Física I, Engenharia Química (2012-2013)

1. Seja a velocidade instantânea $v(t)$ do lançamento vertical, dentro de um meio viscoso e perto da superfície da Terra, dada por

$$v = v(t) = v_{\rm term} \left\{ e^{-t/T} - 1 \right\} \tag{1}$$

onde $v_{\rm term}$ e T representam constantes positivas com unidades m/s e segundos, respectivamente.

a Demonstre que a derivada df/dx da função $f = f(x) = e^{-x}$ é dada por $df/dx = -e^{-x}$.

b Demonstre que a derivada dv/dt da velocidade instantânea na Eq. 1 é igual a

$$dv/dt = -g + Av \tag{2}$$

no caso $v_{\rm term} = -g/A$ e $T = -1/A$.

c Verifique numa calculadora que $1 - e^{-t/T} \approx t/T$ para valores de t tais que $|t| \ll T$. Em seguida e a partir deste resultado, tire uma conclusão sobre o tipo de movimento que se descreve com a velocidade instantânea na Eq. 1 no caso $|t| \ll T$.

d Verifique numa calculadora que $1 - e^{-t/T} \approx 1$ para valores de t tais que $t \gg T$. Em seguida e a partir deste resultado, tire uma conclusão sobre o tipo de movimento que se descreve com a velocidade instantânea na Eq. 1 no caso $t \gg T$.

2. Para objectos esféricos de massa m e raio r, que se movem com velocidade v dentro de um meio viscoso cuja constante de viscosidade se representa por C_1, a força de resistência F_{res} é dada por

$$F_{\text{res}} = -C_1 r v \tag{3}$$

Numa experiência medem-se as velocidades terminais de esferas sólidas de metal que caem livremente dentro de um meio viscoso. As esferas vêm em quatro variedades diferentes, que se distinguem pelos seus diâmetros. São determinados os intervalos de tempo Δt que cada uma das esferas demora para se deslocar sobre uma distância de quatro centimetros dentro do meio viscoso. A tabela a seguir mostra os resultados (Walter Lewin, Physics 801, Lecture 12 "Resistive Forces", "http://www.youtube.com/watch?v=n2im2xIGxPo")

diâmetro (mm)	Δt (s)
3.175	5.66 - 5.93
3.962	3.80±0.10
4.775	2.69±0.20
6.350	1.40 - 1.68

A densidade do metal das esferas é igual a 7.85 g/cm^3 e a aceleração gravítica g = 9.8 m/s^2. Aplicando a fórmula da Eq. 2 do problema (1), determine a constante de viscosidade C_1 do meio viscoso para cada uma das esferas, bem como as incertezas nos resultados.

3. Seja a velocidade instantânea $v(t)$ do lançamento vertical para dentro de um meio viscoso e perto da superfície da Terra dada por

$$v = v(t) = \begin{cases} -\alpha \tan(t/T) & \text{para} \quad -\pi T/2 < t < 0 \\ -\alpha \tanh(t/T) & \text{para} \quad 0 < t \end{cases} \tag{4}$$

onde α e T representam constantes positivas com unidades m/s e segundos, respectivamente.

a Demonstre que a derivada df/dx da função $f = f(x) = \tan(x)$ é dada por $df/dx = 1 + \tan^2(x)$.

b A função $f = f(x) = \tanh(x)$ define-se por

$$\tanh(x) = \frac{e^{+x} - e^{-x}}{e^{+x} + e^{-x}}$$

Demonstre que a derivada df/dx da função $f = f(x) = \tanh(x)$ é dada por $df/dx = 1 - \tanh^2(x)$.

c Demonstre que a derivada dv/dt da velocidade instantânea na Eq. 4 é igual a

$$dv/dt = -g \pm |B|\, v^2 \tag{5}$$

no caso $\alpha = \sqrt{g/|B|}$ e $T = 1/\sqrt{g\,|B|}$

d Verifique numa calculadora que $\tan(t/T) \approx \tanh(t/T) \approx t/T$ para valores de t tais que $|t| \ll T$. Em seguida e a partir deste resultado, tire uma conclusão sobre o tipo de movimento que se descreve com a velocidade instantânea na Eq. 4 no caso $|t| \ll T$.

e Verifique numa calculadora que $\tanh(t/T) \approx 1$ para valores de t tais que $t \gg T$. Em seguida e a partir deste resultado, tire uma conclusão sobre o tipo de movimento que se descreve com a velocidade instantânea na Eq. 4 no caso $t \gg T$.

4. Para objectos esféricos de massa m e raio r, que se movem com velocidade v dentro de água cuja constante de viscosidade se representa por C_2, a força de resistência $F_{\rm res}$ é dada por

$$F_{\rm res} = \pm C_2 r^2 v^2 \ . \qquad (6)$$

Numa experiência medem-se as velocidades terminais de esferas sólidas de metal que caem livremente para dentro da água. As esferas vêm em cinco variedades diferentes, que se distinguem pelos seus diâmetros. A tabela a seguir mostra os resultados (J. P. Owen and W. S. Ryu, European Journal of Physics 26, 1085 (2005), "http://iopscience.iop.org/0143-0807/26/6/016")

raio (mm)	$v_{\rm term}$ (m/s)
0.80	0.62±0.02
1.18	0.72±0.04
1.60	0.86±0.03
2.37	1.01±0.04
3.16	1.17±0.05

A densidade do metal das esferas é igual a 8.02 g/cm^3 e a aceleração gravítica $g = 9.8$ m/s^2. Aplicando a fórmula da Eq. 5 do exercício (3), determine a constante de viscosidade C_2 de água para cada uma das esferas, bem como as incertezas nos resultados.

5. No problema 3, considere-se que a aceleração dv/dt de um objecto caindo verticalmente para dentro de um meio viscoso e perto da superfície da Terra é dada por (Eq. 5)

$$dv/dt = -g + |B|\, v^2 \ .$$

No entanto, uma fórmula mais completa também contempla a impulsão do meio viscoso

$$\begin{aligned} m\, dv/dt &= -mg + m\,|B|\, v^2 + F_{\text{Arquimedes}} \\ &= -mg + m\,|B|\, v^2 + V \rho_{\text{fluido}} g \ , \end{aligned} \tag{7}$$

onde m representa a massa do objecto, V o seu volume e ρ_{fluido} a densidade do meio viscoso.

a Demonstre que esta complementação da fórmula inicial pode ser implementada da seguinte forma:

$$dv/dt = -g' + |B|\, v^2 \ , \tag{8}$$

onde g' é dado por

$$g' = g \frac{\rho_{\text{objecto}} - \rho_{\text{fluido}}}{\rho_{\text{objecto}}} \ . \tag{9}$$

Determine a constante de viscosidade C_2 de água do problema 4 com esta última fórmula.

b Numa experiência medem-se as velocidades v em função do tempo t das esferas sólidas de metal de raio 2.37 mm do problema 4, quando caem livremente para dentro da água. A tabela a seguir mostra os resultados (J. P. Owen and W. S. Ryu, European Journal of Physics 26, 1085 (2005), “http://iopscience.iop.org/0143-0807/26/6/016”):

instante (s)	$\|v\|$ (m/s)
0.000	0.00±0.01
0.033	0.33±0.05
0.067	0.49±0.05
0.099	0.66±0.05
0.134	0.77±0.05
0.166	0.83±0.05
0.199	0.89±0.06
0.232	0.95±0.06
0.264	0.98±0.06
0.298	1.01±0.06

Utilizando a fórmula da velocidade instantânea $v(t)$ do lançamento vertical para dentro de um meio viscoso e perto da superfície da Terra, dada em Eq. 4 do exercício (3), determine as velocidades teóricas nos vários instantes de tempo da tabela e compare-as com os valores obtidos na experiência.

6. Considere uma placa de madeira de pinheiro com uma área de 6.3 m^2 e uma altura de 34.0 cm que flutuava numa piscina. A densidade do pinho é igual a 0.58 g/cm^3.

a Qual a altura da parte da placa que se encontrava acima da superfície da água da piscina?

Com a ajuda de uma grua colocou-se uma pedra por cima da placa de madeira que assim, embora ainda flutuasse, ficou completamente submergida na água.

b Qual a massa da pedra?

O interior da piscina tem uma área de 14.0 m^2. Antes de soltar a pedra da grua, foi marcada a altura da água que se encontrava dentro da piscina. No entanto, por um descuido na desmontagem, a placa mais a pedra capotaram e a pedra afundou-se. Verificou-se que a água da piscina, depois de se acalmar, estava 4.00 cm abaixo da marca acima mencionada.

c Qual a densidade da pedra?

7.

a Um pedaço sólido de cobre flutua em mercúrio de tal forma que 65.46% do seu volume está mergulhado no mercúrio. A densidade do mercúrio é igual a 13.596 g/cm^3. Qual a densidade do cobre?

b Uma esfera oca de cobre flutua em mercúrio de tal forma que 5.13% do seu volume está mergulhado no mercúrio. Qual a fracção da esfera que corresponde à cavidade?

8. Considere uma gota de petróleo que se encontra no fundo de um poço de água à temperatura de 20^0C. A densidade do petróleo é igual a 0.80 g/cm^3.

a Considerando nula a velocidade inicial da gota de petróleo e desprezando a viscosidade da água, determine a equação que descreve a velocidade vertical $v(t)$ da gota e determine ainda o tempo que demora a gota de petróleo para viajar 1.5 km.

b Tomando em consideração a viscosidade da água (regime I) determine a equação que descreve a velocidade vertical $v(t)$ da gota.

c Determine a velocidade terminal de uma gota de petróleo com um diâmetro de 60 micrómetros na sua trajectória a caminho da superfície da água ($\mu = 1.00 \times 10^{-3}$ kg/ms a 20^0C e $C_1 = 6\pi\mu$).

d Determine ainda quanto tempo (em dias) demora a gota de petróleo até chegar à superfície da água a partir de uma profundidade de 1.5 km.

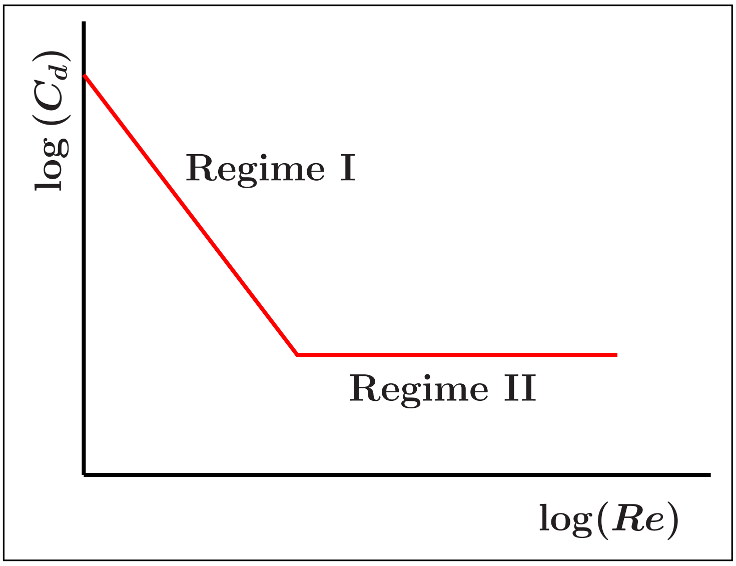
9. Seja o coeficiente de arrasto C_d (definido em relação à força de resistência F_{res}) para um fluido com densidade ρ_{fluido} e viscosidade μ em que um objecto com uma área S se desloca com velocidade v, dado por

$$F_{\text{res}} = \frac{1}{2} C_{\text{d}}\, \rho_{\text{fluido}}\, v^2 S$$

e seja o número de Reynolds Re dado por

$$\text{Re} = \rho_{\text{fluido}}\, vL/\mu$$

onde L representa o comprimento característico do objecto.



A figura mostra esquematicamente como, para uma esfera ($S = \pi r^2$ e $L = 2r$), os logaritmos dos coeficientes de arrasto C_d variam com os dos números de Reynolds Re.

a Demonstre que a relação dada por $C_d = 24/Re$ (para uma esfera em água) resulta numa recta inclinada no gráfico do logaritmo do coeficiente de arrasto C_d em função do logaritmo do número de Reynolds Re.

b Demonstre que para a força de resistência F_{res} a relação $C_d = 24/Re$ resulta na lei de Stokes dada por $F_{\text{res}} = 6\pi\mu r v$ (regime I) e determine ainda a relação entre o coeficiente de viscosidade C_1 e μ.

c Demostre que a relação dada por $C_d = 0.47$ (para uma esfera em água) resulta numa recta horizontal no gráfico do logaritmo do coeficiente de arrasto C_d em função do logaritmo do número de Reynolds Re.

d Demostre que para a força de resistência F_{res} a relação $C_d = 0.47$ resulta em $F_{\text{res}} = \frac{1}{2}\pi C_d \rho_{\text{fluido}} r^2 v^2$ (regime II) e determine ainda a relação entre os coeficientes de pressão C_2 e de arrasto C_d.

10. Nas aulas teóricas tratamos o assunto de *expansão térmica linear* de um objecto com a seguinte expressão (http://cft.fis.uc.pt/eef/FisicaI01/fluids/thermo02.htm).

$$L(T+\Delta T) \; - \; L(T) \; == \; \alpha L \Delta T \; , \tag{10}$$

onde α representa o coeficiente de expansão térmica linear (unidades K^{-1} ou C^{-1}) e onde $L(T)$ e $L(T+\Delta T)$ representam os comprimentos do objecto quando a sua temperatura é dada por T e $T+\Delta T$ respectivamente. No entanto, a partir da relação (10) podemos deduzir a seguinte equação diferencial.

$$\frac{dL(T)}{dT} = \lim_{\Delta T \to 0} \frac{L(T+\Delta T) - L(T)}{\Delta T} = \alpha L(T) \tag{11}$$

a Demonstre que a equação diferencial (11) é resolvida por

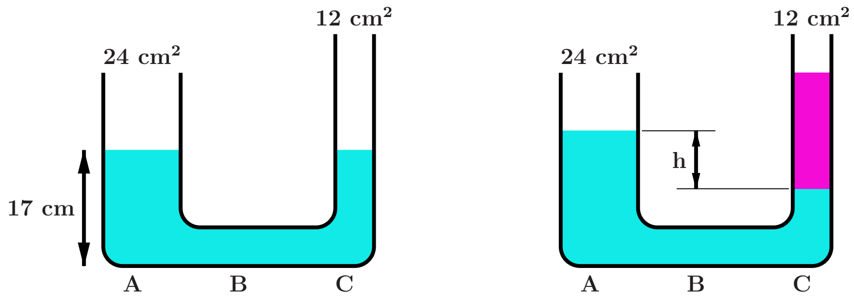
$$L(T) \; = \; L(0)\, e^{\alpha T} \tag{12}$$

onde $L(0)$ representa o comprimento do objecto a uma temperatura de 0^0C.

b Para um carril de comboio de aço ($\alpha = 14.3 \times 10^{-6}$ K^{-1}) com um comprimento de 10 km determine $L(50^0\mathrm{C}) - L(-20^0\mathrm{C})$ utilizando primeiro a fórmula (10) e depois a fórmula (12).

c A partir de uma aproximação da fórmula (12) para $\alpha \Delta T \ll 1$ explique o resultado da alínea **b**.

11. Considere um líquido de densidade 1.20 g/cm^3, colocado até uma altura de 17.0 cm em vasos comunicantes abertos em forma de U cujas secções têm, respectivamente, áreas de 24 cm^2 e 12 cm^2 (ver figura à esquerda).



a Determine a pressão no fundo do tubo em U nos pontos indicados por A, B e C.

A seguir colocam-se, no vaso com uma secção de 12 cm^2, 72 cm^3 de um outro líquido de densidade 0.80 g/cm^3 que não se mistura com o líquido que já se encontrava no tubo em U (ver figura à direita).

b Determine a diferença h das alturas do líquido que já se encontrava nos vasos e a pressão no fundo do tubo que liga os vasos nos pontos indicados por A, B e C.

12. Uma esfera oca de metal de raio 8.95 cm e de massa 6.00 kg (”sofar bomb”) resiste a uma pressão máxima 10^7 Pa.

a Qual a profundidade do mar (densidade 1.025 g/cm^3) onde esta esfera implode?

b Assumindo que podemos aplicar a fórmula (6) do problema 4 e ainda a constante de viscosidade C_2 do mesmo problema, determine o tempo que a esfera demora até chegar à profundidade da alínea (a) a partir da superfície do mar.

