RELATIVIDADE GERAL 2011-2012

1. Considere três referenciais A, B e C, cujas coordenadas são dadas por respectivamente

$$\begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix} , \begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} \text{ e } \begin{pmatrix} t^{(C)} \\ x_1^{(C)} \\ x_2^{(C)} \\ x_3^{(C)} \end{pmatrix} .$$

O referencial A está em movimento relativamente ao referencial B com velocidade constante $\vec{\beta}^{(AB)}$, enquanto o referencial B está em movimento relativamente ao referencial C com velocidade constante $\vec{\beta}^{(BC)}$.
A transformação entre as coordenadas dos referenciais A e B é dado pelo "*boost*" $\Lambda(AB)$:

$$\begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} = \Lambda(AB) \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda(AB)^0{}_0 & \Lambda(AB)^0{}_1 & \Lambda(AB)^0{}_2 & \Lambda(AB)^0{}_3 \\ \Lambda(AB)^1{}_0 & \Lambda(AB)^1{}_1 & \Lambda(AB)^1{}_2 & \Lambda(AB)^1{}_3 \\ \Lambda(AB)^2{}_0 & \Lambda(AB)^2{}_1 & \Lambda(AB)^2{}_2 & \Lambda(AB)^2{}_3 \\ \Lambda(AB)^3{}_0 & \Lambda(AB)^3{}_1 & \Lambda(AB)^3{}_2 & \Lambda(AB)^3{}_3 \end{pmatrix} \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix} \tag{1}$$

onde

$$\Lambda(AB)^0{}_0 = \gamma_{AB} = \frac{1}{\sqrt{1 - \vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)}}} \quad ,$$

$$\Lambda(AB)^0{}_i = \Lambda(AB)^i{}_0 = \gamma_{AB}\beta_i^{(AB)}$$

$$\text{e} \quad \Lambda(AB)^i{}_j = \delta_{ij} + \frac{\beta_i^{(AB)}\beta_j^{(AB)}}{\vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)}}(\gamma_{AB} - 1) \quad ,$$

para $i, j = 1, 2, 3$.
A transformação $\Lambda(BC)$ entre as coordenadas dos referenciais B e C obtem-se pela substituição de A por B e B por C na Equação (1).
Demonstre $\Lambda(BC)\Lambda(AB) = \Lambda(AC)R$, onde $\Lambda(AC)$ represente um Lorentz boost e R uma rotação espacial.

Solutions

Exercício 1

For a Lorentz boost in any direction, given by $\vec{\beta}$, one has to remember that the space components which are perpendicular to the boost, do not suffer any contraction. Hence, when we decompose a three-dimensional vector $\vec{r}$ into its parallel and perpendicular to $\vec{\beta}$ components, by ($\hat{\beta} = \vec{\beta}/\beta$)

$$\vec{r} = \left(\hat{\beta}\cdot\vec{r}\right)\hat{\beta} + \vec{r}_{\perp} \quad ,$$

then the Lorentz boost can be written by ($\gamma = 1/\sqrt{1-\beta^2}$)

$$\begin{aligned} t' &= \gamma\left\{t + \beta\left(\hat{\beta}\cdot\vec{r}\right)\right\} \\ \text{and}\quad \vec{r}\,' &= \gamma\left\{\left(\hat{\beta}\cdot\vec{r}\right) + \beta t\right\}\hat{\beta} + \vec{r}_{\perp} \\ &= \gamma\left\{\left(\hat{\beta}\cdot\vec{r}\right) + \beta t\right\}\hat{\beta} + \vec{r} - \left(\hat{\beta}\cdot\vec{r}\right)\hat{\beta} \quad . \end{aligned}$$

Next, we elaborate on the expressions

$$\begin{aligned} t' &= \gamma t + \gamma\vec{\beta}\cdot\vec{r} \\ \text{and}\quad \vec{r}\,' &= \gamma\vec{\beta}t + \gamma\left(\hat{\beta}\cdot\vec{r}\right)\hat{\beta} + \vec{r} - \left(\hat{\beta}\cdot\vec{r}\right)\hat{\beta} \\ &= \gamma\vec{\beta}t + \vec{r} + (\gamma-1)\left(\hat{\beta}\cdot\vec{r}\right)\hat{\beta} = \gamma\vec{\beta}t + \vec{r} + (\gamma-1)\frac{\left(\vec{\beta}\cdot\vec{r}\right)\vec{\beta}}{\beta^2} \quad , \end{aligned}$$

which in index notation reads

$$\begin{aligned} t' &= \gamma t + \gamma\beta_i x_i \\ \text{and}\quad x_i' &= \gamma\beta_i t + \left\{\delta_{ij} + (\gamma-1)\frac{\beta_i\beta_j}{\beta^2}\right\}x_j \quad . \end{aligned}$$

In order to perform the calculus of the product of the two boosts, we will start by simplifying the matrix (1). For that we define

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \quad . \tag{2}$$

With definition (2) we have

$$\boldsymbol{\alpha}\boldsymbol{\beta}^T = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\beta_3 \\ \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_3\beta_3 \end{pmatrix} \quad . \tag{3}$$

Moreover,

$$\boldsymbol{\alpha}^T\boldsymbol{\beta} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = \vec{\alpha}\cdot\vec{\beta} \quad . \tag{4}$$

