times \vec{R} + \vec{r} \times \vec{A}$ equals zero. Hence there is no net torque in this case. The only movement which the barrel could eventually make is sliding. That depends on the maximum possible friction, given by

$$A_{\text{max}} = 0.75\ |R|$$

If $|A| \leq A_{\text{max}}$ then nothing will happen. But, if the angle is such that $P\sin(\theta) > A_{\text{max}}$, then the barrel starts sliding because, in that case, $|A| = A_{\text{max}} < P\sin(\theta)$, hence the forces are not in equilibrium.

When the angle increases and the barrel does not start sliding, we may imagine that we could arrive at the situation shown in the figures below.



However, here the reaction force cannot come below the center of mass of the barrel. It can at most act in the border of the barrel. Consequently, the application vector $\vec{r}$ is now different. When we take the coordinate system as indicated in the above figures, we calculate for the respective torques:

$$\vec{r} \times \vec{R} = \begin{pmatrix} 30 \text{ cm} \\ 50 \text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ P\cos(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (30 \text{ cm})P\cos(\theta) \end{pmatrix}$$

and

$$\vec{r} \times \vec{A} = \begin{pmatrix} 30 \text{ cm} \\ 50 \text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} P\sin(\theta) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(50 \text{ cm})P\sin(\theta) \end{pmatrix}$$

Those torques can only be in equilibrium if

$$(30 \text{ cm})P\cos(\theta) = (50 \text{ cm})P\sin(\theta) \qquad \Longleftrightarrow \qquad \tan(\theta) = \frac{30 \text{ cm}}{50 \text{ cm}} = 0.6$$

For $\tan(\theta) = 0.6$ the center of mass of the barrel is just vertically above the lower right border edge of the barrel. Hence, when in that situation the friction force A is still smaller than the maximum friction force A_{max}, the barrel can only tumble.
When $\tan(\theta) = 0.6$ and $|A| < A_{\text{max}}$, one has furthermore that

$$0.75 = \frac{A_{\text{max}}}{|R|} > \frac{|A|}{|R|} = \frac{P\sin(\theta)}{P\cos(\theta)} = \tan(\theta) = 0.6$$

Hence the criterium for tumbling is

$$\text{static friction coefficient } > 0.6$$

where 0.6 stems from the dimensions of the barrel.

Exercício 9

After rolling downhill a distance s the height of the wheel has changed by $h = s\sin(\theta)$, whereas its velocity is given by $v = ds/dt$. Its gain in kinetic energy is related to its loss in gravitational potential energy by

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = E_{\text{kin}} = E_{\text{pot}} = mgh$$

The relation between ω and v is given by

$$v = \omega R \quad \Longleftrightarrow \quad I\omega^2 = mR^2 \left(\frac{v}{R}\right)^2 = mv^2$$

So we have the relation

$$mv^2 = E_{\text{kin}} = E_{\text{pot}} = mgh \quad \Longleftrightarrow \quad v^2 = gh$$

We take the time derivative on both sides of the above equation (a stands for the acceleration of the wheel).

$$2va = 2v\frac{dv}{dt} = \frac{dv^2}{dt} = \frac{d\{gh\}}{dt} = \frac{d\{gs\sin(\theta)\}}{dt} = g\sin(\theta)\frac{ds}{dt} = g\sin(\theta)v$$

Hence

$$a = \frac{1}{2}g\sin(\theta)$$

Exercício 10

a: Since it is given that $m_1(r_1 + x_1) = m_2(r_2 + x_2)$ and $m_1 r_1 = m_2 r_2$, we have

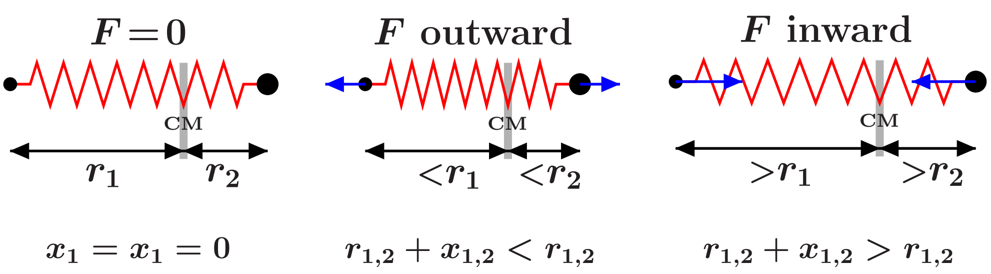
$$m_1 A_1 \sin(\omega t) = m_1 x_1 = m_1(r_1 + x_1) - m_1 r_1 = m_2(r_2 + x_2) - m_2 r_2 = m_2 x_2 = m_2 A_2 \sin(\omega t)$$

By dividing the lefthand side and the righthand side by $\sin(\omega t)$, one obtains $m_1 A_1 = m_2 A_2$.

b: The first part is easy:

$$m_1 a_1 = m_1 \frac{d^2 x_1}{dt^2} = \frac{d^2 m_1 x_1}{dt^2} = \frac{d^2 m_2 x_2}{dt^2} = m_2 \frac{d^2 x_2}{dt^2} = m_2 a_2$$

For the next part, we must consider the forces on both sides of the spring. Those forces must be equal and opposite: F and $-F$. Otherwise, the spring as a whole would start moving and thus the CM would move. But, we consider here the case that the CM does not move.