13. Uma sala fechada de 6 m de comprimento, 5 m de largura e 3 m de altura contem ar composto por 22% oxigénio e 78% nitrogénio. A temperatura da sala é igual a 20^0C e a pressão do ar é de 1 atm.

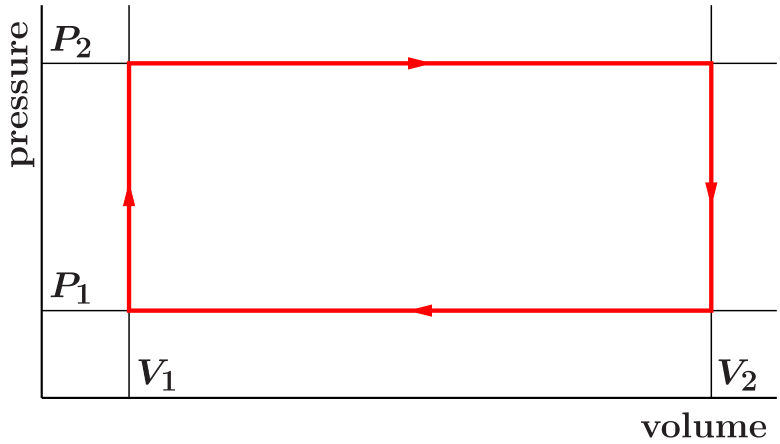
a Uma molécula de oxigénio que consegue evitar colisões com as outras moléculas move-se para a frente e para trás ao longo do comprimento da sala. Quantas vezes por segundo choca com uma das paredes?

b Quantas moléculas estão dentro da sala?

c Se ao longo de uma aula de Física a temperatura da sala aumentar para 27^0C, qual a pressão do ar na sala no final da aula e qual o calor fornecido ao ar?

14. A função principal do motor de um carro (ver a página de Motor_de_combustão_interna em pt.wikipedia.org) é transformar combustível em trabalho capaz de gerar movimento nas rodas. O essencial desse sistema é uma pequena câmara de combustão chamada cilindro. Os carros de passageiros normalmente têm quatro ou seis cilindros. Dentro de cada cilindro estão os pistões. A queima do combustível faz os pistões movimentarem-se, fazendo girar um eixo chamado cambota (http://pt.wikipedia.org/wiki/Cambota). Este eixo leva a energia mecânica até ao sistema de transmissão, que por sua vez distribui o trabalho produzido para as rodas. O resultado dessa reacção em cadeia é o movimento do carro.

Considere um carro de 1000 cc com quatro cilindros. Portanto o volume máximo de cada cilindro durante um ciclo, é de 0.25 litros. A figura seguinte mostra, de forma simplificada, o que se passa durante um ciclo de um cilindro do motor.



Considere os valores $V_1 = 0.1$ e $V_2 = 0.25$ litros e $P_1 = 1.0 \times 10^6$ e $P_2 = 3.0 \times 10^6$ Pascal.

a Se a temperatura no ponto (V_1,P_1) é igual a 300 K, qual a temperatura dos pontos (V_1,P_2), (V_2,P_2) e (V_2,P_1)?

b Determine as quantidades de calor fornecidas ao combustível na transformação isocórica $(V_1, P_1) \rightarrow (V_1, P_2)$ e na transformação isobárica $(V_1, P_2) \rightarrow (V_2, P_2)$.

c Determine as quantidades de calor fornecidas pelo combustível ao exterior (por exemplo ao interior do carro) na transformação isocórica $(V_2, P_2) \rightarrow (V_2, P_1)$ e na transformação isobárica $(V_2, P_1) \rightarrow (V_1, P_1)$.

d Qual o trabalho exercido pelo/sobre o combustível em cada uma das quatro transformações?

e Considere a eficácia de um motor definida como a razão entre o trabalho efectuado e o calor fornecido. Qual a eficácia do motor representado pela diagrama P-V em causa?

15.

a Utilizando os valores de V_1, P_1 e da temperatura deste caso como são dados no exercício (14), determine o número de moles de gás dentro do cilindro, em cada ciclo (constante universal dos gases perfeitos $R = 8.314$ J/K/mol).

b O gás que entra no cilindro do problema (14) em cada ciclo, consiste de uma mistura de n_a moles de ar e n_g moles de gasolina, na razão $n_g/n_a \approx 0.008$.
Considerando então que 0.8% do resultado da alínea anterior corresponde a gasolina e utilizando o resultado da alínea **b** do problema (14), determine o valor energético da gasolina em J/litro. A massa molar da gasolina é igual a 105 g/mol e a densidade da gasolina líquida a 740 kg/m^3.

c Caso os quatro cilindros trabalhem uma hora a 1500 rotações por minuto (rpm), qual o consumo de gasolina em litros?

Nota: no painel de indicadores do carro indicam-se 3000 rpm neste caso, porque na realidade os cilindros fazem duas rotações por ciclo, uma para esvaziar o cilindro dos gases queimados através do tubo de exaustão do carro e para o enchimento do cilindro com uma mistura de ar e vapor do combustível, e outra para o ciclo acima estudado.

d Qual a potência do carro em cavalos-vapor (1 cavalo-vapor = 750 Watt) no caso considerado na alínea **c**.

16. Um tubo de água que tem uma secção de 8 cm^2, está ligado a um tubo de água que tem uma secção de 2 cm^2. O sistema de tubos transporta 12.0 litros de água por minuto. Se a pressão da água no tubo mais largo for igual a 1.00×10^5 Pa, qual a pressão da água no tubo mais estreito?

17. Uma casa tem um tecto plano horizontal de 10 m^2 que consiste de uma placa de 10 cm de altura. A pressão do ar em repouso dentro da casa é igual a 1.00×10^5 Pa. A temperatura do ar fora da casa é igual à temperatura do ar dentro da casa. No entanto o tempo fora da casa está bastante desagradável com rajadas de vento com velocidades até 180 km/h. A densidade de ar é igual a 1.29 kg/m^3.

a Qual a força sobre o tecto devida à pressão do ar no interior da casa?

b Qual a diferença da pressão do ar sobre o tecto entre o interior e exterior da casa, quando a velocidade do vento é horizontal e igual a 180 km/h?

c Qual a força resultante sobre o tecto se ele for de madeira de pinha (densidade 0.58 g/cm^3)?

d Qual a força resultante sobre o tecto se ele for de betão (densidade 2.4 g/cm^3)?

18. Um caçador acerta por engano com três tiros num tanque de água, resultando em três furos de 0.25 cm^2. O tanque está cheio de água até uma altura de 2.20 m acima da superfície de apoio. Os furos têm, respectivamente, alturas de 2.02 m, 1.63 m e 1.15 m.

a Qual a velocidade da água que sai em cada um dos furos?

b Qual a quantidade de água, em litros por minuto, que sai em cada um dos furos?

c Quais as distâncias, d, medidas a partir da base do tanque, onde os feixes de água tocam no chão?

d Demostre que as distâncias d da última alínea podem ser determinadas com a seguinte fórmula (s representa a altura da água dentro do tanque, h a altura do furo).

$$d^2 = 4(s - h)h$$

e Os resultados das alíneas **c** e **d** seriam diferentes, se o acontecimento tivesse ocorrido na Lua?

19. Um meio elástico ideal, está numa oscilação harmónica em torno da posição de equilíbrio com uma frequência de 500 Hz e tem uma velocidade de 0.50 m/s quando passa pela posição de equilíbrio. A constante da elasticidade do meio elástico é igual a 50000 N/m^2.

a Determine a amplitude da oscilação.

b Qual o valor da massa que oscila?

c Quais as energias cinéticas do meio elástico no instante em que passa pela posição de equilíbrio e no instante em que passa pela posição do deslocamento máximo.

d Demonstre que a diferença entre a energia cinética máxima do movimento oscilatório e a energia cinética num instante t em que o deslocamento da oscilação é dado por $u(t)$ é dada por

$$E_{\text{cin}}(\text{máxima}) - E_{\text{cin}}(t) = \frac{1}{2} m\omega^2 u^2(t)$$

20. Considere um meio elástico ideal com constante da elasticidade dada por C_{el}. Designa-se por u a distância do deslocamento a partir da situação do equilíbrio do meio elástico.

a Demostre que o trabalho ΔW externo que é preciso para aumentar o deslocamento de u para $u + \Delta u$ é dado por

$$\Delta W = W(u + \Delta u) - W(u) = F_{\text{el}}\Delta u = C_{\text{el}}u\Delta u$$

e ainda que a partir desta relação surge a seguinte equação diferencial.

$$\frac{dW}{du} = C_{\text{el}}u$$

b Demonstre que $W(u) = \frac{1}{2}C_{\text{el}}u^2$ é solução da equação diferencial da alínea **a**.

O trabalho $W(u)$ realizado por uma força externa sobre o meio elástico para concretizar um deslocamento u do meio elástico fica depois armazado no meio elástico, representando energia potencial $U_{\text{el}}(u)$ que pode ser devolvida ao exterior pelo meio elástico.

c Demonstre que no caso da oscilação considerada no exercício (19), a soma da energia potencial $U_{\text{el}}(u)$ com a energia cinética $E_{\text{cin}}(u)$ é constante e obtenha deste resultado uma conclusão sobre a conservação da energia neste caso.

21. Um objecto pontual de massa m pendurado por uma corda de comprimento ℓ e sem massa realiza, em torno da posição de equilíbrio, pequenas oscilações harmónicas descritas por

$$\alpha(t) = \alpha_{\text{max}} \sin(\omega t) \quad \text{com} \quad \alpha_{\text{max}} \ll 1.$$

onde α representa o ângulo entre a corda e a vertical.

a Utilizando a seguinte relação

$$\cos(\alpha) \approx 1 - \frac{1}{2}\alpha^2 \quad \text{para} \quad \alpha \ll 1.$$

demonstre que a altura $h(t)$ da posição da massa no instante t, acima da posição de equilíbrio, é dada por

$$h(t) = \frac{1}{2}\ell\alpha_{\text{max}}^2 \sin^2(\omega t)$$

b Demonstre que a soma da energia gravítica e da energia cinética da massa é independente do instante t.

22. Uma onda que se propaga num meio elástico é descrita pela seguinte equação

$$u(x,t) = (15 \text{ cm}) \sin(3.60x - 270t)$$

Determine a amplitude, a frequência angular, o número de onda, o comprimento de onda, a frequência e a velocidade da onda.

23. Uma flauta transversal pode ser considerada uma cavidade de ressonância com os dois lados abertos. Quando todos os furos estão tapados a flauta produz o dó central (262 Hz).

a Qual o comprimento da flauta transversal sabendo que a temperatura do ar é igual a 20° C e a velocidade de som no ar a esta temperatura é de 343 m/s.

b Qual a distância entre o lado onde se sopra e o primeiro furo aberto da flauta transversal para produzir um lá (440 Hz)?

24. Um condutor de um automóvel que circula numa estrada horizontal, com uma velocidade v_0, trava a fundo para evitar um acidente.

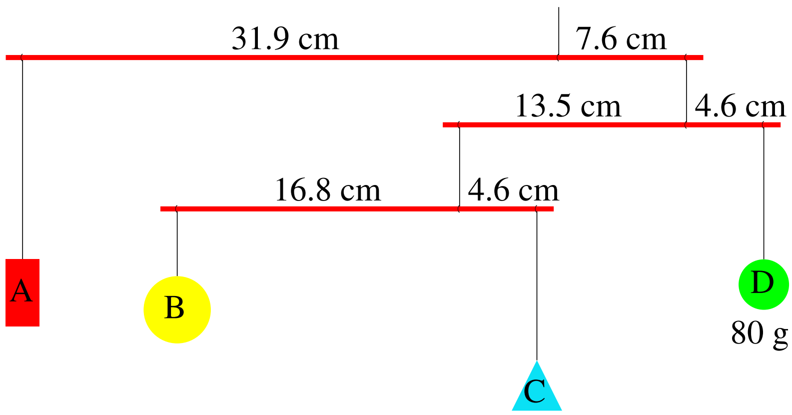
a Deduza uma expressão para a distância d da travagem, considerando o coeficiente de atrito dinâmico (ou cinético) representado por α.

b Para um valor de $\alpha = 0.50$ determine as distâncias d para os seguintes valores da velocidade inicial: 20, 50, 80 120, 140 e 180 km/h.

25. A saída do canhão de um lançador de berlindes de vidro localiza-se a uma altura de 1.00 m. Um berlinde, lançado horizontalmente, alcança 2.50 m. Determine o ângulo que o canhão tem de fazer com a horizontal para que os berlindes caiam num cesto cujo centro se encontra a uma distância horizontal de 1.65 m do lançador e a uma altura de 1.42 m? (http://www.youtube.com/watch?v=EZIb8nWiP78 e http://www.youtube.com/watch?v=ObjV0d6zoJ4)

26. Um carro está parado numa rampa móvel cuja inclinação é lentamente aumentada. Quando a inclinação da rampa atinge 37° com a horizontal o carro começa a deslizar para baixo. Determine o coeficiente de atrito estático entre o carro e a superfície da rampa.

27.



Um móbile é uma escultura abstrata, composta de materiais leves, suspensos no espaço por meio de fios e de barras, impulsionados pela força natural das correntes de ar. O móbile da figura acima está em equilíbrio. Os dados necessários estão indicados na figura. As massas dos fios e das barras são desprezáveis.

Determine as massas dos objectos A, B e C.

28. Uma barra de metal de 1.00 m de comprimento e com peso desprezável liga duas massas pontuais de 1.0 kg e 2.0 kg.

Determine o momento de inércia quando a barra gira em torno

a da massa de 2.0 kg.

b do ponto médio da barra.

c do centro de massa do sistema.

29. Uma barra de metal de 1.00 m de comprimento está numa posição horizontal, ligada no meio a um eixo vertical, de tal forma que pode girar livremente (sem atrito) em torno deste eixo. As massas da barra e do eixo são desprezáveis. Duas massas pontuais idênticas de 0.25 kg podem deslizar livremente sobre esta barra mas não podem sair da barra nos seus extremos. Inicialmente as duas massas estão presas por imans, que são montados dentro da barra a uma distância de 10.0 cm do eixo, ficando uma massa num lado e a outra massa no outro lado do eixo. Nesta situação faz-se girar a barra em torno do eixo até se atingir uma velocidade angular de $90°/s$. Depois o sistema não sofre a acção de mais forças externas.

a Determine a força que a barra excerce sobre uma das massas.

b Diga o que acontece quando o imam é desligado.

c Determine a velocidade angular final do sistema.

30. Uma viga de madeira está encostada contra uma parede sem atrito. O centro de massa da viga encontra-se no meio da viga. Numa experiência científica procura-se saber qual o ângulo mínimo que a viga pode fazer com a horizontal para que a viga não deslize. Determine o coeficiente de atrito estático entre a viga e o chão se este ângulo mínimo é igual a $32°$.

31. Considere uma barra delgada, com largura desprezável e comprimento ℓ. Pelo facto da massa m da barra estar distribuida homogeneamente ao longo do seu comprimento, o momento de inércia da barra é $I = \frac{1}{12}m\ell^2$ para rotações em torno de um eixo perpendicular à barra e que passa pelo seu centro de massa.
Inicialmente a barra está colocada verticalmente sobre um piso perfeitamente liso. No entanto, após uma pequena perturbação, a barra começa a caír.

a Considerando que um extremo da barra desliza livremente sobre o chão e desprezando a resistência do ar, demonstre que o centro de massa da barra cai verticalmente.

b Demonstre que a velocidade com que o outro extremo cai no chão é dada pela expressão $\sqrt{3g\ell}$, onde g representa a aceleração gravítica.

32. Considere um comboio de alta velocidade que emite um sinal sonoro de 55 Hz (A_1 na escala musical) e que se aproxima, com uma velocidade retilínio uniforme de 360 km/h, de uma passagem de nível onde se encontra uma pessoa (A) à espera junto das cancelas fechadas.

a. Determine a fracção Δx da distância $x - \Delta x$ da pessoa (A) onde o comboio se encontra quando emite o fim de uma oscilação completa do som, caso o comboio tenha emitido o início da oscilação completa em causa a uma distância x da pessoa (A).

b. Determine a diferença de tempo entre as chegadas no ouvido da pessoa (A) do início e do fim da oscilação completa referida na alínea **a**. A velocidade do som no ar é igual a 340 m/s.

c. Qual a frequência do som ouvido pela pessoa (A)? O seu valor significa que a pessoa (A) recebe um som mais agudo, mais grave ou igual ao som emitido?