The product $\boldsymbol{\alpha}\boldsymbol{\beta}^T$ (3) has the following property

$$\left(\boldsymbol{\alpha}\boldsymbol{\beta}^T\right)\boldsymbol{\sigma} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\beta_3 \\ \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_3\beta_3 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = (\beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \left(\boldsymbol{\beta}^T\boldsymbol{\sigma}\right)\boldsymbol{\alpha} \quad . \tag{5}$$

The product $\boldsymbol{\beta}^T\boldsymbol{\sigma}$ (4) has the well-known property

$$\boldsymbol{\beta}^T\boldsymbol{\sigma} = \vec{\beta}\cdot\vec{\sigma} = \vec{\sigma}\cdot\vec{\beta} = \boldsymbol{\sigma}^T\boldsymbol{\beta} \quad . \tag{6}$$

Moreover, property (5) follows from the fact that $\boldsymbol{\beta}^T\boldsymbol{\sigma}$ is a scalar (6) and the associativity property of matrix multiplication:

$$\left(\boldsymbol{\alpha}\boldsymbol{\beta}^T\right)\boldsymbol{\sigma} = \boldsymbol{\alpha}\left(\boldsymbol{\beta}^T\boldsymbol{\sigma}\right) = \left(\boldsymbol{\beta}^T\boldsymbol{\sigma}\right)\boldsymbol{\alpha} \quad .$$

Furthermore, we define

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

Using the above definitions and $\gamma^2\beta^2 = \gamma^2 - 1$, we may write

$$\begin{pmatrix} \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} = I + \frac{\gamma - 1}{\vec{\beta}\cdot\vec{\beta}} \begin{pmatrix} \beta_1\beta_1 & \beta_1\beta_2 & \beta_1\beta_3 \\ \beta_2\beta_1 & \beta_2\beta_2 & \beta_2\beta_3 \\ \beta_3\beta_1 & \beta_3\beta_2 & \beta_3\beta_3 \end{pmatrix} = I + (\gamma - 1)\,\frac{\boldsymbol{\beta}\boldsymbol{\beta}^T}{\boldsymbol{\beta}^T\boldsymbol{\beta}} = I + \frac{\gamma^2}{\gamma + 1}\boldsymbol{\beta}\boldsymbol{\beta}^T,$$

$$\begin{pmatrix} \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \end{pmatrix} = \gamma\,\boldsymbol{\beta}^T \quad ,$$

$$\begin{pmatrix} \Lambda^0{}_1 \\ \Lambda^0{}_2 \\ \Lambda^0{}_3 \end{pmatrix} = \gamma\,\boldsymbol{\beta} \quad .$$

Putting things together, we find

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\,\boldsymbol{\beta}^T \\ \gamma\,\boldsymbol{\beta} & I + \dfrac{\gamma^2}{\gamma + 1}\boldsymbol{\beta}\boldsymbol{\beta}^T \end{pmatrix} \quad . \tag{7}$$

1. $\Lambda(BC)\Lambda(AB)$

Next, we devote ourselves to the calculus of $\Lambda(BC)\Lambda(AB)$. We write

$$\Lambda(BC)\Lambda(AB) = \tag{8}$$

$$= \begin{pmatrix} \gamma_{BC} & \gamma_{BC}\ \boldsymbol{\beta}_{BC}^T \\ \gamma_{BC}\ \boldsymbol{\beta}_{BC} & I + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T \end{pmatrix} \begin{pmatrix} \gamma_{AB} & \gamma_{AB}\ \boldsymbol{\beta}_{AB}^T \\ \gamma_{AB}\ \boldsymbol{\beta}_{AB} & I + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{BC}\gamma_{AB} + \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} & \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{AB}^T + \gamma_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T \\ \begin{array}{c} \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC} + \gamma_{AB}\boldsymbol{\beta}_{AB} + \\ + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\gamma_{AB}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} \end{array} & \begin{array}{c} \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T + \\ +I + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \\ + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T \end{array} \end{pmatrix} .$$

From formula (8) we read

$$(8a) \qquad [\Lambda(BC)\Lambda(AB)]^0{}_0 = \gamma_{BC}\gamma_{AB} + \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}$$

$$(8b) \qquad [\Lambda(BC)\Lambda(AB)]^0{}_i = \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{AB}^T + \gamma_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T$$

$$(8c) \qquad [\Lambda(BC)\Lambda(AB)]^i{}_0 = \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC} + \gamma_{AB}\boldsymbol{\beta}_{AB} + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\gamma_{AB}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}$$

$$(8d) \qquad [\Lambda(BC)\Lambda(AB)]^i{}_j = I + \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T +$$

$$+ \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T$$

$$= I + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\gamma_{AB}\left\{1 + \frac{\gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}}{(\gamma_{BC}+1)(\gamma_{AB}+1)}\right\}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T \quad .$$