On one side the spring has an extension with respect to the equilibrium position, which equals x_1. On the other side that extension equals x_2. In total the string is streched, or shrinked, by an amount $x_1 + x_2$. So, according to Hooke's law, the elastic force which opposes deformation, equals

$$m_1 a_1 = m_2 a_2 = F = -C_{\text{el}}(x_1 + x_2)$$

c:

$$
\begin{aligned}
-m_1\omega^2 x_1 &= m_1\frac{d^2x_1}{dt^2} = m_1 a_1 = -C_{\text{el}}\,(x_1+x_2) \\
&= -C_{\text{el}}\left(x_1+\frac{m_1}{m_2}x_1\right) = -C_{\text{el}}\,\frac{m_1+m_2}{m_2}\,x_1
\end{aligned}
$$

Hence,

$$
\omega^2 = C_{\text{el}}\,\frac{m_1+m_2}{m_1 m_2} = \frac{C_{\text{el}}}{\mu}
$$

d:

$$
v_1 = \frac{dx_1}{dt} = \omega A_1\cos(\omega t) \quad \text{and} \quad v_2 = \frac{dx_2}{dt} = \omega A_2\cos(\omega t)
$$

Hence,

$$
\begin{aligned}
E_{\text{cin}}(t) &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1\omega^2A_1^2\cos^2(\omega t) + \frac{1}{2}m_2\omega^2A_2^2\cos^2(\omega t) \\
&= \frac{1}{2}\omega^2\left(m_1A_1^2 + m_2A_2^2\right)\cos^2(\omega t) = \frac{1}{2}\omega^2\left(\frac{\mu^2}{m_1}A^2 + \frac{\mu^2}{m_2}A^2\right)\cos^2(\omega t) \\
&= \frac{1}{2}\omega^2\mu^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)A^2\cos^2(\omega t) = \frac{1}{2}\omega^2\mu^2\,\frac{m_1+m_2}{m_1m_2}A^2\cos^2(\omega t) \\
&= \frac{1}{2}\omega^2\mu^2\frac{1}{\mu}A^2\cos^2(\omega t) = \frac{1}{2}\,\mu\,\omega^2A^2\cos^2(\omega t)
\end{aligned}
$$

e: In exercise 20 we have learned that the elastic potential energy is given by (x_1+x_2 is the extension of the spring)

$$
\begin{aligned}
E_{\text{potencial elástico}}(t) &= \frac{1}{2}C_{\text{el}}\,(x_1+x_2)^2 = \frac{1}{2}C_{\text{el}}\left(x_1+\frac{m_1}{m_2}x_1\right)^2 = \frac{1}{2}C_{\text{el}}\left(1+\frac{m_1}{m_2}\right)^2 x_1^2 \\
&= \frac{1}{2}C_{\text{el}}\left(\frac{m_1+m_2}{m_2}\right)^2 A_1^2\sin^2(\omega t) = \frac{1}{2}\,\mu\,\omega^2\left(\frac{m_1+m_2}{m_2}\right)^2\left(\frac{\mu}{m_1}A\right)^2\sin^2(\omega t) \\
&= \frac{1}{2}\,\mu\,\omega^2\left(\frac{m_1+m_2}{m_1m_2}\right)^2\mu^2A^2\sin^2(\omega t) = \frac{1}{2}\,\mu\,\omega^2\left(\frac{1}{\mu}\right)^2\mu^2A^2\sin^2(\omega t) \\
&= \frac{1}{2}\,\mu\,\omega^2A^2\sin^2(\omega t)
\end{aligned}
$$

f:

$$
\begin{aligned}
E_{\text{total}}(t) &= E_{\text{cin}}(t) + E_{\text{potencial elástico}}(t) \\
&= \frac{1}{2}\,\mu\,\omega^2A^2\left(\cos^2(\omega t)+\sin^2(\omega t)\right) = \frac{1}{2}\,\mu\,\omega^2A^2
\end{aligned}
$$

Notice, that E_{total} does not depend on t. The total energy is conserved.