Solutions

Exercício 1

a: The function e^x has the property that its derivative de^x/dx, its double derivative d^2e^x/dx^2, its triple derivative d^3e^x/dx^3, etcetera, are all equal to e^x (see http://cft.fis.uc.pt/eef/FisicaI01/-vertical/vibondas.pdf pages 7, 8 and 9).
It can furthermore be expanded in a series according to

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(see http://cft.fis.uc.pt/eef/FisicaI01/vertical/vibondas.pdf page 10).
A direct calculation of its derivative is shown at page 11 of http://cft.fis.uc.pt/eef/FisicaI01/-vertical/vibondas.pdf using the definition of derivative.
Here, we deal with the function $f = f(x) = e^{-x}$.
There exist several methods to determine its derivative.
First, we could observe that e^{-x} takes decreasing values for increasing values of x. As a consequence, its derivative must be negative.

1. $f = f(x) = e^{-x} = \left(e^x\right)^{-1} = \dfrac{1}{e^x}$

   One could define $h = h(x) = e^x$ and $f = h^{-1}$
   We find then (chain rule http://en.wikipedia.org/wiki/Chain_rule)

$$f' = \frac{df}{dx} = \left(\frac{df}{dh}\right)\left(\frac{dh}{dx}\right) = \left(-h^{-2}\right)\left(e^x\right) = -\left(e^x\right)^{-2}\left(e^x\right) = -\left(e^x\right)^{-1} = -e^{-x}$$

2. One could define $g = g(x) = -x$ and $f = e^g$
   We find then (chain rule http://en.wikipedia.org/wiki/Chain_rule)

$$f' = \frac{df}{dx} = \left(\frac{df}{dg}\right)\left(\frac{dg}{dx}\right) = \left(e^g\right)(-1) = \left(e^{-x}\right)(-1) = -e^{-x}$$

3. Directly from the definition of the derivative

$$f' = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{-(x+\Delta x)} - e^{-x}}{\Delta x}$$

   Now, $e^{-(x+\Delta x)} = e^{-x-\Delta x} = e^{-x}\, e^{-\Delta x}$.
   Hence

$$f' = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{e^{-x}\, e^{-\Delta x} - e^{-x}}{\Delta x} = \lim_{\Delta x \to 0} e^{-x}\, \frac{e^{-\Delta x} - 1}{\Delta x}$$

   For $e^{-\Delta x}$ we use the expansion of the previous page, which results in

$$e^{-\Delta x} = 1 + (-\Delta x) + \frac{(-\Delta x)^2}{2!} + \frac{(-\Delta x)^3}{3!} + \cdots$$

By substitution we obtain

$$f' = \frac{df}{dx} = \lim_{\Delta x \to 0} \; e^{-x} \; \frac{(-\Delta x) + \frac{(-\Delta x)^2}{2!} + \frac{(-\Delta x)^3}{3!} + \cdots}{\Delta x}$$

From the division by Δx we find

$$f' = \frac{df}{dx} = \lim_{\Delta x \to 0} \; e^{-x} \; \left\{ -1 + \frac{\Delta x}{2!} - \frac{(\Delta x)^2}{3!} + \cdots \right\}$$

Taking the limit gives the final result

$$f' = \frac{df}{dx} = e^{-x} \; \left\{ -1 + \frac{0}{2!} - \frac{0}{3!} + \cdots \right\} = -e^{-x}$$

b: We want to show that the expression

$$v = v(t) = v_{\text{term}} \left\{ e^{-t/T} - 1 \right\} = v_{\text{term}} e^{-t/T} - v_{\text{term}}$$

is a solution of the differential equation

$$dv/dt = -g + Av$$

for $v_{\text{term}} = -g/A$ and $T = -1/A$.
So, we start by taking the time-derivative of v, given by dv/dt.

$$\frac{dv}{dt} = \frac{d}{dt} \left\{ v_{\text{term}} e^{-t/T} - v_{\text{term}} \right\} = \frac{d}{dt} \left\{ v_{\text{term}} e^{-t/T} \right\} - \frac{d}{dt} \left\{ v_{\text{term}} \right\} = v_{\text{term}} \; \frac{d}{dt} \left\{ e^{-t/T} \right\} - 0$$

We define $h = h(t) = -\frac{t}{T} = -\frac{1}{T} \, t$.
Then, using the chain rule

$$\frac{d}{dt} \left\{ e^{-t/T} \right\} = \frac{d}{dt} \left\{ e^{h} \right\} = \left(\frac{d}{dh} \left\{ e^{h} \right\} \right) \left(\frac{dh}{dt} \right) = \left(e^{h} \right) \left(-\frac{1}{T} \right) = -\frac{1}{T} \; e^{h} = -\frac{1}{T} \; e^{-t/T}$$

We substitute that result in the expression for dv/dt:

$$\frac{dv}{dt} = v_{\text{term}} \; \left(-\frac{1}{T} \; e^{-t/T} \right) = -\frac{1}{T} \; v_{\text{term}} \; e^{-t/T} = -\frac{1}{T} \; (v + v_{\text{term}}) = -\frac{1}{T} \; (v(t) + v_{\text{term}})$$

Here, $v = v(t)$ varies with time t, whereas T and v_{term} are constants.
We thus found

$$\frac{dv}{dt} = -\frac{1}{T} \; (v(t) + v_{\text{term}}) = -\frac{v_{\text{term}}}{T} - \frac{1}{T} \, v(t)$$

We want that result to be equal to the differential equation

$$dv/dt = -g + Av = -g + Av(t)$$

where g and A are constants.
So we obtain the equation

$$-g + Av(t) = dv/dt = -\frac{v_{\text{term}}}{T} - \frac{1}{T} \, v(t)$$

which is equivalent to

$$Av(t) + \frac{1}{T}\, v(t) = g - \frac{v_{\text{term}}}{T} \quad \text{or} \quad \left\{A + \frac{1}{T}\right\}\, v(t) = \left\{g - \frac{v_{\text{term}}}{T}\right\}$$

In the equation on the righthand side $A + \frac{1}{T}$ and $g - \frac{v_{\text{term}}}{T}$ are constants. But $v(t)$ is a function which is different for different values of the variable t. However, the equation

$$\left\{A + \frac{1}{T}\right\}\, v(t) = \left\{g - \frac{v_{\text{term}}}{T}\right\}$$

must be true for any possible value of the variable t. Otherwise, $v(t)$ is NOT a solution of the equation $dv/dt = -g + Av$.
The ONLY possible way that a solution can be achieved for all possible values of t for the equation

$$\left\{A + \frac{1}{T}\right\}\, v(t) = \left\{g - \frac{v_{\text{term}}}{T}\right\}$$

is by setting

$$\left\{A + \frac{1}{T}\right\} = 0 \quad \text{and} \quad \left\{g - \frac{v_{\text{term}}}{T}\right\} = 0$$

because then one has

$$0 \times v(t) = 0$$

which is always true for any value of $v(t)$.
So we end up with

$$T = -\frac{1}{A} \quad \text{and} \quad v_{\text{term}} = -\frac{g}{A}$$

as the only possible way that

$$v(t) = v_{\text{term}} \left\{e^{-t/T} - 1\right\}$$

is a solution of the differential equation

$$dv/dt = -g + Av$$

Consequently, a solution of the differential equation $dv/dt = -g + Av$ is given by

$$v = v(t) = -\frac{g}{A}\, \left\{e^{At} - 1\right\}$$

However, we prefer to write

$$v = v(t) = v_{\text{term}} \left\{e^{-t/T} - 1\right\}$$

and remember that $v_{\text{term}} = -g/A$ and $T = -1/A$.
Notice that A is a negative constant, whereas T is a positive constant.
A MUST be a negative constant, because the expression Av represents the acelleration of the resistive force of a viscous medium which must always work against the velocity. Hence, for v positive Av must be negative, whereas for v negative Av must be positive.

c: We want to verify that $1 - e^{-t/T} \approx t/T$ for values of t such $|t| \ll T$ or $\dfrac{|t|}{T} \ll 1$ by the use of a calculator.

Let us choose several values for $\frac{t}{T}$ and calculate $1 - e^{-t/T}$:

t/T	$1 - e^{-t/T}$	difference (%)
-1.0	-1.718	71.8
-0.5	-0.6487	29.7
-0.1	-0.10517	5.17
-0.05	-0.051271	2.54
-0.01	-0.01005016	0.50
-0.005	-0.00501252	0.25
-0.001	-0.001000500	0.05
0	0	0
+0.001	0.000999500	0.05
+0.005	0.00498752	0.25
+0.01	0.009950	0.50
+0.05	0.04877	2.46
+0.1	0.09516	4.84
+0.5	0.393	21.3
+1.0	0.63212	36.8

We find that $1 - e^{-t/T} \approx t/T$ is a reasonable approximation for $-0.1T < t < +0.1T$ with differences of at most some 5%, whereas for $-0.01T < t < +0.01T$ the differences are at most 0.5%. In the interval $-0.001T < t < +0.001T$ the differences are even at most 0.05%.
The object reaches its maximum height for $t = 0$. Then its velocity vanishes and, consequently, also the friction force Av. Just before and just after reaching its maximum height the motion of the object is described by

$$v = v(t) = v_{\rm term} \left\{ e^{-t/T} - 1 \right\} \approx v_{\rm term} \left\{ -\frac{t}{T} \right\} = -\frac{v_{\rm term}}{T}\, t = -gt$$

That is the velocity for a freely falling object, $v = v(t) = v_0 - gt$ with $v_0 = v(t = 0) = 0$, without any frictional forces. It is exactly what we expected, since at $t = 0$ the velocity of our object vanishes and near $t = 0$ its velocity is very low, such that the frictional force Av can be neglected. In that case the differential equation for its motion is given by $dv/dt = -g$ which is the dynamical equation for a freely falling object near the surface of the Earth and which is solved by $v = v(t) = v_0 - gt$.

d: We want to verify that $1 - e^{-t/T} \approx 1$ for values of t such $t \gg T$ or $\frac{t}{T} \gg 1$ by the use of a calculator.

Let us choose several values for $\frac{t}{T}$ and calculate $1-e^{-t/T}$. The difference with 1 is given by $e^{-t/T}$

t/T	$1-e^{-t/T}$	$e^{-t/T}$
1.0	0.63	0.37
2.0	0.86	0.14
3.0	0.95	0.05
4.0	0.98	0.02
5.0	0.99	0.007
6.0	0.9975	0.002
7.0	0.9991	0.0009
8.0	0.9997	0.0003
9.0	0.9999	0.0001
10.0	0.99995	4.5×10^{-5}
20.0	0.999999998	2.1×10^{-9}
50.0	1.0	1.9×10^{-22}
100.0	1.0	3.7×10^{-44}

We find that $1-e^{-t/T}\approx 1$ is a reasonable approximation for $t>3T$ with differences of at most some 5%, whereas for $t>5T$ the differences are less than 0.7%. For $t>10T$ the differences are even smaller than 0.0045%.
Hence, for $t \gg T$ we may approximate $v=v(t)=v_{\text{term}}\left\{e^{-t/T}-1\right\}$ by

$$v=v(t)=-v_{\text{term}}$$

That is a constant downward velocity, the terminal velocity of the object. It is exactly what we expected, since the friction between the object and the medium causes a resistive acelleration Av, the modulus of which increases when the speed of the object increases. However, when the object is freely falling there only act 2 forces on it, namely the downward gravitational force and the upward resistive force of the medium. When, at a certain instant, these 2 forces cancel each other, the acceleration vanishes, hence the velocity does not increase anymore. From that instant on, the object continues to fall down with constant speed, given by v_{term}.

Exercício 2

Formula (2) of exercise (1) reads

$$dv/dt=-g+Av$$

for $v_{\text{term}}=-g/A$ and $T=-1/A$.

Hence, the resistive force, which follows from (m is the mass of the metal sphere)

$$mdv/dt=-mg+mAv$$

is here given by $F_{\text{res}}=mAv$.
In this exercise it is given by $F_{\text{res}}=-C_1rv$. So, evidently, $mA=-C_1r$.
For the terminal velocities that leads to

$$v_{\text{term}}=-g/A=mg/C_1r \tag{13}$$

Let us elaborate the formula

$$\begin{aligned} v_{\text{term}} &= \frac{mg}{C_1 r} = \frac{(\text{volume metal sphere}) \times (\text{density of the metal}) \times g}{C_1 r} \\ &= \frac{\left(\frac{4}{3}\pi r^3\right) \times \rho_{\text{metal}} \times g}{C_1 r} = \frac{4\pi r^2 \rho_{\text{metal}} g}{3C_1} = \frac{\pi d^2 \rho_{\text{metal}} g}{3C_1} \end{aligned}$$

where $d = 2r$ stands for the diameter of the metal sphere.
For C_1 follows thus

$$C_1 = \frac{\pi d^2 \rho_{\text{metal}}\ g}{3 v_{\text{term}}} \tag{14}$$

Now, not the terminal velocity is measured by the experiment, but the time Δt it takes a metal sphere to fall 4 cm in the viscous medium. Hence, $v_{\text{term}} = (0.04\ \text{m}) / \Delta t$. We substitute that in Eq. (14) to find

$$C_1 = \frac{\pi d^2 \rho_{\text{metal}}\ g \Delta t}{3 \times 0.04} \tag{15}$$

Here, we have the constants $\pi = 3.14159\ldots$, $\rho_{\text{metal}} = 7.85 \times 10^3$ kg/m^3 and $g = 9.8$ m/s^2, which can be subtituted already, to give

$$C_1 = \left(\frac{\pi \rho_{\text{metal}}\ g}{3 \times 0.04\ \text{m}}\right)\ d^2 \Delta t = \left(2.014 \times 10^6\ \text{kg m}^{-3}\ \text{s}^{-2}\right)\ d^2 \Delta t$$

Given the fact that the diameters of the metal spheres are given in mm and not in m, we may elaborate the above expression a little bit more, which yields

$$C_1 = \left(2.014\ \text{kg m}^{-3}\ \text{s}^{-2}\right)\ \left(10^3 \times d\right)^2 \Delta t \tag{16}$$

The results of our calculus are collected in the table below

diameter (mm)	Δt (s)	C_1 (kg/ms)	error (kg/ms)
3.175	5.66	115	-
3.175	5.93	120	-
3.962	3.80±0.10	120	3.1
4.775	2.69±0.20	124	9.2
6.350	1.40	114	-
6.350	1.68	136	-

The error for the spheres with radii 3.962 mm and 4.775 mm are estimated by the following reasoning. The main error comes from the measurement of the time. The error equals a fraction $0.10/3.80 = 0.026$ of the measured time for the sphere with radius 3.962 mm, whereas the error equals a fraction $0.20/2.69 = 0.074$ of the measured time for the sphere with radius 4.775 mm. So, we expect similar fractions for the errors in the respective values for C_1. We find $C_1 = 120$ kg/ms for the sphere with radius 3.962 mm, hence the error is expected to be a fraction of 0.026 of that value, which equals 0.026×120 kg/ms $= 3.1$ kg/ms. A similar reasoning gives for the other sphere an error of 0.074×124 kg/ms $= 9.2$ kg/ms.