2. $\boldsymbol{\beta}_{AC}$

Before we continue, we first must translate $\boldsymbol{\beta}_{AB}$ into $\boldsymbol{\beta}_{AC}$. For that it is necessary to know how velocities transform from B to C. Using Eq. (7), we deduce

$$\begin{pmatrix} t^{(C)} \\ \vec{x}^{\,(C)} \end{pmatrix} = \begin{pmatrix} \gamma_{BC} & \gamma_{BC}\ \boldsymbol{\beta}_{BC}^T \\ \gamma_{BC}\ \boldsymbol{\beta}_{BC} & I + \dfrac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T \end{pmatrix} \begin{pmatrix} t^{(B)} \\ \vec{x}^{\,(B)} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{BC} t^{(B)} + \gamma_{BC}\ \boldsymbol{\beta}_{BC}^T \vec{x}^{\,(B)} \\ \gamma_{BC}\ \boldsymbol{\beta}_{BC} t^{(B)} + \left\{ I + \dfrac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T \right\} \vec{x}^{\,(B)} \end{pmatrix} \quad . \tag{9}$$

From the result (9) we find

$$\frac{d\,t^{(C)}}{dt^{(B)}} = \gamma_{BC} + \gamma_{BC}\ \boldsymbol{\beta}_{BC}^T \frac{d\,\vec{x}^{\,(B)}}{dt^{(B)}} = \gamma_{BC} + \gamma_{BC}\ \boldsymbol{\beta}_{BC}^T \vec{v}^{(B)}$$

and

$$\frac{d\,\vec{x}^{(C)}}{dt^{(B)}} = \gamma_{BC}\ \boldsymbol{\beta}_{BC} + \left\{ I + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T \right\} \frac{d\,\vec{x}^{\,(B)}}{dt^{(B)}} = \gamma_{BC}\ \boldsymbol{\beta}_{BC} + \left\{ I + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T \right\} \vec{v}^{(B)}$$

Hence, the transformation which we are looking for, results in

$$\frac{d\,\vec{x}^{(C)}}{dt^{(c)}} = \frac{d\,t^{(B)}}{dt^{(C)}}\frac{d\,\vec{x}^{(C)}}{dt^{(B)}} = \frac{\dfrac{d\,\vec{x}^{(C)}}{dt^{(B)}}}{\dfrac{d\,t^{(C)}}{dt^{(B)}}} = \frac{\gamma_{BC}\ \boldsymbol{\beta}_{BC} + \left\{ I + \dfrac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T \right\} \vec{v}^{(B)}}{\gamma_{BC} + \gamma_{BC}\ \boldsymbol{\beta}_{BC}^T \vec{v}^{(B)}} \quad . \tag{10}$$

In particular, for $\boldsymbol{\beta}_{AC}$, which is the velocity of reference frame A with respect to reference frame C, we obtain, in terms of the velocity $\boldsymbol{\beta}_{AB}$ of reference frame A with respect to reference frame B, the following expression

$$\boldsymbol{\beta}_{AC} = \frac{\gamma_{BC}\ \boldsymbol{\beta}_{BC} + \left\{ I + \dfrac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T \right\} \boldsymbol{\beta}_{AB}}{\gamma_{BC} + \gamma_{BC}\ \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}}$$

$$= \frac{(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB} + \left\{ \gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right) \right\} \boldsymbol{\beta}_{BC}}{\gamma_{BC}\,(\gamma_{BC}+1) \left\{ 1 + \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB} \right\}} \quad . \tag{11}$$

Notice that, in the above, we made use of (see Eq. 5)

$$\left(\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\right)\boldsymbol{\beta}_{AB} = \boldsymbol{\beta}_{BC}\left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right) = \left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)\boldsymbol{\beta}_{BC} \quad . \tag{12}$$

Notice, furthermore, (see Eq. 6) that $\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{BC}$ and $\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}$ are scalars.

3. γ_{AC}

Next, we need to determine γ_{AC} in terms of γ_{BC}, $\boldsymbol{\beta}_{BC}$ and $\boldsymbol{\beta}_{AB}$.
Let us start with (see Eq. 4),

$$\left|\vec{\beta}_{AC}\right|^2 = \boldsymbol{\beta}_{AC}^T \boldsymbol{\beta}_{AC} =$$

$$= \left(\frac{(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB} + \left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{BC}}{\gamma_{BC}\,(\gamma_{BC}+1) \left\{1 + \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right\}} \right)^T$$

$$\left(\frac{(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB} + \left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{BC}}{\gamma_{BC}\,(\gamma_{BC}+1) \left\{1 + \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right\}} \right) \quad . \tag{13}$$

The numerator of the product (13) has 4 terms. We collect them one by one:

$$(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB}^T\,(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB} = (\gamma_{BC}+1)^2\,\boldsymbol{\beta}_{AB}^T \boldsymbol{\beta}_{AB} \quad , \tag{14}$$

$$(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB}^T \left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{BC} =$$

$$= (\gamma_{BC}+1) \left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{AB}^T \boldsymbol{\beta}_{BC}$$

$$= (\gamma_{BC}+1) \left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB} \quad , \tag{15}$$

$$\left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{BC}^T\,(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB} =$$

$$= (\gamma_{BC}+1) \left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB} \quad , \tag{16}$$

$$\left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\}^2 \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{BC} =$$

$$= \left\{(\gamma_{BC}+1) + \gamma_{BC} \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\}^2 \gamma_{BC}^2 \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{BC}$$

$$= \left\{(\gamma_{BC}+1) + \gamma_{BC} \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\}^2 \left(\gamma_{BC}^2 - 1\right) \quad , \tag{17}$$

The sum of the 4 terms of the numerator of the product (13), Eqs. (14-17), gives, also using the fact that $\gamma^2\beta^2 = \gamma^2 - 1$

$$\text{numerator} = (\gamma_{BC}+1) \Big[(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB}^T \boldsymbol{\beta}_{AB} +$$

$$+2 \left\{\gamma_{BC}\,(\gamma_{BC}+1) + \gamma_{BC}^2 \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\} \boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB} +$$

$$+ (\gamma_{BC}-1) \left\{(\gamma_{BC}+1) + \gamma_{BC} \left(\boldsymbol{\beta}_{BC}^T \boldsymbol{\beta}_{AB}\right)\right\}^2 \Big]$$

$$= (\gamma_{BC}+1)\left[(\gamma_{BC}+1)\,\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB} + (\gamma_{BC}+1)\left(\gamma_{BC}^2-1\right)\right.$$

$$\left. + 2\gamma_{BC}^2(\gamma_{BC}+1)\,\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} \;+ \gamma_{BC}^2(\gamma_{BC}+1)\left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)^2\right]$$

$$= (\gamma_{BC}+1)^2\left[\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB} + \left(\gamma_{BC}^2-1\right) + 2\gamma_{BC}^2\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} \;+ \gamma_{BC}^2\left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)^2\right] \quad . \tag{18}$$

The denominator of the product (13) gives

$$\text{denominator} = \gamma_{BC}^2(\gamma_{BC}+1)^2\left(1+\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)^2 \quad . \tag{19}$$

Hence, putting things together, also using $\gamma^{-2} = 1 - \boldsymbol{\beta}^T\boldsymbol{\beta}$, we have for Eq. (13)

$$\gamma_{AC}^{-2} = 1 - \left|\vec{\beta}_{AC}\right|^2 = 1 - \boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{AC} = 1 - \frac{\text{numerator}}{\text{denominator}} =$$

$$= 1 - \frac{\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB} + 2\gamma_{BC}^2\left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right) + \gamma_{BC}^2\left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)^2 + \gamma_{BC}^2 - 1}{\gamma_{BC}^2\left(1+\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)^2}$$

$$= \frac{1-\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB}}{\gamma_{BC}^2\left(1+\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)^2} = \frac{1}{\gamma_{BC}^2\gamma_{AB}^2\left(1+\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right)^2} \quad . \tag{20}$$

Hence,

$$\gamma_{AC} = \gamma_{BC}\gamma_{AB}\left(1+\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\right) \quad , \tag{21}$$

which is exactly the $[\Lambda(BC)\Lambda(AB)]^0{}_0$ term of expression (8a).

4. $\boldsymbol{\gamma_{AC}\beta_{AC}}$

Next, using Eqs. (11) and (21), we determine

$$
\begin{aligned}
&\gamma_{AC}\boldsymbol{\beta}_{AC} = \\
&\quad = \gamma_{BC}\gamma_{AB}\left(1+\boldsymbol{\beta}_{BC}^{T}\boldsymbol{\beta}_{AB}\right)\frac{\left(\gamma_{BC}+1\right)\boldsymbol{\beta}_{AB}+\left\{\gamma_{BC}\left(\gamma_{BC}+1\right)+\gamma_{BC}^{2}\left(\boldsymbol{\beta}_{BC}^{T}\boldsymbol{\beta}_{AB}\right)\right\}\boldsymbol{\beta}_{BC}}{\gamma_{BC}\left(\gamma_{BC}+1\right)\left\{1+\boldsymbol{\beta}_{BC}^{T}\boldsymbol{\beta}_{AB}\right\}} \\
&\quad = \gamma_{AB}\boldsymbol{\beta}_{AB}+\left\{\gamma_{BC}\gamma_{AB}+\frac{\gamma_{BC}^{2}\gamma_{AB}}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}^{T}\boldsymbol{\beta}_{AB}\right\}\boldsymbol{\beta}_{BC} \\
&\quad = \gamma_{AB}\boldsymbol{\beta}_{AB}+\left\{\gamma_{BC}\gamma_{AB}+\frac{\gamma_{BC}(\gamma_{AC}-\gamma_{BC}\gamma_{AB})}{\gamma_{BC}+1}\right\}\boldsymbol{\beta}_{BC} \\
&\quad = \gamma_{AB}\boldsymbol{\beta}_{AB}+\frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC} \quad ,
\end{aligned}
\tag{22}
$$

which are exactly the $[\Lambda(BC)\Lambda(AB)]^{i}{}_{0}$ terms of expression (8c).

5. $\gamma_{AC}\boldsymbol{\beta}^T_{AC}$

One expects now that the $[\Lambda(BC)\Lambda(AB)]^0{}_i$ terms are equal to $\gamma_{AC}\boldsymbol{\beta}^T_{AC}$. However, as one can easily see from (8b) and (8c), that is not the case. The reason is that a general Lorentz transformation is the combination of a Lorentz boost and a rotation R:

$$\begin{aligned}\Lambda(BC)\Lambda(AB) &= \begin{pmatrix} \gamma_{AC} & \gamma_{AC}\,\boldsymbol{\beta}^T_{AC} \\ \gamma_{AC}\,\boldsymbol{\beta}_{AC} & I + \dfrac{\gamma^2_{AC}}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\boldsymbol{\beta}^T_{AC} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{AC} & \gamma_{AC}\,\boldsymbol{\beta}^T_{AC}R \\ \gamma_{AC}\,\boldsymbol{\beta}_{AC} & R + \dfrac{\gamma^2_{AC}}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\boldsymbol{\beta}^T_{AC}\,R \end{pmatrix} \;. \end{aligned} \tag{23}$$

It leaves the $[\Lambda(BC)\Lambda(AB)]^0{}_0$ and $[\Lambda(BC)\Lambda(AB)]^i{}_0$ nicely in peace, exactly as we found. But, the $[\Lambda(BC)\Lambda(AB)]^0{}_i$ and $[\Lambda(BC)\Lambda(AB)]^i{}_j$ terms are affected.