For the other two spheres we have two measurements. We could (not exactly standard procedure) say that the average for the sphere with radius 3.175 mm equals $C_1 = 117.5$ kg/ms with an error of 2.5 kg/ms. Similarly, for the sphere with radius 6.350 mm we obtain $C_1 = 125$ kg/ms with an error of 11 kg/ms.
When we round things off we find that all measurements agree on $C_1 = 117$ - 120 kg/ms.

diameter (mm)	C_1 (kg/ms) from - to
3.175	115 - 120
3.962	117 - 123
4.775	115 - 133
6.350	114 - 136

Exercício 3

a: The derivatives of the sine and cosine functions, $\sin(x)$ and $\cos(x)$ respectively, are given by

$$\frac{d\sin(x)}{dx} = \cos(x) \quad \text{and} \quad \frac{d\cos(x)}{dx} = -\sin(x)$$

The tangent function $\tan(x)$ is given by

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \sin(x)\left\{\cos(x)\right\}^{-1}$$

Let us first determine the derivative of $\{\cos(x)\}^{-1}$ by using the chain rule with $f = f(x) = \cos(x)$

$$\begin{aligned}
\frac{d\left\{\cos(x)\right\}^{-1}}{dx} &= \frac{df^{-1}}{dx} = \left(\frac{df^{-1}}{df}\right)\left(\frac{df}{dx}\right) = \left(-f^{-2}\right)(-\sin(x)) = \left(f^{-2}\right)(\sin(x)) \\
&= \left(\cos^{-2}(x)\right)(\sin(x)) = \frac{\sin(x)}{\cos^2(x)}
\end{aligned}$$

For the derivative of the tangent function we apply the product rule

$$\begin{aligned}
\frac{d\tan(x)}{dx} &= \frac{d\sin(x)\left\{\cos(x)\right\}^{-1}}{dx} \\
&= \left(\frac{d\sin(x)}{dx}\right)\left(\left\{\cos(x)\right\}^{-1}\right) + (\sin(x))\left(\frac{d\left\{\cos(x)\right\}^{-1}}{dx}\right) \\
&= (\cos(x))\left(\left\{\cos(x)\right\}^{-1}\right) + (\sin(x))\left(\frac{\sin(x)}{\cos^2(x)}\right) \\
&= 1 + \frac{\sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x)
\end{aligned}$$

b: We define $f = f(x) = e^{+x} - e^{-x}$, $g = g(x) = e^{+x} + e^{-x}$ and $\tanh(x) = fg^{-1}$. We first notice that $df/dx = g$ and $dg/dx = f$. Then we have

$$\begin{aligned}
\frac{d\tanh(x)}{dx} &= \frac{dfg^{-1}}{dx} = \left(\frac{df}{dx}\right) g^{-1} + f\left(\frac{dg^{-1}}{dx}\right) \\
&= (g)\ g^{-1} + f\left(\frac{dg^{-1}}{dg}\frac{dg}{dx}\right) \\
&= 1 + f\left(-g^{-2}\ f\right) = 1 - f^2 g^{-2} = 1 - \tanh^2(x)
\end{aligned}$$

c:

$$\text{For}\quad v = v(t) = \begin{cases} -\alpha\tan(t/T) & \text{for} \quad -\pi T/2 < t < 0 \\ -\alpha\tanh(t/T) & \text{for} \quad 0 < t \end{cases}$$

one has

$$\frac{dv}{dt} = \begin{cases} -\frac{\alpha}{T}\left\{1 + \tan^2(t/T)\right\} = -\frac{\alpha}{T} - \frac{\alpha}{T}\tan^2(t/T) = -\frac{\alpha}{T} - \frac{v^2}{\alpha T} & \text{for} \quad -\pi T/2 < t < 0 \\ -\frac{\alpha}{T}\left\{1 - \tanh^2(t/T)\right\} = -\frac{\alpha}{T} + \frac{\alpha}{T}\tanh^2(t/T) = -\frac{\alpha}{T} + \frac{v^2}{\alpha T} & \text{for} \quad 0 < t \end{cases}$$

For $-\pi T/2 < t < 0$, or $-\pi/2 < t/T < 0$, $\tan(t/T)$ is negative, hence the velocity $-\alpha\tan(t/T)$ is positive. In that case we want the resistive force of the medium to be in the negative direction. Consequently, for the dynamical equation we write

$$dv/dt = -g - |B|\, v^2$$

For $t > 0$, or $t/T > 0$, $e^{t/T} > e^{-t/T}$, hence $\tanh(t/T)$ is positive and the velocity $-\alpha\tanh(t/T)$ is negative. In that case we want the resistive force of the medium to be in the positive direction. Consequently, for the dynamical equation we write

$$dv/dt = -g + |B|\, v^2$$

We obtain then the equations

$$\frac{dv}{dt} = \begin{cases} -\frac{\alpha}{T} - \frac{v^2}{\alpha T} = -g - |B|\, v^2 & \text{for} \quad -\pi T/2 < t < 0 \\ -\frac{\alpha}{T} + \frac{v^2}{\alpha T} = -g + |B|\, v^2 & \text{for} \quad 0 < t \end{cases}$$

which can be casted in the form

$$\begin{cases} |B|\, v^2 - \frac{v^2}{\alpha T} = \frac{\alpha}{T} - g & \text{for} \quad -\pi T/2 < t < 0 \\ \frac{v^2}{\alpha T} - |B|\, v^2 = \frac{\alpha}{T} - g & \text{for} \quad 0 < t \end{cases}$$

and

$$\begin{cases} \left\{|B| - \frac{1}{\alpha T}\right\}\ v^2 = \left\{\frac{\alpha}{T} - g\right\} & \text{for} \quad -\pi T/2 < t < 0 \\ \left\{\frac{1}{\alpha T} - |B|\right\}\ v^2 = \left\{\frac{\alpha}{T} - g\right\} & \text{for} \quad 0 < t \end{cases}$$

Now $v = v(t)$ is a function of time t and we want the above equation to be true for any possible value of $t > -\pi T/2$. Consequently, the equation must be true for all possible values of v.

As in problem 1 there exists only one possibility to achieve that, namely

$$\begin{cases} 0 \times v^2 = 0 & \text{for} \quad -\pi T/2 < t < 0 \\ 0 \times v^2 = 0 & \text{for} \quad 0 < t \end{cases}$$

Hence

$$\alpha T = \frac{1}{|B|} \quad \text{and} \quad \frac{\alpha}{T} = g \quad \Longleftrightarrow \quad \alpha^2 = \frac{g}{|B|} \quad \text{and} \quad T^2 = \frac{1}{g\,|B|}$$

Consequently, a solution of the differential equation $dv/dt = -g \pm |B|\, v^2$ is given by

$$v = v(t) = \begin{cases} -\sqrt{\dfrac{g}{|B|}}\, \tan\left(t\sqrt{g\,|B|}\right) & \text{for} \quad -\pi T/2 < t < 0 \\ -\sqrt{\dfrac{g}{|B|}}\, \tanh\left(t\sqrt{g\,|B|}\right) & \text{for} \quad 0 < t \end{cases}$$

However, we prefer to write

$$v = v(t) = \begin{cases} -\alpha \tan(t/T) & \text{for} \quad -\pi T/2 < t < 0 \\ -\alpha \tanh(t/T) & \text{for} \quad 0 < t \end{cases}$$

and remember that $\alpha = \sqrt{g/\,|B|}$ and $T = 1/\sqrt{g\,|B|}$.

Exercício 4

Formula (5) of exercise (3) reads

$$dv/dt = -g \pm |B|\, v^2$$

for $\alpha = \sqrt{g/\,|B|}$ and $T = 1/\sqrt{g\,|B|}$.

From the solution of exercise (3)e we have learned that $v_{\text{term}} = \alpha$.

The resistive force, which follows from (m is the mass of the metal sphere)

$$m dv/dt = -mg \pm m\,|B|\, v^2$$

is here given by $F_{\text{res}} = \pm m\,|B|\, v^2$.
In this exercise it is given by $F_{\text{res}} = \pm C_2 r^2 v^2$. So, evidently, $m\,|B| = C_2 r^2$.
For the terminal velocities that leads to

$$v_{\text{term}} = \alpha = \sqrt{g/\,|B|} = \sqrt{mg/C_2 r^2} \tag{17}$$

Let us elaborate the formula

$$\begin{aligned} v_{\text{term}} &= \sqrt{\frac{mg}{C_2 r^2}} = \sqrt{\frac{(\text{volume metal sphere}) \times (\text{density of the metal}) \times g}{C_2 r^2}} \\ &= \sqrt{\frac{\left(\frac{4}{3}\pi r^3\right) \times \rho_{\text{metal}} \times g}{C_2 r^2}} = \sqrt{\frac{4\pi r \rho_{\text{metal}}\, g}{3 C_2}} \end{aligned}$$

For C_2 follows thus

$$C_2 = \frac{4\pi r \rho_{\text{metal}}\, g}{3v_{\text{term}}^2} \tag{18}$$

Here, we have the constants $\pi = 3.14159\ldots$, $\rho_{\text{metal}} = 8.02 \times 10^3$ kg/m^3 and $g = 9.8$ m/s^2, which can be subtituted already, to give

$$C_2 = \left(\frac{4\pi\rho_{\text{metal}}\, g}{3}\right) \frac{r}{v_{\text{term}}^2} = \left(3.29 \times 10^5 \text{ kg m}^{-2}\text{s}^{-2}\right) \frac{r}{v_{\text{term}}^2}$$

Given the fact that the diameters of the metal spheres are given in mm and not in m, we may elaborate the above expression a little bit more, which yields

$$C_2 = \left(329 \text{ kg m}^{-2}\text{s}^{-2}\right) \frac{10^3 \times r}{v_{\text{term}}^2} \tag{19}$$

The results of our calculus are collected in the table below

diameter (mm)	v_{term} (m/s)	C_2 (kg/m^3)
0.80	0.62±0.02	685±44
1.18	0.72±0.04	749±83
1.60	0.86±0.03	712±50
2.37	1.01±0.04	765±61
3.16	1.17±0.05	760±65

The errors are estimated by following a similar reasoning as for the errors in exercise (2). The main error comes from the measurement of the terminal velocity. But, the terminal velocity comes squared in formula (19). For that reason we have to double the error, as follows

$$\text{error in } C_2 = 2 \times \frac{\text{error in } v_{\text{term}}}{v_{\text{term}}} \times C_2 \tag{20}$$

Since the errors in C_2 are of the same magnitude we could determine just the average value for C_2. We obtain $C_2 = 734$ kg/m^3.
For the error one should sum the squares of the errors and take the square root divided by the number of measurements. We obtain $C_2 = 734 \pm 28$ kg/m^3.
If we take the differences in the errors into account, then we must determine a weighted average: The smaller the error, the higher the weight. When we take for the weight $C_2/$ (error in C_2), then the average equals $C_2 = 730 \pm 26$ kg/m^3. Indeed not a big difference with the ordinary average.

Exercício 5

a: When we consider the Archimedes force, buoyancy, then for a falling object, where the resistive force is upward, we must change Eq. 5 to

$$mdv/dt = -mg + m\,|B|\, v^2 + F_{\text{Archimedes}}\ . \tag{21}$$

For the buoyancy we have

$$\begin{aligned}
F_{\text{Archimedes}} &= \text{weight of the displaced fluid} = \text{mass of the displaced fluid} \times g \\
&= \text{volume of the immersed object} \times \text{density of the fluid} \times g \\
&= \frac{\text{mass of the immersed object}}{\text{density of the immersed object}} \times \text{density of the fluid} \times g \\
&= \text{mass of the immersed object} \times \frac{\text{density of the fluid}}{\text{density of the immersed object}} \times g \\
&= m \frac{\rho_{\text{fluid}}}{\rho_{\text{object}}}\, g
\end{aligned}$$

So, we find then for Eq. 21 the following

$$mdv/dt = -mg + m\,|B|\,v^2 + m\frac{\rho_{\text{fluid}}}{\rho_{\text{object}}}\, g\ . \tag{22}$$

We divide by m and join the terms which contain g.

$$dv/dt = -g\left(1 - \frac{\rho_{\text{fluid}}}{\rho_{\text{object}}}\right) + |B|\,v^2 = -g\frac{\rho_{\text{object}} - \rho_{\text{fluid}}}{\rho_{\text{object}}} + |B|\,v^2$$

So, when we define

$$g' = g\frac{\rho_{\text{object}} - \rho_{\text{fluid}}}{\rho_{\text{object}}}$$

then we obtain Eq. 8.

b: From exercise (4) we remember $\rho_{\text{object}} = 8.02\ \text{g/cm}^3$, $\rho_{\text{water}} = 1.00\ \text{g/cm}^3$ and furthermore $g = 9.8\ \text{m/s}^2$. So, we obtain

$$g' = g\frac{\rho_{\text{object}} - \rho_{\text{fluid}}}{\rho_{\text{object}}} = \left(9.8\ \text{m/s}^2\right)\frac{8.02 - 1.00}{8.02} = 8.578\ \text{m/s}^2$$

From the solution of exercise (4) we remember that the terminal velocity of the metal sphere with radius 2.37 mm equals 1.01±0.04 m/s.
We want to apply the formula of Eq. 4 do exercício (3), which reads (with $v_{\text{term}} = \alpha$)

$$v = v(t) = -\alpha\tanh(t/T) = -\alpha\frac{e^{t/T} - e^{-t/T}}{e^{t/T} + e^{-t/T}} \tag{23}$$

for an object falling in a fluid. So, we need to know T. Now, T was also given in exercise (3), namely $T = 1/\sqrt{g'\,|B|}$ but here with g substituted by g', whereas $|B| =$ follows from $\alpha = \sqrt{g'/\,|B|}$ and $v_{\text{term}} = \alpha$.
One has thus

$$T = \frac{1}{\sqrt{g'\,|B|}} = \frac{1}{\sqrt{g'^2/v_{\text{term}}^2}} = \frac{v_{\text{term}}}{g'} = \frac{1.01\ \text{m/s}}{8.578\ \text{m/s}^2} = 0.118\ \text{s}$$

Hence, all one has to do is applying the formula of Eq. 23, or alternatively

$$|v(t)| = \alpha\frac{e^{2t/T} - 1}{e^{2t/T} + 1}$$

for $\alpha = v_{\text{term}} = 1.01$ m/s and $T = 0.118$ s.

instante (s)	$\lvert v\rvert$ (m/s)	formula (m/s)
0.000	0.00±0.01	0.00
0.033	0.33±0.05	0.28
0.067	0.49±0.05	0.52
0.099	0.66±0.05	0.69
0.134	0.77±0.05	0.82
0.166	0.83±0.05	0.90
0.199	0.89±0.06	0.94
0.232	0.95±0.06	0.97
0.264	0.98±0.06	0.99
0.298	1.01±0.06	1.00

We observe that the values of the formula are well within the error bars except for $t = 0.166$ s.

Exercício 6

Let us assume that the water in the swimming pool has a height h_{pool} before the wooden block has entered in the water. Hence, the volume of the water in the swimming pool equals

$$V_{\text{pool}} = \left(14.0\ \text{m}^2\right) \times h_{\text{pool}} \tag{24}$$

a: We assume that the wooden block floats after it has entered in the water. When it floats there is equilibrium in the forces which act on the wooden block. Hence, its weight must be equal to the buoyancy:

$$\text{weight wooden block} = F_{\text{Archimedes}} \tag{25}$$

We elaborate Eq. 25 in the following.

$$\begin{aligned}
\text{weight wooden block} &= F_{\text{Archimedes}} \\
\text{weight wooden block} &= \text{weight displaced water} \\
m_{\text{wooden block}} \times g &= m_{\text{displaced water}} \times g \\
\text{volume wooden block} \times \text{density wood} \times g &= \text{volume displaced water} \times \text{density water} \times g \\
\frac{\text{volume displaced water}}{\text{volume wooden block}} &= \frac{\text{density wood}}{\text{density water}} = \frac{0.58\ \text{g/cm}^3}{1.00\ \text{g/cm}^3} = 0.58 \\
\text{volume displaced water} &= 0.58 \times \text{volume wooden block}
\end{aligned}$$

So we find that a fraction of 58% of the wooden block is immersed in the water and therefor 42% sticks out of the water.
Now, 42% of the height (34.0 cm) of the block of wood equals $0.42 \times (34.0\ \text{cm}) = 14.3$ cm. Hence, 14.3 cm of the height of the block of wood is above the water.

b: The additionally displaced water due to the weight of the stone equals 42% of the volume of the wooden block, which is given by

$$0.42 \times \text{volume wooden block} = 0.42 \times \left\{\left(6.3\ \text{m}^2\right) \times (0.340\ \text{m})\right\} = 0.900\ \text{m}^3$$

The weight of the stone equals the weight of the additionally displaced water. Hence, the mass of the stone equals the mass of the additionally displaced water, which is given by

$$\text{volume additionally displaced water} \times \text{density water} = \left(0.900 \text{ m}^3\right) \times \left(1.00 \times 10^3 \text{ kg/m}^3\right) = 900 \text{ kg}$$

Consequently, the mass of the stone equals 900 kg.
Next, we ask ourselves how much has risen the level of the water in the swimming pool.
Since the wooden block is completely immersed in the water, the amount of displaced water is equal to the volume of the wooden block. So, using expression (24), the level of the water is now given by

$$\begin{aligned}
h_{\text{pool}} + d_1 &= \frac{V_{\text{pool}} + \text{volume wood}}{(14.0 \text{ m}^2)} \\
&= \frac{V_{\text{pool}}}{(14.0 \text{ m}^2)} + \frac{\text{volume wood}}{(14.0 \text{ m}^2)} \\
&= h_{\text{pool}} + \frac{(6.3 \text{ m}^2) \times (0.340 \text{ m})}{(14.0 \text{ m}^2)} \\
&= h_{\text{pool}} + 0.153 \text{ m} \\
d_1 &= 15.3 \text{ cm}
\end{aligned}$$

So, in total the water in the swimming pool has risen 15.3 cm with respect to the situation that only water was in the pool.