6. $\boldsymbol{R}$

We deduce from Eq. (23) for R, also using expressions (8), (8a-d)

$$\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} = \Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB) = \tag{24}$$

$$= \begin{pmatrix} \gamma_{AC} & \gamma_{AC}\,\boldsymbol{\beta}_{AC}^T \\ \gamma_{AC}\,\boldsymbol{\beta}_{AC} & I + \frac{\gamma_{AC}^2}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\boldsymbol{\beta}_{AC}^T \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \gamma_{BC}\gamma_{AB} + \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} & \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{AB}^T + \gamma_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T \\ \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC} + \gamma_{AB}\boldsymbol{\beta}_{AB} + & I + \gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T + \\ +\frac{\gamma_{BC}^2}{\gamma_{BC}+1}\gamma_{AB}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} & +\frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \\ & +\frac{\gamma_{BC}^2}{\gamma_{BC}+1}\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T \end{pmatrix} .$$

Now, the inverse of a boost with velocity $\boldsymbol{\beta}$ is the same expression, but with $\boldsymbol{\beta}$ replaced by $-\boldsymbol{\beta}$. Consequently, also using relation (21),

$$\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} = \Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB) = \tag{25}$$

$$= \begin{pmatrix} \gamma_{AC} & -\gamma_{AC}\,\boldsymbol{\beta}_{AC}^T \\ -\gamma_{AC}\,\boldsymbol{\beta}_{AC} & I + \frac{\gamma_{AC}^2}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\boldsymbol{\beta}_{AC}^T \end{pmatrix}$$

$$\begin{pmatrix} \gamma_{AC} & \gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T \\ \gamma_{AC}\boldsymbol{\beta}_{AC} & I + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \\ & +\gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T \end{pmatrix} .$$

We may start by verifying the $(0,0)$ element of the rotation matrix, which should give 1. We obtain from the matrix product the following result:

$$[\Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB)]^0{}_0 = \gamma_{AC}^2 - \gamma_{AC}^2\boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{AC} = 1 \quad , \tag{26}$$

which is what we expected.

The $(0,i)$ element of the matrix product (25) must vanish. We find

$$[\Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB)]^i{}_0 = -\gamma_{AC}^2\boldsymbol{\beta}_{AC} + \left(I + \frac{\gamma_{AC}^2}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\boldsymbol{\beta}_{AC}^T\right)\gamma_{AC}\boldsymbol{\beta}_{AC} = \tag{27}$$

$$= -\gamma_{AC}\boldsymbol{\beta}_{AC}\left(-\gamma_{AC}+1+\frac{\gamma_{AC}^2}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{AC}\right) = -\gamma_{AC}\boldsymbol{\beta}_{AC}\left(-\gamma_{AC}+1+\frac{\gamma_{AC}^2-1}{\gamma_{AC}+1}\right) = 0 \quad .$$

Also the $(0,i)$ element of the matrix product (25) must vanish:

$$[\Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB)]^0{}_i = \gamma_{AC}\left(\gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T\right) +$$

$$-\gamma_{AC}\boldsymbol{\beta}_{AC}^T\left(I + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right) \tag{28}$$

Before we determine the $(0,i)$ element of the matrix product (25), we perform the following calculus, making use of expressions (21) for γ_{AC} and (22) for $\gamma_{AC}\ \boldsymbol{\beta}_{AC}$.

$$\gamma_{AC}\ \boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{AB} = \left(\gamma_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}^T\right)\ \boldsymbol{\beta}_{AB} =$$

$$= \gamma_{AB}\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB} + \frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} = \frac{1}{\gamma_{AB}}\left(\gamma_{AB}^2 - 1 + \frac{(\gamma_{AC}-\gamma_{BC}\gamma_{AB})(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\right)$$

$$= \frac{1}{\gamma_{AB}}\left(\frac{(\gamma_{AC}+\gamma_{AB})^2}{\gamma_{BC}+1} - \gamma_{AC}\gamma_{AB} - 1\right) \tag{29}$$

and

$$\gamma_{AC}\ \boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{BC} = \left(\gamma_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}^T\right)\ \boldsymbol{\beta}_{BC} =$$

$$= \gamma_{AB}\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{BC} + \frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{BC} = \frac{1}{\gamma_{BC}}\left(\gamma_{AC} - \gamma_{BC}\gamma_{AB} + \frac{(\gamma_{BC}^2-1)(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\right)$$

$$= \gamma_{AC} - \frac{\gamma_{AB}}{\gamma_{BC}} \quad . \tag{30}$$

With the use of Eqs. (22), (29) and (30), we obtain for the $(0,i)$ element of the matrix product (25) the following.

$$[\Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB)]^0{}_i = \gamma_{AC}\gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AC}\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T - \gamma_{AC}\boldsymbol{\beta}_{AC}^T +$$

$$
-\frac{\gamma_{AC}\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T-\frac{\gamma_{AC}\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T-\gamma_{AC}\gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\boldsymbol{\beta}_{AC}^T\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T
$$

$$
=\gamma_{AC}\gamma_{BC}\boldsymbol{\beta}_{BC}^T+\frac{\gamma_{AC}\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T-\gamma_{AB}\boldsymbol{\beta}_{AB}^T-\frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}^T+
$$

$$
-\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\frac{1}{\gamma_{AB}}\left(\frac{(\gamma_{AC}+\gamma_{AB})^2}{\gamma_{BC}+1}-\gamma_{AC}\gamma_{AB}-1\right)\boldsymbol{\beta}_{AB}^T-\frac{\gamma_{BC}^2}{\gamma_{BC}+1}\left(\gamma_{AC}-\frac{\gamma_{AB}}{\gamma_{BC}}\right)\boldsymbol{\beta}_{BC}^T+
$$

$$
-\gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\left(\gamma_{AC}-\frac{\gamma_{AB}}{\gamma_{BC}}\right)\boldsymbol{\beta}_{AB}^T
$$

$$
=\left[\gamma_{AC}\gamma_{BC}-\frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}-\frac{\gamma_{BC}^2}{\gamma_{BC}+1}\left(\gamma_{AC}-\frac{\gamma_{AB}}{\gamma_{BC}}\right)\right]\boldsymbol{\beta}_{BC}^T+
$$

$$
+\left[\frac{\gamma_{AC}\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}-\gamma_{AB}-\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\frac{1}{\gamma_{AB}}\left(\frac{(\gamma_{AC}+\gamma_{AB})^2}{\gamma_{BC}+1}-\gamma_{AC}\gamma_{AB}-1\right)+\right.
$$

$$
\left.-\gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\left(\gamma_{AC}-\frac{\gamma_{AB}}{\gamma_{BC}}\right)\right]\boldsymbol{\beta}_{AB}^T
$$

$$
=\frac{\gamma_{BC}}{\gamma_{BC}+1}\left[\gamma_{AC}(\gamma_{BC}+1)-\gamma_{AC}-\gamma_{AB}-(\gamma_{AC}\gamma_{BC}-\gamma_{AB})\right]\boldsymbol{\beta}_{BC}^T+
$$

$$
+\frac{\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\Big[\gamma_{AC}(\gamma_{AC}+\gamma_{BC})(\gamma_{BC}+1)-(\gamma_{BC}+1)\,(\gamma_{AB}+1)-(\gamma_{AC}+\gamma_{AB})^2+
$$

$$
+(\gamma_{AC}\gamma_{AB}+1)(\gamma_{BC}+1)-(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB})(\gamma_{AC}\gamma_{BC}-\gamma_{AB})\Big]\,\boldsymbol{\beta}_{AB}^T
$$

$$
=0 \quad , \tag{31}
$$

exactly as we expected.

So, we are then ready to tackle $R=[\Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB)]^i{}_j$. But, first we make an important observation, namely, from the fact that $[\Lambda(AC)^{-1}\Lambda(BC)\Lambda(AB)]^0{}_i$ vanishes, we may deduce

$$
\gamma_{AC}\boldsymbol{\beta}_{AC}^T\left(I+\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T+\frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T+\gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right)
$$

$$
=\gamma_{AC}\left(\gamma_{BC}\boldsymbol{\beta}_{BC}^T+\frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T\right) \quad . \tag{32}
$$

It is relation (32) which reduces by factors the length of the formulas. Without it, one would obtain lengthy expressions which are difficult to manage and may lead to several possibilities of mistakes in signs and coefficients.

Using Eqs. (22), (25) and (32), we arrive at

$$R = -\gamma_{AC}\boldsymbol{\beta}_{AC}\left(\gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T\right) +$$

$$+\left(I + \frac{\gamma_{AC}^2}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\boldsymbol{\beta}_{AC}^T\right)$$

$$\left(I + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right)$$

We split the product of the second term in two terms. One term equals I times the expression in the last line and the other $\frac{\gamma_{AC}}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\gamma_{AC}\boldsymbol{\beta}_{AC}^T$ times that expression. The latter term can be handled with formula (32). We obtain then

$$R = -\gamma_{AC}\boldsymbol{\beta}_{AC}\left(\gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T\right) +$$

$$+I + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T +$$

$$+\frac{\gamma_{AC}^2}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\left(\gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T\right)$$

Next, we add the first term and the last term, using $-\gamma_{AC} + \frac{\gamma_{AC}^2}{\gamma_{AC}+1} = -\frac{\gamma_{AC}}{\gamma_{AC}+1}$.

$$R = I - \frac{\gamma_{AC}}{\gamma_{AC}+1}\boldsymbol{\beta}_{AC}\left(\gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T\right) +$$

$$+\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$$

Here, we substitute in the second term, expression (22) for $\gamma_{AC}\boldsymbol{\beta}_{AC}$.

$$R = I - \frac{1}{\gamma_{AC}+1}\left(\gamma_{AB}\boldsymbol{\beta}_{AB} + \frac{\gamma_{BC}(\gamma_{AC}+\gamma_{AB})}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\right)\left(\gamma_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{AB}(\gamma_{AC}+\gamma_{BC})}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}^T\right) +$$

$$+\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$$

Subsequently, we elaborate the second term.