c: Now the stone is at the bottom of the pool and the wooden block is back to the situation of alinea **a** when the displaced water was only 58% of its volume.
In the new situation we have thus

$$\begin{aligned}
h_{\text{pool}} + d_2 &= \frac{V_{\text{pool}} + 0.58 \times \text{volume wood} + \text{volume stone}}{(14.0 \text{ m}^2)} \\
&= \frac{V_{\text{pool}}}{(14.0 \text{ m}^2)} + 0.58 \times \frac{\text{volume wood}}{(14.0 \text{ m}^2)} + \frac{\text{volume stone}}{(14.0 \text{ m}^2)} \\
&= h_{\text{pool}} + 0.58 \times d_1 + \frac{\text{volume stone}}{(14.0 \text{ m}^2)} \\
\frac{\text{volume stone}}{(14.0 \text{ m}^2)} &= d_2 - 0.58 \times d_1
\end{aligned}$$

It is given that d_2 is 4.00 cm below d_1. Hence,

$$\begin{aligned}
\frac{\text{volume stone}}{(14.0 \text{ m}^2)} &= d_1 - (0.0400 \text{ m}) - 0.58 \times d_1 = 0.42 \times d_1 - (0.0400 \text{ m}) \\
&= 0.42 \times (0.153 \text{ m}) - (0.0400 \text{ m}) = 0.0242 \text{ m} \\
\text{volume stone} &= \left(14.0 \text{ m}^2\right) \times (0.0242 \text{ m}) = 0.34 \text{ m}^3
\end{aligned}$$

The density of stone is thus given by

$$\text{density stone} = \frac{\text{mass stone}}{\text{volume stone}} = \frac{900 \text{ kg}}{0.34 \text{ m}^3} = 2.62 \times 10^3 \text{ kg/m}^3$$

Exercício 7

a: It is given that the piece of copper floats on the liquid mercury. When it floats there is equilibrium in the forces which act on the piece of copper. Hence, its weight must be equal to the buoyancy:

$$\text{weight piece of copper} = F_{\text{Archimedes}} \tag{26}$$

We elaborate Eq. 26 in the following.

$$\begin{aligned}
\text{weight piece of copper} &= F_{\text{Archimedes}} \\
\text{weight piece of copper} &= \text{weight displaced mercury} \\
m_{\text{piece of copper}} \times g &= m_{\text{displaced mercury}} \times g \\
\text{volume piece of copper} \times \text{density copper} \times g &= \text{volume displaced mercury} \times \text{density mercury} \times g \\
\frac{\text{density copper}}{\text{density mercury}} &= \frac{\text{volume displaced mercury}}{\text{volume piece of copper}} = 0.6546 \\
\text{density copper} &= 0.6546 \times \text{density mercury} \\
&= 0.6546 \times \left(13.596 \text{ g/cm}^3\right) = 8.900 \text{ g/cm}^3
\end{aligned}$$

So we find that the density of copper equals 8.900 g/cm^3.

b: The total volume of the sphere is made of the interior cavity and a spherical surface of copper. Hence, we may write

$$\text{volume sphere} = \text{volume cavity} + \text{volume copper} = V_{\text{cavity}} + V_{\text{copper}}$$

For its mass we assume that the cavity has no mass (or negligible mass). Hence,

$$\text{mass sphere} = 0 + \text{volume copper} \times \text{density copper} = V_{\text{copper}} \times \rho_{\text{copper}}$$

Next we study the buoyancy.

$$\begin{aligned}
\text{weight sphere} &= F_{\text{Archimedes}} \\
\text{weight sphere} &= \text{weight displaced mercury} \\
\text{mass sphere} \times g &= \text{mass displaced mercury} \times g \\
\text{mass sphere} &= \text{mass displaced mercury} \\
V_{\text{copper}} \times \rho_{\text{copper}} &= 0.0513 \times V_{\text{sphere}} \times \rho_{\text{mercury}} \\
\frac{V_{\text{copper}}}{V_{\text{sphere}}} &= 0.0513 \times \frac{\rho_{\text{mercury}}}{\rho_{\text{copper}}} = 0.0513 \times \frac{13.596 \text{ g/cm}^3}{8.900 \text{ g/cm}^3} = 0.078
\end{aligned}$$

We find that 7.8% of the sphere is made of copper, hence 92.2% of the sphere consists of cavity.

Exercício 8

a: Given that the density of petroleum is smaller than the density of water, we may expect that a petroleum droplet moves towards the surface of the water. Its acceleration a is given by

$F = ma$ (Newton). The acceleration must be due to the resultant force of its buoyancy and its weight.

$$\begin{aligned}
m_{\text{droplet}} \times a_{\text{droplet}} &= F_{\text{Archimedes}} - \text{weight petroleum droplet} \\
&= \text{weight displaced water} - \text{weight petroleum droplet} \\
&= \text{mass displaced water} \times g - \text{mass petroleum droplet} \times g \\
&= (V_{\text{droplet}} \times \rho_{\text{water}}) \times g - (V_{\text{droplet}} \times \rho_{\text{petroleum}}) \times g \\
&= V_{\text{droplet}} \times (\rho_{\text{water}} - \rho_{\text{petroleum}}) \times g \\
&= V_{\text{droplet}} \times \rho_{\text{petroleum}} \frac{\rho_{\text{water}} - \rho_{\text{petroleum}}}{\rho_{\text{petroleum}}} \times g \\
&= m_{\text{droplet}} \frac{\rho_{\text{water}} - \rho_{\text{petroleum}}}{\rho_{\text{petroleum}}} \times g \\
a_{\text{droplet}} &= \frac{\rho_{\text{water}} - \rho_{\text{petroleum}}}{\rho_{\text{petroleum}}} \times g \\
&= \frac{\left(1.0 \text{ g/cm}^3\right) - \left(0.8 \text{ g/cm}^3\right)}{\left(0.8 \text{ g/cm}^3\right)} \times g = 0.25g = 2.45 \text{ m/s}^2
\end{aligned}$$

Consequently, if we do not consider the resistive force, then we have a uniformly accelerated motion, which can be given by

$$v(t) = v_0 + at = 0 + \left(2.45 \text{ m/s}^2\right) t = \left(2.45 \text{ m/s}^2\right) t$$

The distance it travels is given by

$$y(t) = y_0 + \frac{1}{2}at^2 = 0 + \frac{1}{2}\left(2.45 \text{ m/s}^2\right) t^2 = \left(1.225 \text{ m/s}^2\right) t^2$$

For 1.5 km it takes the droplet the following time

$$1500 \text{ m} = \left(1.225 \text{ m/s}^2\right) (t_{1.5 \text{ km}})^2 \quad \Longrightarrow \quad t_{1.5 \text{ km}} = \sqrt{\frac{1500 \text{ m}}{1.225 \text{ m/s}^2}} = 35.0 \text{ s}$$

We find that without viscosity the droplet would arrive within 35 seconds at the surface of a 1.5 km deep ocean.

b: The motion of the droplet is like the motion of a falling object in a viscous medium, but just upside-down. In regime I we have the expression (see e.g. Eq. 1 of exercise (1)

$$v(t) = v_{\text{term}} \left\{ e^{-t/T} - 1 \right\}$$

At $t = 0$ the speed equals

$$v(0) = v_{\text{term}} \left\{ e^0 - 1 \right\} = v_{\text{term}} \{1 - 1\} = 0$$

as demanded by the initial conditions of the problem.

c: For the terminal velocity we found in exercise (1) the following relation $v_{\text{term}} = -g/A$, where A was given by the acceleration $+Av$ for resistive force. Furthermore in exercise (2), defining the resistive force by $-C_1 rv$, we obtained in Eq. 13 the relation for the terminal velocity and C_1, given by

$$v_{\text{term}} = -g/A = mg/C_1 r$$

which was further elaborated, to result in

$$v_{\text{term}} = \frac{\pi d^2 \rho_{\text{petroleum}} g}{3C_1}$$

Then, in exercise (5), we found that g should be replaced by

$$g' = \frac{\rho_{\text{water}} - \rho_{\text{petroleum}}}{\rho_{\text{petroleum}}} \times g$$

Hence,

$$\begin{aligned} v_{\text{term}} &= \frac{\pi d^2 \rho_{\text{petroleum}} g'}{3C_1} \\ &= \frac{\pi d^2 \rho_{\text{petroleum}} \frac{\rho_{\text{water}} - \rho_{\text{petroleum}}}{\rho_{\text{petroleum}}} \times g}{3C_1} \\ &= \frac{\pi d^2 \left(\rho_{\text{water}} - \rho_{\text{petroleum}}\right) g}{3C_1} \end{aligned}$$

Finally, here we have been given the relation $C_1 = 6\pi\mu$, hence

$$v_{\text{term}} = \frac{\pi d^2 \left(\rho_{\text{water}} - \rho_{\text{petroleum}}\right) g}{3 \times 6\pi\mu} = \frac{d^2 \left(\rho_{\text{water}} - \rho_{\text{petroleum}}\right) g}{18\mu}$$

With $d = 60\ \mu\text{m} = 60 \times 10^{-6}$ m and $\mu = 1.00 \times 10^{-3}$ kg/ms a 20^0C, we obtain

$$v_{\text{term}} = \frac{(60 \times 10^{-6}\ \text{m})^2 \left\{\left(1.0 \times 10^3\ \text{kg/m}^3\right) - \left(0.8 \times 10^3\ \text{kg/m}^3\right)\right\} \left(9.8\ \text{m/s}^2\right)}{18\,(1.00 \times 10^{-3}\ \text{kg/ms})} = 0.000392\ \text{m/s}$$

We find that the terminal speed of such small petroleum droplet equals about 0.4 mm/s (see e.g. http://www.lantecp.com/application_oil_water_separator.htm).

d: Without resistive force, the droplet would have had a velocity of 2.45 m/s after one second (see alinea **a**) which is way larger than its terminal speed. We may thus assume that the droplet reaches its terminal velocity within a small fraction of a second. So the droplet goes to the surface with a speed of 0.4 mm/s. Then it will take the following time to reach the surface of the ocean for such droplet that is released at 1.5 km below the surface.

$$\text{travel time} = \frac{1500\ \text{m}}{0.000392\ \text{m/s}} = 3.83 \times 10^6\ \text{s} = 1063\ \text{hours} = 44.3\ \text{days}$$

This indicates the problem of petroleum droplets at large depths in the ocean.

Exercício 9

a: If $C_d = 24/\text{Re}$, then $\log(C_d) = \log(24) - \log(\text{Re})$, which is represented by a straight line in the graph of $\log(\text{Re})$ against $\log(C_d)$ that passes through the point $\log(C_d) = \log(24)$ for $\log(\text{Re}) = 0$ and which has a slope of tangent equal to -1.

b:

$$\begin{aligned} F_{\text{res}} &= \frac{1}{2} C_{\text{d}}\, \rho_{\text{fluid}}\, v^2 S = \frac{1}{2}\frac{24}{\text{Re}}\, \rho_{\text{fluid}}\, v^2 S \\ &= \frac{1}{2}\frac{24}{\rho_{\text{fluid}}\, vL/\mu}\, \rho_{\text{fluid}}\, v^2 S = \frac{12\mu v S}{L} = \frac{12\mu v \pi r^2}{2r} = 6\pi\mu r v \end{aligned}$$

In terms of C_1 one has (regime I) $F_{\text{res}} = C_1 r v$. Hence, $C_1 = 6\pi\mu$.

c: If $C_d = 0.47$, then $\log(C_d) = \log(0.47)$, which is represented by a horizontal straight line in the graph of $\log(\text{Re})$ against $\log(C_d)$ which has the same value $\log(C_d) = \log(0.47)$ for all $\log(\text{Re})$.

d:

$$\begin{aligned} F_{\text{res}} &= \frac{1}{2} C_{\text{d}}\, \rho_{\text{fluid}}\, v^2 S = \frac{1}{2}\, 0.47\, \rho_{\text{fluid}}\, v^2 S \\ &= = \frac{1}{2}\, 0.47\, \rho_{\text{fluid}}\, v^2 \pi r^2 = \frac{1}{2}\pi\, 0.47\, \rho_{\text{fluid}}\, r^2 v^2 = \frac{1}{2}\pi\, C_{\text{d}}\, \rho_{\text{fluid}}\, r^2 v^2 \end{aligned}$$

In terms of C_2 one has $F_{\text{res}} = C_2 r^2 v^2$. Hence, $C_2 = \frac{1}{2}\pi\, C_{\text{d}}\rho_{\text{fluid}}$, where $\frac{1}{2}\pi\, C_{\text{d}} = 0.738$.
Notice that for water ($\rho_{\text{water}} = 1.0 \times 10^3$ kg/m^3) one has $C_2 = 0.738\rho_{\text{water}}$. In exercise (4) we found from experiment the value $C_2 = 730 \pm 26$ kg/m^3. Hence a perfect agreement.

Exercício 10

a: The only way to prove that a function is a solution of the differential equation Eq. 12, is to determine its derivative and figure out if it gives the right result. Hence, we perform

$$\frac{d}{dT}L(T) = \frac{d}{dT}L(0)\, e^{\alpha T} = L(0)\, \frac{d}{dT}e^{\alpha T} = L(0)\, \alpha e^{\alpha T} = \alpha L(0)\, e^{\alpha T} = \alpha L(T)$$

It works!

b: First, using Eq. 10 with $\Delta T = 70$ K, we find

$$L(50^0\text{C}) - L(-20^0\text{C}) = \alpha L(0)\Delta T = \left(14.3 \times 10^{-6}\ \text{K}^{-1}\right)\left(10.0 \times 10^3\ \text{m}\right)(70\ \text{K}) = 10.01\ \text{m}$$

Then, using Eq. 12, we find

$$\begin{aligned} &L(50^0\text{C}) - L(-20^0\text{C}) = L(0)e^{\alpha\,(50^0\text{C})} - L(0)e^{\alpha\,(-20^0\text{C})} \\ &= \left(10.0 \times 10^3\ \text{m}\right) e^{\left(14.3 \times 10^{-6}\ \text{K}^{-1}\right)(50^0\text{C})} - \left(10.0 \times 10^3\ \text{m}\right) e^{\left(14.3 \times 10^{-6}\ \text{K}^{-1}\right)(-20^0\text{C})} \\ &= 10.01\ \text{m} \end{aligned}$$

We find with both methods the same result!

c:

$$\begin{aligned} L(T+\Delta T) - L(T) &= L(0)e^{\alpha\,(T+\Delta T)} - L(0)e^{\alpha T} \\ &= L(0)e^{\alpha T}\left\{e^{\alpha\Delta T} - 1\right\} \\ &\approx L(0)e^{\alpha T}\left\{1+\alpha\Delta T - 1\right\} \\ &= L(T)\alpha\Delta T = \alpha L(T)\Delta T \end{aligned}$$

for $\alpha\Delta T \ll 1$ as we have verified in alinea **c** of exercise (1). The latter condition is in all practical cases satisfied since most substances have melted long before $\Delta T = 5000K$.

Exercício 11

Since the liquid is in equilibrium, we may assume that the pressure at equal height in the liquid is the same everywhere. If that would not be the case, there would be horizontal flow from the high pressure region to the low pressure region. Hence,

$$P_A = P_B = P_C$$

In the vertical direction things are different. The weight of the liquid causes pressure which increases the deeper you go below the surface of the liquid.

a: In the theory classes we have learned that at d metres below the surface of a liquid, the pressure $P(d)$ is given by

$$P_{\text{liquid}}(d) = d\rho_{\text{liquid}}g$$

Consequently,

$$P_{\text{liquid}}(17\text{ cm}) = (0.17\text{ m})\left(1.20\times10^3\text{ kg/m}^3\right)\left(9.8\text{ m/s}^2\right) = 2.00\times10^3\text{ Pa}$$

Since the tubes are open from above, they are in contact with the air. In that case the total pressure at the bottom of the tube is given by

$$P(\text{bottom tube}) = P_{\text{liquid}}(17\text{ cm}) + P(\text{air at the surface of the liquid})$$

b: First we must determine the pressure of the new liquid on the surface of the original liquid. For that we need to know the height of the new liquid. That can be determined from

$$h_{\text{new liquid}} = \frac{\text{volume new liquid}}{\text{area tube}} = \frac{72\text{ cm}^3}{12\text{ cm}^2} = 6.0\text{ cm}$$

So, the pressure which the new liquid excerts on the surface of the original liquid is given by

$$P_{\text{new liquid}}(6.0\text{ cm}) = (0.060\text{ m})\left(0.80\times10^3\text{ kg/m}^3\right)\left(9.8\text{ m/s}^2\right) = 470\text{ Pa}$$

Since the pressure at equal heights must be the same, the pressure of the original liquid in the wider tube at the height of the surface of the original liquid in the narrower tube must be equal. Hence,

$$\begin{aligned} h\rho_{\text{liquid}}g &= 470 \text{ Pa} \\ h &= \frac{470 \text{ Pa}}{\rho_{\text{liquid}}g} = \frac{470 \text{ Pa}}{\left(1.20 \times 10^3 \text{ kg/m}^3\right)\left(9.8 \text{ m/s}^2\right)} = 0.04 \text{ m} \end{aligned}$$

We find that the level difference of the surfaces of original liquid in the two tubes, equals 4.0 cm.

Exercício 12

As in problem (11) we may apply here

$$P_{\text{liquid}}(d) = d\rho_{\text{liquid}}g$$

Hence,

$$d = \frac{10^7 \text{ Pa}}{\rho_{\text{seawater}}g} = \frac{10^7 \text{ Pa}}{\left(1.025 \times 10^3 \text{ kg/m}^3\right)\left(9.8 \text{ m/s}^2\right)} = 996 \text{ m}$$

So, the sphere will implode at a depth of close to 1 km below the surface of the sea. Notice that the air pressure on the surface of the sea is negligible in this calculus, because it is about 1% of the total pressure of 10^7 Pa.

b: Part of the calculus has been performed in exercise (4), where we obtained the expression

$$v_{\text{term}} = \sqrt{\frac{4\pi r\rho_{\text{sphere}}\, g}{3C_2}}$$

Now, here we must substitute g for the corrected expression obtained in exercise (5a) and given in Eq. 9. We obtain then

$$v_{\text{term}} = \sqrt{\frac{4\pi r\left(\rho_{\text{sphere}} - \rho_{\text{seawater}}\right)\, g}{3C_2}}$$

For C_2 let us use the value of exercise (9d), which was found to be $C_2 = 730 \pm 26$ kg/m^3. Now, there is still a detail to be solved, which is the density of the sphere.

$$\text{density sphere} = \frac{\text{mass sphere}}{\text{volume sphere}} = \frac{6.00 \text{ kg}}{\frac{4}{3}\pi\left(0.0895 \text{ m}\right)^3} = 2.00 \times 10^3 \text{ kg/m}^3$$

For the terminal velocity we find now

$$v_{\text{term}} = \sqrt{\frac{4\pi\left(0.0895 \text{ m}\right)\left\{\left(2.00 \times 10^3 \text{ kg/m}^3\right) - \left(1.025 \times 10^3 \text{ kg/m}^3\right)\right\}\left(9.8 \text{ m/s}^2\right)}{3\left(730 \text{ kg/m}^3\right)}} = 2.22 \text{ m/s}$$

Assuming that this speed is obtained in a short distance, we calculate for the time it takes to travel 996 m for the sphere

$$\text{travel time} = (996/2.22) = 449 \text{ s} = 7.5 \text{ min}$$

This was a great invention during WWII.

Exercício 13

a: In the theory classes we have learned that the kinetic energy of one molecule with mass m and speed v at temperature T is given by

$$\frac{1}{2}mv^2 = \frac{3}{2}kT \tag{27}$$

From Eq. 27 we may deduce that the speed of a molecule is given by

$$v = \sqrt{\frac{3kT}{m}}$$

So, we need the mass of one molecule of oxygen m and the Boltzmann's constant k. They may be looked up in WikepediA. First, I typed "oxygen molecule mass kg" in Google, which resulted in a list of sites. The first was the site "The_mass_of_one_oxygen_molecule_is" of "wiki.answers.com/Q", which gave me

$$\text{mass of 1 oxygen molecule} = 5.35607818 \times 10^{-26}\ \text{kg}$$

Then, I googled "Boltzmann's constant" which gave the result

$$\text{Boltzmann constant} = 1.3806503 \times 10^{-23}\ \text{J/K}$$

Furthermore, we need the temperature in Kelvin. So I googled "Celsius to Kelvin" and got from "http://en.wikipedia.org/wiki/Kelvin" the result

$$^0\text{K} = {}^0\text{C} + 273.15$$

Now, we have all we need to calculate the speed of an oxygen molecule at 20^0C, so it remains to be put in the formula and our calculator.

$$\text{speed} = \sqrt{\frac{3\,(1.3806503 \times 10^{-23}\ \text{J/K})\,(20 + 273.15\ \text{K})}{(5.35607818 \times 10^{-26}\ \text{kg})}} = 476\ \text{m/s} = 1714\ \text{km/h}$$

The oxygen molecule travels with a supersonic speed through the room.
How often it collides with one of the walls?
For that we need the length of the room, which is given to be equal to 6 m. So, the molecule travels 12 m in coming and going through the room while every time hitting both of the walls. For one wall we have thus

$$\text{number of collisions per second} = \frac{\text{speed of the molecule}}{2 \times (\text{length of the room})} = \frac{476\ \text{m/s}}{12\text{m}} = 39.7\ \text{times/s}$$

We find that the molecule hits one of the walls almost 40 times per second.

b: The number of molecules can be determined using the ideal gas law.

$$PV = NkT \quad \Longrightarrow \quad N = \frac{PV}{kT}$$

The pressure is found, using "http://www.asknumbers.com/atm-to-pascal.aspx", to be

$$1 \text{ atm} = 101325 \text{ Pascal}$$

The volume V of the room can be determined by

$$V = \text{length} \times \text{width} \times \text{height} = 6 \times 5 \times 3 \text{ m}^3 = 90 \text{ m}^3$$

The temperature follows from "http://en.wikipedia.org/wiki/Kelvin"

$$^0\text{K} = {}^0\text{C} + 273.15$$

which gives $T = (20 + 273.15)$ K $= 293.15$ K. Then, I googled "Boltzmann's constant" which gave the result

$$\text{Boltzmann constant} = 1.3806503 \times 10^{-23} \text{ J/K}$$

Hence,

$$N = \frac{PV}{kT} = \frac{(101325 \text{ Pascal})\,(90 \text{ m}^3)}{(1.3806503 \times 10^{-23} \text{ J/K})\,(293.15 \text{ K})} = 2.25 \times 10^{27}$$

Just try to imagine how many molecules that are!
If you would represent each of the molecules by one grain of sand (http://cft.fis.uc.pt/eef/-braga06/quarks/htm/zand.htm), then you would need a layer of 450 metres of sand over all the surface of our planet Earth, including the oceans, to have as many grains of sand as there are air molecules in a small class room.

Since the volume of the room is constant, we may apply the formula of Gay-Lussac

$$P_{27^0\text{C}} = \frac{T_{27^0\text{C}}}{T_{20^0\text{C}}}\, P_{20^0\text{C}} = \frac{300.15 \text{ K}}{293.15 \text{ K}}\, (101325 \text{ Pascal}) = 103744 \text{ Pa} \qquad (28)$$

Since the volume remains constant, there is no work involved in this transformation. Consequently, we only have to determine the change in kinetic energy of the air.

$$\begin{aligned}
\Delta E_{\text{kinetic}} &= E_{\text{kinetic}}(27^0\text{C}) - E_{\text{kinetic}}(20^0\text{C}) = \frac{3}{2} P_{27^0\text{C}} V - \frac{3}{2} P_{20^0\text{C}} V \\
&= \frac{3}{2}\left(P_{27^0\text{C}} - P_{20^0\text{C}}\right)\, V = \frac{3}{2}\{(103744 \text{ Pa}) - (101325 \text{ Pa})\}\,\left(90 \text{ m}^3\right) \\
&= = 0.33 \times 10^6 \text{ J}
\end{aligned}$$

We find that about 0.33 MJ of heat is needed to heat up the air in the room from 20^0C to 27^0C. The heat output of a human body is not well registered. The estimates range from as low as 50 Watt to as high as 500 Watt. If we take 100 Watt, then, in 1 hour, 20 students produce $20 \times 3600 \times 100 = 7.2$ MJ. Apparently, most of this heat disappears from the room by radiation or heat flow to the exterior.

Exercício 14

a: The first process $(V_1, P_1, T_{11}) \rightarrow (V_1, P_2, T_{12})$ is isochoric, hence the volume is constant. In that case we may apply the formula of Gay-Lussac

$$T_{12} = \frac{P_2}{P_1}\, T_{11} = \frac{3.0 \times 10^6 \text{ Pa}}{1.0 \times 10^6 \text{ Pa}}\, (300 \text{ K}) = 900 \text{ K} \qquad (29)$$

The second process $(V_1, P_2, T_{12}) \rightarrow (V_2, P_2, T_{22})$ is isobaric, hence the pressure is constant. In that case we may apply the formula of Boyle

$$T_{22} = \frac{V_2}{V_1} T_{12} = \frac{0.25 \text{ litres}}{0.10 \text{ litres}} (900 \text{ K}) = 2250 \text{ K} \tag{30}$$

The third process $(V_2, P_2, T_{22}) \rightarrow (V_2, P_1, T_{21})$ is again isochoric, hence the volume is constant. In that case we may apply the formula of Gay-Lussac

$$T_{21} = \frac{P_1}{P_2} T_{22} = \frac{1.0 \times 10^6 \text{ Pa}}{3.0 \times 10^6 \text{ Pa}} (2250 \text{ K}) = 750 \text{ K} \tag{31}$$

The fourth process $(V_2, P_1, T_{21}) \rightarrow (V_1, P_1, T_{11})$ is again isobaric, hence the pressure is constant. In that case we may apply the formula of Boyle

$$T_{11} = \frac{V_1}{V_2} T_{21} = \frac{0.10 \text{ litres}}{0.25 \text{ litres}} (750 \text{ K}) = 300 \text{ K} \tag{32}$$

The latter value is, of course, no surprise because that is the situation from where the cicle started. **b**: During the transformation $(V_1, P_1, T_{11}) \rightarrow (V_1, P_2, T_{12})$ the volume remains constant. Consequently, no work is done. The heat which is given to the gas is just used for the kinetic energy. For that we may use the formula

$$E_{\text{kinetic}} = \frac{3}{2} PV \tag{33}$$

In the transformation under consideration only the pressure changes, in which case it is easy to determine the change in kinetic energy, namely

$$\Delta E_{\text{kinetic}} = \frac{3}{2} \Delta P V_1 = \frac{3}{2} (P_2 - P_1) V_1 = \frac{3}{2} \left(3.0 \times 10^6 - 1.0 \times 10^6\right) 0.1 \times 10^{-3} = 300 \text{ J}$$

Consequently, the total heat for this transformation is given by

$$Q_{11 \rightarrow 12} = \Delta E_{\text{kinetic}} = 300 \text{ J} \tag{34}$$

During the transformation $(V_1, P_2, T_{12}) \rightarrow (V_2, P_2, T_{22})$ the pressure remains constant. The volume changes, hence work is done. The heat which is given to the gas is used for the work delivered and for the kinetic energy. For the work we may use the formula

$$W = P_2 \Delta V = P_2 (V_2 - V_1) = 3.0 \times 10^6 \left(0.25 \times 10^{-3} - 0.10 \times 10^{-3}\right) = 450 \text{ J}$$

For the change in kinetic energy, we have here only change in volume, hence

$$\Delta E_{\text{kinetic}} = \frac{3}{2} P_2 \Delta V = \frac{3}{2} P_2 (V_2 - V_1) = \frac{3}{2} \times 3.0 \times 10^6 \left(0.25 \times 10^{-3} - 0.10 \times 10^{-3}\right) = 675 \text{ J}$$

The total heat for this transformation is given by

$$Q_{12 \rightarrow 22} = W + \Delta E_{\text{kinetic}} = 450 + 675 = 1125 \text{ J} \tag{35}$$

c: During the transformation $(V_2, P_2, T_{22}) \rightarrow (V_2, P_1, T_{21})$ the volume remains constant. Consequently, no work is done. The heat which is released by the gas is just used to decrease the kinetic energy. For that we may use formula (33). In the transformation under consideration only the pressure changes, in which case it is easy to determine the change in kinetic energy, namely

$$\Delta E_{\text{kinetic}} = \frac{3}{2}\Delta P V_2 = \frac{3}{2}\left(P_1 - P_2\right)V_1 = \frac{3}{2}\left(1.0\times10^6 - 3.0\times10^6\right)0.25\times10^{-3} = -750 \text{ J}$$

Consequently, the total heat (negative because it is released by the gas) for this transformation is given by

$$Q_{22\to21} = \Delta E_{\text{kinetic}} = -750 \text{ J} \tag{36}$$

During the transformation $(V_2, P_1, T_{21}) \to (V_1, P_1, T_{11})$ the pressure remains constant. The volume changes, hence work is done. The heat which is released by the gas is used for the work excerted on the gas and for the decrease in kinetic energy. For the work we may use the formula

$$W = P_1\Delta V = P_1\left(V_1 - V_2\right) = 1.0\times10^6\left(0.10\times10^{-3} - 0.25\times10^{-3}\right) = -150 \text{ J}$$

For the change in kinetic energy, we have here only change in volume, hence

$$\Delta E_{\text{kinetic}} = \frac{3}{2}P_1\Delta V = \frac{3}{2}P_1\left(V_1 - V_2\right) = \frac{3}{2}\times1.0\times10^6\left(0.10\times10^{-3} - 0.25\times10^{-3}\right) = -225 \text{ J}$$

The total heat (negative because it is released by the gas) for this transformation is given by

$$Q_{21\to11} = W + \Delta E_{\text{kinetic}} = -150 - 225 = -375 \text{ J} \tag{37}$$

d:

$$\begin{aligned} W_{11\to12} &= 0 \text{ J} \\ W_{12\to22} &= 450 \text{ J} \\ W_{22\to21} &= 0 \text{ J} \\ W_{21\to11} &= -150 \text{ J} \end{aligned} \tag{38}$$

Notice that our engine has a positive net work of 300 J which is given by the expression.

$$W = P_2\left(V_2 - V_1\right) + P_1\left(V_1 - V_2\right) = \left(P_2 - P_1\right)\left(V_2 - V_1\right) \tag{39}$$

which is just the area enclosed by the four lines in the figure of the P-V diagram. This is generally true, also for engines which have closed loops which contain curved lines.
d: We had to furnish heat to the engine in the transformations $(V_1, P_1) \to (V_1, P_2)$ and $(V_1, P_2) \to (V_2, P_2)$, whereas the net work is the difference between the work done by the engine in the transformation $(V_1, P_2) \to (V_2, P_2)$ and the work which had to be done on the engine in the transformation $(V_2, P_1) \to (V_1, P_1)$.

$$\text{efficiency} = \frac{\text{net work done by the engine}}{\text{heat used by engine}} = \frac{W_{12\to22} + W_{21\to11}}{Q_{11\to12} + Q_{12\to22}} = \frac{300 \text{ J}}{1425 \text{ J}} = 0.21 \tag{40}$$

Our engine is thus not very efficient.

Let us also study the energy balance

transformation	work (J)	kinetic energy (J)	heat (J)
$11 \rightarrow 12$	0	300	300
$12 \rightarrow 22$	450	675	1125
$22 \rightarrow 21$	0	-750	-750
$21 \rightarrow 11$	-150	-225	-375
totals	300	0	300

The difference in heat consumed and released is used for net work!

The kinetic energy could not give a positive or negative balance, because when you are in point (V_1, P_1) the kinetic energy is just given by the ideal gas law, which says $E(\text{kinetic}) = 3V_1P_1/2$. Consequently, if you add all the contributions to the kinetic energy in each of the steps of the cycle of the engine, then it must give a vanishing result:

$$\begin{aligned}
\Delta E_{\text{kinetic}} &= \Delta E_{\text{kinetic}}(11 \rightarrow 12) + \Delta E_{\text{kinetic}}(12 \rightarrow 22) + \Delta E_{\text{kinetic}}(22 \rightarrow 21) + \Delta E_{\text{kinetic}}(21 \rightarrow 11) \\
&= \frac{3}{2}(P_2 - P_1)\ V_1 + \frac{3}{2}P_2\ (V_2 - V_1) + \frac{3}{2}(P_1 - P_2)\ V_2 + \frac{3}{2}P_1\ (V_1 - V_2) \\
&= \frac{3}{2}(P_2V_1 - P_1V_1 + P_2V_2 - P_2V_1 + P_1V_2 - P_2V_2 + P_1V_1 - P_1V_2) \\
&= 0
\end{aligned}$$

Exercício 15

a: From exercise (14) we learn that $V_1 = 0.1 \times 10^{-3}$ m^3, $P_1 = 1.0 \times 10^6$ Pa e $T_{11} = 300$ K. Using the ideal gas law we have thus for the number of moles (n)

$$n = \frac{P_1V_1}{RT_{11}} = \frac{(0.1 \times 10^{-3}\ \text{m}^3)\,(1.0 \times 10^6\ \text{Pa})}{(8.314\ \text{J/K/mol})\,(300\ \text{K})} = 0.040\ \frac{\text{m}^3 Pa}{\text{J/K/mol K}} = 0.040\ \text{mol} \tag{41}$$

b: The amount of gasoline that enters the cylinder during each cycle equals 0.008×0.040 mol $= 3.2 \times 10^{-4}$ mol. The mass of 1 mol of gasoline is given to be 105 grams. Hence, 3.2×10^{-4} mol corresponds to $(3.2 \times 10^{-4}$ mol$) \times (0.105$ kg/mol$) = 3.37 \times 10^{-5}$ kg.