$$R = I - \frac{\gamma_{BC}\gamma_{AB}}{\gamma_{AC}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \frac{\gamma_{AB}^2(\gamma_{AC}+\gamma_{BC})}{(\gamma_{AC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T +$$

$$-\frac{\gamma_{BC}^2(\gamma_{AC}+\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T - \frac{\gamma_{BC}\gamma_{AB}(\gamma_{AC}+\gamma_{BC})(\gamma_{AC}+\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T +$$

$$+\frac{\gamma_{AB}^2}{\gamma_{AB}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$$

In the next step, we join terms of the same matrices $\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T$, $\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T$, $\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$ and $\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T$.

$$R = I - \frac{\gamma_{BC}\gamma_{AB}}{\gamma_{AC}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T + \left\{-\frac{\gamma_{AB}^2(\gamma_{AC}+\gamma_{BC})}{(\gamma_{AC}+1)(\gamma_{AB}+1)} + \frac{\gamma_{AB}^2}{\gamma_{AB}+1}\right\}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T +$$

$$+\left\{-\frac{\gamma_{BC}^2(\gamma_{AC}+\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)} + \frac{\gamma_{BC}^2}{\gamma_{BC}+1}\right\}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T +$$

$$+\left\{-\frac{\gamma_{BC}\gamma_{AB}(\gamma_{AC}+\gamma_{BC})(\gamma_{AC}+\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)} + \gamma_{BC}\gamma_{AB}\frac{1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}}{(\gamma_{BC}+1)\,(\gamma_{AB}+1)}\right\}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$$

and elaborate on the coefficients.

$$R = I - \frac{\gamma_{BC}\gamma_{AB}}{\gamma_{AC}+1}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \frac{\gamma_{AB}^2(\gamma_{BC}-1)}{(\gamma_{AC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T +$$

$$-\frac{\gamma_{BC}^2(\gamma_{AB}-1)}{(\gamma_{AC}+1)(\gamma_{BC}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T + \frac{\gamma_{BC}\gamma_{AB}(1+2\gamma_{AC}+\gamma_{BC}+\gamma_{AB}-\gamma_{BC}\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$$

7. The rotation

Now, we study the details of the rotation. Thereto, we determine the coefficients of $\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$ and $\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T + \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$, and join the remaining two, $\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T$ and $\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T$, with a common denominator.

$$R = I - \frac{\gamma_{BC}\gamma_{AB}(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\left(\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right)+$$

$$+\frac{\gamma_{BC}\gamma_{AB}(\gamma_{AC}-\gamma_{BC}\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\left(\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T + \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right)+$$

$$-\frac{\gamma_{AB}^2(\gamma_{BC}^2-1)\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \gamma_{BC}^2(\gamma_{AB}^2-1)\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}$$

Finally, we substitute the relations $\gamma_{BC}\gamma_{AB}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB} = \gamma_{AC} - \gamma_{BC}\gamma_{AB}$ (see Eq. 21), $\gamma_{BC}^2\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{BC} = \gamma_{BC}^2 - 1$ and $\gamma_{AB}^2\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB} = \gamma_{AB}^2 - 1$

$$R = I - \frac{\gamma_{BC}\gamma_{AB}(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\left(\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right)+$$

$$+\frac{\gamma_{BC}^2\gamma_{AB}^2}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\left\{\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T + \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{AB}^T + \right.$$

$$\left. -\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T\right\}$$

and write the result in a more compact form.

$$R = I - \frac{\gamma_{BC}\gamma_{AB}(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB})}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\left(\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right)+$$

$$+\frac{\gamma_{BC}^2\gamma_{AB}^2}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\left(\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T\right)^2 \quad . \tag{33}$$

In order to understand that the result (33) represents a rotation, we write the explicit matrix form for $\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T$.

$$\boldsymbol{\beta}_{AB}\boldsymbol{\beta}_{BC}^T - \boldsymbol{\beta}_{BC}\boldsymbol{\beta}_{AB}^T =$$

$$= \begin{pmatrix} \beta_1^{(AB)}\beta_1^{(BC)} & \beta_1^{(AB)}\beta_2^{(BC)} & \beta_1^{(AB)}\beta_3^{(BC)} \\ \beta_2^{(AB)}\beta_1^{(BC)} & \beta_2^{(AB)}\beta_2^{(BC)} & \beta_2^{(AB)}\beta_3^{(BC)} \\ \beta_3^{(AB)}\beta_1^{(BC)} & \beta_3^{(AB)}\beta_2^{(BC)} & \beta_3^{(AB)}\beta_3^{(BC)} \end{pmatrix} - \begin{pmatrix} \beta_1^{(BC)}\beta_1^{(AB)} & \beta_1^{(BC)}\beta_2^{(AB)} & \beta_1^{(BC)}\beta_3^{(AB)} \\ \beta_2^{(BC)}\beta_1^{(AB)} & \beta_2^{(BC)}\beta_2^{(AB)} & \beta_2^{(BC)}\beta_3^{(AB)} \\ \beta_3^{(BC)}\beta_1^{(AB)} & \beta_3^{(BC)}\beta_2^{(AB)} & \beta_3^{(BC)}\beta_3^{(AB)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\left(\beta_1^{(BC)}\beta_2^{(AB)} - \beta_2^{(BC)}\beta_1^{(AB)}\right) & \left(\beta_3^{(BC)}\beta_1^{(AB)} - \beta_1^{(BC)}\beta_3^{(AB)}\right) \\ \left(\beta_1^{(BC)}\beta_2^{(AB)} - \beta_2^{(BC)}\beta_1^{(AB)}\right) & 0 & -\left(\beta_2^{(BC)}\beta_3^{(AB)} - \beta_3^{(BC)}\beta_2^{(AB)}\right) \\ -\left(\beta_3^{(BC)}\beta_1^{(AB)} - \beta_1^{(BC)}\beta_3^{(AB)}\right) & \left(\beta_2^{(BC)}\beta_3^{(AB)} - \beta_3^{(BC)}\beta_2^{(AB)}\right) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\left[\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right]_3 & \left[\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right]_2 \\ \left[\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right]_3 & 0 & -\left[\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right]_1 \\ -\left[\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right]_2 & \left[\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right]_1 & 0 \end{pmatrix} = \left(\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right) \cdot \vec{A} \quad , \tag{34}$$