The density of gasoline is given to be 740 kg/m^3. Hence 3.37×10^{-5} kg of gasoline corresponds to $(3.37 \times 10^{-5}\ \text{kg}) / \left(740\ \text{kg/m}^3\right) = 4.55 \times 10^{-8}\ \text{m}^3 = 4.55 \times 10^{-5}$ litres of gasoline.

According to our calculus in exercise (14)b it produces 1425 J of heat. Hence, its energy density (see Densidade_de_energia page of pt.wikipedia.org/wiki) equals

$$\text{energy density gasoline} = \frac{1425\ \text{J}}{4.55 \times 10^{-5}\ \text{litres}} = 31.3 \times 10^6\ \text{J/litre} = 31.3\ \text{MJ/litre}$$

c: Each cylinder uses 4.55×10^{-5} litres of gasoline in each cycle, according to the previous alinea. The four cylinders use four times that quantity per cycle.

It is given that they work for one hour (60 minutes) at 1500 cycles per minute. That makes $60 \times 1500 = 9.0 \times 10^4$ cycles in total. Hence the four cylinders consume

(4 cylinders) $\times$ (9.0×10^4 cycles/cylinder) $\times$ (4.55×10^{-5} litres/cycle) = 16.4 litres of gasoline during their hour of work.

d: The power is calculated by the amount of work the engine produces during one second.

In each cycle one cylinder produces a net work which is given by the area enclosed by the four lines in the figure of exercise (14). It has been determined in exercise (14)d to be equal to $450 - 150 = 300$ J. The four cylinders produce four times that quantity per cycle.

The engine makes 1500 cycles per minute (60 seconds). That makes $1500/60 = 25$ cycles per second. Hence, in one second the four cylinders produce (4 cylinders) $\times$ (25 cycles/cylinder) $\times$ (300 J/cycle) $= 3.0 \times 10^4$ J.

The power of the engine is thus 3.0×10^4 J/s $= 3.0 \times 10^4$ Watts (W).

When we assume that 1 horsepower is given by 750 Watt, then 3.0×10^4 Watts corresponds to (3.0×10^4 Watts) / (750 Watts/horsepower) = 40 horsepower.

Exercício 16

For the velocities we have

$$\begin{aligned} v &= \frac{\text{quantity per unit of time}}{\text{cross section}} = \frac{12.0 \text{ litres/minute}}{8 \text{ cm}^2 \text{ or } 2 \text{ cm}^2} \\ &= \frac{12.0 \times 10^3 \text{ cm}^3/60 \text{ s}}{8 \text{ cm}^2 \text{ or } 2 \text{ cm}^2} = \frac{12.0 \times 10^3 \text{ cm/s}}{8 \times 60} \text{ or } \frac{12.0 \times 10^3 \text{ cm/s}}{2 \times 60} \\ &= 25 \text{ cm/s} = 0.25 \text{ m/s or } 100 \text{ cm/s} = 1.00 \text{ m/s} \end{aligned} \tag{42}$$

Then we apply Bernoulli's law:

$$p_{\text{narrow}} + \frac{1}{2}\rho_{\text{water}} v_{\text{narrow}}^2 = p_{\text{broad}} + \frac{1}{2}\rho_{\text{water}} v_{\text{broad}}^2$$

Hence, using the result of Eq. 42, we obtain

$$\begin{aligned} p_{\text{narrow}} &= p_{\text{broad}} + \frac{1}{2}\rho_{\text{water}} v_{\text{broad}}^2 - \frac{1}{2}\rho_{\text{water}} v_{\text{narrow}}^2 \\ &= p_{\text{broad}} + \frac{1}{2}\rho_{\text{water}} \left\{ v_{\text{broad}}^2 - v_{\text{narrow}}^2 \right\} \\ &= \left(1.00 \times 10^5 \text{ Pa}\right) + \frac{1}{2}\left(1.0 \times 10^3 \text{ kg/m}^3\right) \left\{ (0.25 \text{ m/s})^2 - (1.00 \text{ m/s})^2 \right\} \\ &= \left(1.00 \times 10^5 \text{ Pa}\right) + \frac{1}{2}\left(1.0 \times 10^3\right) (0.0625 - 1.00) \left(\text{kg m/s}^2\right)/\text{m}^2 \\ &= = \left(1.00 \times 10^5 \text{ Pa}\right) - 0.469 \times 10^3 \text{ N/m}^2 = 0.9953 \times 10^5 \text{ Pa} \end{aligned}$$

Exercício 17

a: Force equals pressure times area. So, we find

$$F_{\text{inside}} = \left(10 \text{ m}^2\right)\left(1.00 \times 10^5 \text{ Pa}\right) = 1.00 \times 10^6 \text{ N}$$

b: Here, we first must determine the pressure for $v = 180$ km/h $= 50$ m/s.

$$\begin{aligned} P_{\text{outside}} + \frac{1}{2}\rho_{\text{air}} v_{\text{outside}}^2 &= P_{\text{inside}} + \frac{1}{2}\rho_{\text{air}} v_{\text{inside}}^2 \\ &= P_{\text{inside}} \\ P_{\text{inside}} - P_{\text{outside}} &= \frac{1}{2}\rho_{\text{air}} v_{\text{outside}}^2 \\ &= \frac{1}{2}\left(1.29 \text{ kg/m}^3\right)(50 \text{ m/s})^2 = 1.61 \times 10^3 \text{ Pa} \end{aligned}$$

c: The weight of the roof equals volume (10 m^2 $\times$ 0.1 m $= 1$ m^3) times the density (0.58×10^3 kg/m^3) times $g = 9.8$ m/s^2. We find

$$W = 5.68 \times 10^3 \text{ N}$$

The difference in pressure gives an upward force (pressure times area) which is equal to $F = 16.1 \times 10^3$ N. The resultant force equals $F = 10.4 \times 10^3$ N upward.

d: Concrete is more heavy. We obtain $W = 23.5 \times 10^3$ N. So, now the resultant force equals $F = 7.4 \times 10^3$ N downward.

Exercício 18

a: At the altitude of a hole on the outside of the water tank one has the pressure of the air, whereas on the inside one has additionally the pressure of the water. Furthermore, besides very close to the hole, on the inside the water is at rest. We apply Bernoulli's principle (s hight of the water inside the tank, h hight of the hole).

$$\begin{aligned} P_{\text{outside}} + \frac{1}{2}\rho_{\text{water}} v_{\text{outside}}^2 &= P_{\text{inside}} + \frac{1}{2}\rho_{\text{water}} v_{\text{inside}}^2 \\ &= P_{\text{inside}} \\ P_{\text{air}} + \frac{1}{2}\rho_{\text{water}} v_{\text{outside}}^2 &= P_{\text{air}} + (s-h)\rho_{\text{water}} g \\ \frac{1}{2}\rho_{\text{water}} v_{\text{outside}}^2 &= (s-h)\rho_{\text{water}} g \\ v_{\text{outside}}^2 &= 2(s-h)g \end{aligned}$$

$s = 2.20$ m and $g = 9.8$ m/s^2.

h (m)	v (m/s)
2.02	1.88
1.63	3.34
1.15	4.54

b: The quantity of water which comes through the hole is given by the product of the section of the hole (0.25 cm^2 $= 0.25 \times 10^{-4}$ m^2) and the velocity. So, the only difficulty is converting m^3/s in litres/minute.

$$1 \text{ m}^3/\text{s} = 1000 \text{ litres}/\left(\frac{1}{60} \text{ minute}\right) = 6 \times 10^4 \text{ litres/minute}$$

So, we obtain ($A = 0.25 \times 10^{-4}$ m^2 for area)

h (m)	v (m/s)	Av (litres/minute)
2.02	1.88	2.82
1.63	3.34	5.01
1.15	4.54	6.91

c e **d**: We assume that the water comes out horizontally. So, we have the movement of free fall with an initial horizontal velocity. The time it takes to fall is given by (h hight of the hole)

$$h = \frac{1}{2}gt^2 \quad \Leftrightarrow \quad t^2 = \frac{2h}{g}$$

The horizontal displacement is given by

$$d = vt \quad \Leftrightarrow \quad d^2 = v^2t^2 = 2(s-h)g \times \frac{2h}{g} = 4(s-h)h$$

We obtain (s=2.20 m)

h (m)	v (m/s)	Av (litres/minute)	d (m)
2.02	1.88	2.82	1.21
1.63	3.34	5.01	1.93
1.15	4.54	6.91	2.20

e: Given that the formula for v does not depend on the air pressure (there is no air on the Moon) and the formula for d does not contain the constant of gravity g, the answer is "no". At the Moon one would have obtained exactly the same distances d.

Exercício 20

a: We determine the work over an infinitesimal distance Δu by assuming that the elastic force is constant over an infinitesimal distance. Hence,

$$\Delta W = W(u+\Delta u) - W(u) = F_{\text{elastic}}\Delta u = C_{\text{el}}u\Delta u \quad \Longleftrightarrow \quad \frac{W(u+\Delta u) - W(u)}{\Delta u} = C_{\text{el}}u$$

In the limit for $\Delta u \to 0$ this turns into

$$\frac{dW(u)}{du} = C_{\text{el}}u$$

b: Trivial

c: As we have seen in Ex. 21d

$$E_{\text{cin}}(t) = E_{\text{cin}}(\text{máxima}) - \frac{1}{2}m\omega^2 u^2(t) = E_{\text{cin}}(\text{máxima}) - \frac{1}{2}C_{\text{el}}u^2(t)$$

Hence,

$$E_{\text{cin}}(t) + E_{\text{potencial elástico}}(t) = E_{\text{cin}}(\text{máxima}) - \frac{1}{2}C_{\text{el}}u^2(t) + \frac{1}{2}C_{\text{el}}u^2(t) = E_{\text{cin}}(\text{máxima})$$

The sum of the two forms of energy kinetic and potential is constant, independent of t.

Exercício 21

a:

$$\begin{aligned} h(t) &= \ell - \ell\cos(\alpha(t)) = \ell\,(1 - \cos(\alpha(t))) \\ &= \ell\left(\frac{1}{2}(\alpha(t))^2\right) = \frac{1}{2}\ell\alpha_{\text{max}}^2 \sin^2(\omega t) \end{aligned}$$

b:

$$E_{\text{grav}}(t) = mgh(t) = \frac{1}{2}mg\ell\alpha_{\text{max}}^2 \sin^2(\omega t)$$

A velocidade é dada por

$$v(t) = \frac{d\ell\alpha(t)}{dt} = \ell\alpha_{\text{max}}\omega\cos(\omega t)$$

Hence, for the kinetic energy we have

$$E_{\text{cin}}(t) = \frac{1}{2}m(v(t))^2 = \frac{1}{2}m\ell^2\alpha_{\text{max}}^2\omega^2\cos^2(\omega t$$

From the differential equation of the pendulum

$$\frac{d^2\alpha(t)}{dt^2} = -\frac{g}{\ell}\alpha(t)$$

we learn that

$$\omega^2 = \frac{g}{\ell}$$

Hence

$$E_{\text{cin}}(t) == \frac{1}{2}m\ell^2\alpha_{\text{max}}^2\frac{g}{\ell}\cos^2(\omega t) = \frac{1}{2}mg\ell\alpha_{\text{max}}^2\cos^2(\omega t)$$

Consequently

$$E_{\text{grav}}(t) + E_{\text{cin}}(t) = \frac{1}{2}mg\ell\alpha_{\text{max}}^2\left(\sin^2(\omega t) + \cos^2(\omega t)\right) = \frac{1}{2}mg\ell\alpha_{\text{max}}^2$$

independent of t.

Exercício 23

a:

$$\lambda = \frac{v}{f} = \frac{343 \text{ m/s}}{262 \text{ Hz}} = 1.31 \text{ m} \quad \Longrightarrow \quad \ell = \frac{\lambda}{2} == 0.66 \text{ m}$$

b:

$$d = \frac{\lambda}{2} = \frac{v}{2f} = \frac{343 \text{ m/s}}{2 \times 440 \text{ Hz}} = 0.39 \text{ m}$$

Exercício 24

For a horizontal street the magnitude of the normal force (N) equals the magnitude of the weight (mg) of the car. Hence, the magnitude of the friction force A is given by

$$A = \alpha N = \alpha m g$$

The friction force is the only horizontal force when a car driver uses his brakes, hence the acceleration a is given by

$$ma = -A = -\alpha m g \quad \Longleftrightarrow \quad a = -\alpha g$$

The velocity of the car, while being stopped, is given by

$$v(t) = v_0 + at = v_0 - \alpha g t$$

Let $t = \tau$ be the instant that the car reaches zero velocity (that is when the car stops), then

$$0 = v(\tau) = v_0 - \alpha g \tau \quad \Longleftrightarrow \quad \tau = \frac{v_0}{\alpha g}$$

The displacement of the car during the time the car driver tries to stop his car, is given by

$$x(t) = v_0 t - \frac{1}{2}\alpha g t^2$$

Hence, at the instant that the car is stopped the car has displaced a distance of

$$x(\tau) = v_0 \tau - \frac{1}{2}\alpha g \tau^2 = \frac{v_0^2}{2\alpha g}$$

In the following table we find some values for the distances which it takes to halt the car when it had a certain initial velocity before the car driver started to use his brakes for $\alpha = 0.5$ (with slippery road conditions, like sand on the road, water or ice, α is a lot lower).

v (km/h)	v (m/s)	brake distance (m)	safety distance (m)
20	5.56	3.1	4
50	13.9	20	25
80	22.2	50	64
120	33.3	113	144
140	38.9	154	196
180	50	255	324

The safety distance is calculated by the following formula.

$$\left(\frac{\text{(speed in km/h)}}{10}\right)^2$$

Exercício 25

First, we solve the issue of the initial velocity. When the marble is launched horizontally from a height given by h, we have

$$\vec{v}(t) = \begin{pmatrix} v_0 \\ -gt \end{pmatrix} \quad \text{and} \quad \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \begin{pmatrix} v_0 t \\ h - \frac{1}{2}gt^2 \end{pmatrix}$$

Let us assume that the marble lands on the ground at the instant $t = \tau$ at a horizontal distance of $x(\tau) = d$ from the launcher. That gives

$$\begin{pmatrix} d \\ 0 \end{pmatrix} = \vec{r}(\tau) = \begin{pmatrix} v_0 \tau \\ h - \frac{1}{2}g\tau^2 \end{pmatrix}$$

From the y coordinate we learn

$$0 = y(\tau) = h - \frac{1}{2}g\tau^2 \quad \Longleftrightarrow \quad \tau = \sqrt{\frac{2h}{g}}$$

whereas, from the x coordinate we obtain

$$d = x(\tau) = v_0\tau = v_0\sqrt{\frac{2h}{g}} \quad \Longleftrightarrow \quad v_0 = d\sqrt{\frac{g}{2h}}$$

Next, we assume that the launcher makes an angle α with the horizontal direction. Then one has

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \begin{pmatrix} v_0 \cos(\alpha) t \\ h + v_0 \sin(\alpha) t - \frac{1}{2}gt^2 \end{pmatrix}$$

Let us denote the position vector of the center of the basket by

$$\begin{pmatrix} D \\ H \end{pmatrix}$$

Then, when the marble passes through that position at instant $t = \tau'$, we have

$$\begin{pmatrix} D \\ H \end{pmatrix} = \vec{r}(\tau') = \begin{pmatrix} v_0 \cos(\alpha)\tau' \\ h + v_0 \sin(\alpha)\tau' - \frac{1}{2}g(\tau')^2 \end{pmatrix}$$

From the x coordinate we learn

$$D = v_0 \cos(\alpha)\tau' \quad \Longleftrightarrow \quad \tau' = \frac{D}{v_0 \cos(\alpha)}$$

whereas the y coordinate leads to

$$H = h + v_0 \sin(\alpha)\tau' - \frac{1}{2}g(\tau')^2 = h + v_0 \sin(\alpha)\frac{D}{v_0 \cos(\alpha)} - \frac{1}{2}g\left(\frac{D}{v_0 \cos(\alpha)}\right)^2$$

We may, moreover, substitute v_0 to find

$$H = h + D\tan(\alpha) - \frac{1}{2}g\left(\frac{D}{d\sqrt{\frac{g}{2h}}\cos(\alpha)}\right)^2 = h + D\tan(\alpha) - \frac{hD^2}{d^2}\frac{1}{\cos^2(\alpha)}$$

Now

$$\tan^2(\alpha) + 1 = \frac{\sin^2(\alpha)}{\cos^2(\alpha)} + 1 = \frac{\sin^2(\alpha) + \cos^2(\alpha)}{\cos^2(\alpha)} = \frac{1}{\cos^2(\alpha)}$$

Hence,

$$H = h + D\tan(\alpha) - \frac{hD^2}{d^2}\left(1 + \tan^2(\alpha)\right)$$

So

$$\frac{hD^2}{d^2}\tan^2(\alpha) - D\tan(\alpha) + \left(H - h + \frac{hD^2}{d^2}\right) = 0$$

With

$$A = \frac{hD^2}{d^2} \quad , \quad B = -D \quad \text{and} \quad C = H - h + \frac{hD^2}{d^2}$$

one has solutions

$$\tan(\alpha) = \frac{1}{2A}\left(-B \pm \sqrt{B^2 - 4AC}\right)$$

At this stage we substitute the numerical values of all variables:

$$d = 2.50 \text{ m} \ , \quad D = 1.65 \text{ m} \ , \quad h = 1.0 \text{ m} \quad \text{and} \quad H = 1.42 \text{ m}$$

We find then

$$A = 0.4356 \text{ m} \ , \quad B = -1.65 \text{ m} \ , \quad C = 0.8556 \text{ m} \quad \text{and} \quad \sqrt{B^2 - 4AC} = 1.10982 \text{ m}$$

For $\tan(\alpha)$ and for α we find the solutions

$$\tan(\alpha) = 3.1678 \ \text{ and } \ 0.62004 \quad \Longleftrightarrow \quad \alpha = 72.48^\circ \ \text{ or } \ 31.80^\circ$$

In the two video it is explained that the values must be corrected for air friction. However, if we take $d = 2.35$ m, instead of $d = 2.50$ m, we find

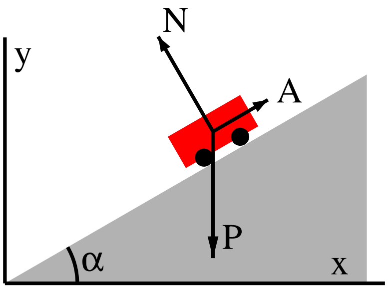
$$\alpha = 69.3^\circ \ \text{ or } \ 35.0^\circ$$

which are exactly the values which launch the marble in the basket. So, maybe the measurement of d was not so accurate.

Exercício 26

We consider three forces for such case: The weight $\vec{P}$ of the car, the normal reaction force $\vec{N}$ of the surface of the car ramp, which force is always perpendicular to the surface, and, furthermore, the friction force $\vec{A}$, which force is always parallel to the surface. The magnitude of the latter force is related to the normal reaction force of the surface. Namely, its magnitude is restricted to

$$A \leq N \times (\text{static friction coefficient})$$



We take the x axis and y axis as indicated in the above figure. Then

$$\vec{P} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} \quad , \quad \vec{N} = \begin{pmatrix} -N\sin(\alpha) \\ N\cos(\alpha) \end{pmatrix} \quad \text{and} \quad \vec{A} = \begin{pmatrix} A\cos(\alpha) \\ A\sin(\alpha) \end{pmatrix}$$

The sum of the three forces must vanish for equilibrium, according to the first law of Newton. Hence,

$$0 = \vec{P} + \vec{N} + \vec{A} = \begin{pmatrix} -N\sin(\alpha) + A\cos(\alpha) \\ -mg + N\cos(\alpha) + A\sin(\alpha) \end{pmatrix}$$

The upper line (x coordinates) gives

$$0 = -N\sin(\alpha) + A\cos(\alpha) \quad \Longleftrightarrow \quad A = N\tan(\alpha)$$

Furthermore

$$A \leq N \times (\text{static friction coefficient})$$

So, when the car starts slipping down for $\alpha = 37°$, then

$$(\text{static friction coefficient}) = \tan(37°) = 0.58$$

Let us also have a look at the y components. With $A = N\tan(\alpha)$ we find

$$0 = -mg + N\cos(\alpha) + A\sin(\alpha) = -mg + N\cos(\alpha) + N\frac{\sin^2(\alpha)}{\cos(\alpha)} = -mg + N\frac{1}{\cos(\alpha)}$$

Consequently,

$$N = mg\cos(\alpha)$$

as had to be expected.

Exercício 27

Let us start with the masses B and C. With respect to the point where their bar is hanging from, the mass B gives a positive torque (anticlockwise) with magnitude given by

$$0.168\ m_B g$$

whereas mass C gives a negative torque (clockwise) with magnitude given by

$$0.046\ m_C g$$

For equilibrium those two torques must be in balance. Hence

$$0.168\, m_B g = 0.046\, m_C g \quad \Longleftrightarrow \quad m_C = \frac{0.168}{0.046} m_B$$

Similarly, the two masses are in balance with mass D:

$$0.135\,(m_B + m_C)g = 0.046\, m_D g \quad \Longleftrightarrow \quad m_B + m_C = \frac{0.046}{0.135}\, m_D = 27.26 \text{ g}$$

From that one can determine the masses of B and C:

$$\frac{0.214}{0.046} m_B = m_B + \frac{0.168}{0.046} m_B = m_B + m_C = 27.26 \text{ g} \quad \Longleftrightarrow \quad m_B = \frac{0.046}{0.214}\, 27.26 \text{ g} = 5.86 \text{ g}$$

hence

$$m_C = (27.26 \text{ g}) - (5.86 \text{ g}) = 21.4 \text{ g}$$

Similarly, we find for mass A and the sum of the masses of B, C and D the relation

$$0.319\, m_A = 0.076\,(m_B + m_C + m_D)$$

from which we deduce

$$m_A = \frac{0.076}{0.319}\,(m_B + m_C + m_D) = \frac{0.076}{0.319}\,((27.26 \text{ g}) + (80 \text{ g})) = 25.55 \text{ g}$$

Exercício 28

The moments of inertia for each case are most easily obtained by selecting the origin at the rotation axis. **a**:

$$I = (2.0 \text{ kg}) \times 0^2 + (1.0 \text{ kg}) \times (1.00 \text{ m})^2 = 1.00 \text{ kg m}^2$$

b:

$$I = (2.0 \text{ kg}) \times (0.50 \text{ m})^2 + (1.0 \text{ kg}) \times (0.50 \text{ m})^2 = 0.75 \text{ kg m}^2$$

c: For determining the center of mass we must first choose a coordinate system, say that the mass of 2.0 kg is in the origin. Then

$$x_{\text{CM}} = \frac{(2.0 \text{ kg}) \times 0 + (1.0 \text{ kg}) \times (1.00 \text{ m})}{(2.0 \text{ kg}) + (1.0 \text{ kg})} = 0.333 \text{ m}$$

Hence, the center of mass is at a distance of 0.333 m from the mass of 2.0 kg. Next, we choose that position for the origin

$$I = (2.0 \text{ kg}) \times (0.33 \text{ m})^2 + (1.0 \text{ kg}) \times (0.67 \text{ m})^2 = 0.67 \text{ kg m}^2$$

Exercício 29

a:

$$F = m\omega^2 r = (0.25\ \text{kg}) \times (0.5\pi\ \text{s}^{-1})^2 \times (0.10\ \text{m}) = 0.062\ \text{N}$$

b: When that force is removed, the point masses cannot be hold in a circular motion and thus fly towards the extremes of the bar where they are stopped from flying away.

c: Initially

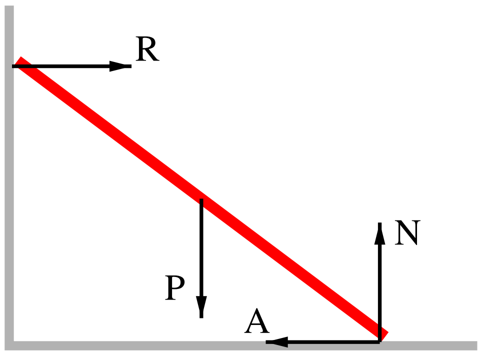
$$L = I\omega = 2mr^2\omega = 2 \times (0.25\ \text{kg}) \times (0.10\ \text{m})^2 \times (0.5\pi\ \text{s}^{-1}) = 7.85 \times 10^{-3}\ \text{Js}$$

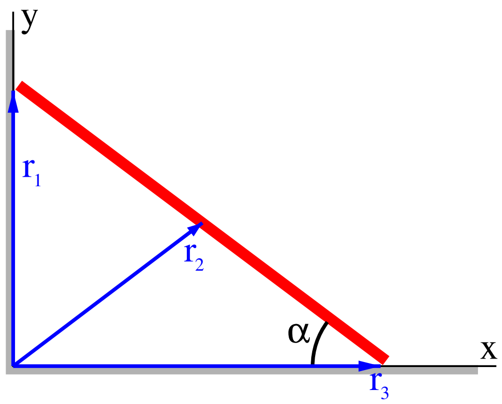
Since the system does not suffer from external forces the final angular momentum is the same. Hence

$$\omega_{\text{final}} = \frac{L}{2mr^2} = \frac{7.85 \times 10^{-3}\ \text{Js}}{2 \times (0.25\ \text{kg}) \times (0.50\ \text{m})^2} = 0.063\ \text{rad/s} = 3.6\ \text{deg/s}$$

Exercício 30

In the figure below we show the four forces which are involved in this case: The weight of the wooden bar $\vec{P}$, the normal reaction force of the floor $\vec{N}$, the friction force between the wooden bar and the floor $\vec{A}$ and the normal reaction force of the wall $\vec{R}$. In the case of point particles all forces apply in the same point. However, in the case of extended objects the forces have different points of application. In the righthand figure we have indicated the positions where the four forces apply: $\vec{r}_1$ where the normal reaction force of the wall $\vec{R}$ applies, $\vec{r}_2$ where the weight of the wooden bar $\vec{P}$ applies and $\vec{r}_3$ where the normal reaction force of the floor $\vec{N}$ and the friction force between the wooden bar and the floor $\vec{A}$ apply.





We have chosen the xy coordinates as indicated in the righthand figure. The resulting vector representations for the four forces are then given by

$$\vec{P} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} \quad , \quad \vec{N} = \begin{pmatrix} 0 \\ N \end{pmatrix} \quad , \quad \vec{R} = \begin{pmatrix} R \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{A} = \begin{pmatrix} -A \\ 0 \end{pmatrix}$$

Let us denote the heigth where the wooden bar touches the wall by h and the distance from the origin of our coordinate system to where the wooden bar touches the floor by d. Then

$$\vec{r}_1 = \begin{pmatrix} 0 \\ h \end{pmatrix} \quad , \quad \vec{r}_2 = \begin{pmatrix} \frac{1}{2}d \\ \frac{1}{2}h \end{pmatrix} \quad \text{and} \quad \vec{r}_3 = \begin{pmatrix} d \\ 0 \end{pmatrix}$$

The angle α is given by

$$\tan(\alpha) = \frac{h}{d}$$

When there is equilibrium, then the forces cancel each other (first law of Newton). Hence:

$$0 = \vec{P} + \vec{N} + \vec{R} + \vec{A} = \begin{pmatrix} R - A \\ -mg + N \end{pmatrix}$$

From which we obtain

$$A = R \quad \text{and} \quad N = P = mg$$

Consequently, the forces $\vec{R}$ and $\vec{A}$ are undetermined.

But, unlike point particles, extended objects can rotate. They do so when a torque is acting on them. In equilibrium, when nothing rotates, there is no torque acting on the wooden bar. Hence, for the torques of the four forces one has

$$\begin{aligned} 0 &= \vec{r}_1 \times \vec{R} + \vec{r}_2 \times \vec{P} + \vec{r}_3 \times \vec{N} + \vec{r}_3 \times \vec{A} \\ &= \begin{pmatrix} 0 \\ 0 \\ -hR \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}dmg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ dN \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -hR - \frac{1}{2}dmg + dN \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -hA + \frac{1}{2}dN \end{pmatrix} \end{aligned}$$

Notice that the torque vectors associated with anticlockwise rotation are in the positive z direction, whereas those associated with clockwise rotation are in the negative z direction.

We find thus:

$$\frac{A}{N} = \frac{1}{2}\frac{d}{h} = \frac{1}{2\tan(\alpha)}$$

Furthermore we know that

$$\frac{1}{2\tan(\alpha)} = \frac{A}{N} \leq \text{(static friction coefficient)}$$

Consequently, if the wooden bar starts slipping away at angles smaller than 32°, then

$$\text{(static friction coefficient)} = \frac{1}{2\tan(32^\circ)} = 0.80$$

Exercício 31

a: Since there are no horizontal forces acting on the bar, its center of mass does not move in the horizontal direction. Consequently, it can only move in the vertical direction.

b: The total kinetic energy of the bar is given by the velocity v of the center of mass and by the angular velocity ω of the rotation of the bar arond the center of mass:

$$E_{\text{kinetic}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

Since the mass is homogeneously distributed along the bar, the center of mass of the bar is at $\frac{1}{2}\ell$ from the place where one end of the bar is on the floor. Consequently, the relation between ω and v is given by

$$v = \tfrac{1}{2}\ell\omega$$

Hence,

$$I\omega^2 = \tfrac{1}{12}m\ell^2 \left(\frac{v}{\frac{1}{2}\ell}\right)^2 = \tfrac{1}{3}mv^2$$

For the kinetic energy we obtain then

$$E_{\text{kinetic}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{6}mv^2 = \tfrac{2}{3}mv^2$$

Before the bar started falling it had a gravitational potential given by

$$E_{\text{gravity}} = mgh = mg\tfrac{1}{2}\ell$$

That energy is all converted into kinetic energy just before the bar touches the floor. Hence,

$$\tfrac{2}{3}mv^2 = mg\tfrac{1}{2}\ell$$

from which we solve

$$v^2 = \tfrac{3}{2}\tfrac{1}{2}g\ell = \tfrac{3}{4}g\ell \quad \Longleftrightarrow \quad v = \sqrt{\tfrac{3}{4}g\ell}$$

The vertical velocity of the upper end of the bar is twice the velocity of the center of mass of the bar, because it is at twice the distance from the end which is on the floor during the motion.

Exercício 32

a: One complete oscillation is performed in a time $T = 1/f$, where f represents the frequency of the sound. In that time the train moves a distance given by

$$\Delta x = v_{\text{train}}T = \frac{v_{\text{train}}}{f} = \frac{360 \ \text{km/h}}{55 \ \text{Hz}} = \frac{100 \ \text{m/s}}{55 \ \text{s}^{-1}} = 1.82 \ \text{m}$$

b: Let us define the time interval between the arrival of the onset of the complete oscillation and the arrival of the end of it, by Δt. The time it takes for a wave to propagate the distance x is given by x/v_{sound}. The time it takes for a wave to propagate the distance $x - \Delta x$ is given by $(x - \Delta x)/v_{\text{sound}}$. But, the onset of the emission of the complete oscillation started a time T earlier than the end of the emission of the complete oscillation. Hence,

$$\begin{aligned}\Delta t &= T + \frac{x - \Delta x}{v_{\text{sound}}} - \frac{x}{v_{\text{sound}}} = T - \frac{\Delta x}{v_{\text{sound}}} = \frac{1}{f} - \frac{1}{f}\frac{v_{\text{train}}}{v_{\text{sound}}} \\ &= \frac{1}{f}\left(1 - \frac{v_{\text{train}}}{v_{\text{sound}}}\right) = \frac{1}{55 \ \text{s}^{-1}}\left(1 - \frac{100 \ \text{m/s}}{340 \ \text{m/s}}\right) = 0.0128 \ \text{s}\end{aligned}$$

c: The person's eardrum makes one complete oscillation in 0.0128 s. Hence, the frequency the person observes equals

$$f_{\text{person}} = \frac{1}{0.0128 \ \text{s}} = 78 \ \text{Hz}$$

which is a higher pitch, actually $(D^{\#}/E^{b})_2$, than what the train emits.