where the generators, A_1, A_2 and A_3, of rotations are given by

$$A_1 = \frac{d}{d\alpha} R(\hat{x}, \alpha)\Big|_{\alpha = 0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \frac{d}{d\vartheta} R(\hat{y}, \vartheta)\Big|_{\vartheta = 0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad A_3 = \frac{d}{d\varphi} R(\hat{z}, \varphi)\Big|_{\varphi = 0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \tag{35}$$

for

$$R(\hat{x}, \alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad R(\hat{y}, \vartheta) = \begin{pmatrix} \cos(\vartheta) & 0 & \sin(\vartheta) \\ 0 & 1 & 0 \\ -\sin(\vartheta) & 0 & \cos(\vartheta) \end{pmatrix},$$

$$\text{and} \quad R(\hat{z}, \varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{36}$$

For a rotation $R(\vec{n}\,)$ with rotation axis $\hat{n}$ and rotation angle n, one has the following expression (see Eq. 67)

$$R(\vec{n}\,) = \exp\{\vec{n} \cdot \vec{A}\} = I + \sin(n)(\hat{n} \cdot \vec{A}) + (1 - \cos(n))(\hat{n} \cdot \vec{A})^2 \quad . \tag{37}$$

Hence, from Eq. (34), we find that the rotation axis of rotation (33) is given by

$$\hat{n} = \frac{\vec{\beta}_{BC} \times \vec{\beta}_{AB}}{\left|\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right|} = \frac{1}{\left|\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right|} \begin{pmatrix} \beta_2^{(BC)}\beta_3^{(AB)} - \beta_3^{(BC)}\beta_2^{(AB)} \\ \beta_3^{(BC)}\beta_1^{(AB)} - \beta_1^{(BC)}\beta_3^{(AB)} \\ \beta_1^{(BC)}\beta_2^{(AB)} - \beta_2^{(BC)}\beta_1^{(AB)} \end{pmatrix} \quad . \tag{38}$$

Comparison of Eq. (33) with the formula given in Eq. (37), gives for the rotation angle n the following relations:

$$\sin(n) = -\frac{\gamma_{BC}\gamma_{AB}(1 + \gamma_{AC} + \gamma_{BC} + \gamma_{AB})}{(\gamma_{AC} + 1)(\gamma_{BC} + 1)(\gamma_{AB} + 1)} \left|\vec{\beta}_{BC} \times \vec{\beta}_{AB}\right| \quad , \tag{39}$$

and

$$1-\cos(n)=\frac{\gamma_{BC}^2\gamma_{AB}^2}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\left|\vec{\beta}_{BC}\times\vec{\beta}_{AB}\right|^2 \quad . \tag{40}$$

In order to verify whether the relations (39) and (40) satisfy the condition $\sin^2(n)+\cos^2(n)=1$, we need to determine $\left|\vec{\beta}_{BC}\times\vec{\beta}_{AB}\right|$.

$$\left|\vec{\beta}_{BC}\times\vec{\beta}_{AB}\right|^2=\left(\vec{\beta}_{AB}\cdot\vec{\beta}_{AB}\right)\left(\vec{\beta}_{BC}\cdot\vec{\beta}_{BC}\right)-\left(\vec{\beta}_{AB}\cdot\vec{\beta}_{BC}\right)^2$$

$$=\left(\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB}\right)\left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{BC}\right)-\left(\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{BC}\right)^2=\left(\boldsymbol{\beta}_{AB}^T\boldsymbol{\beta}_{AB}\right)\left(\boldsymbol{\beta}_{BC}^T\boldsymbol{\beta}_{BC}\right)-\left(\frac{\gamma_{AC}}{\gamma_{BC}\gamma_{AB}}-1\right)^2$$

$$=\frac{1}{\gamma_{AB}^2\gamma_{BC}^2}\left\{(\gamma_{AB}^2-1)(\gamma_{BC}^2-1)-(\gamma_{AC}-\gamma_{BC}\gamma_{AB})^2\right\} \quad . \tag{41}$$

One obtains then for the rotation angle

$$\sin^2(n)=\frac{(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB})^2\left\{(\gamma_{AB}^2-1)(\gamma_{BC}^2-1)-(\gamma_{AC}-\gamma_{BC}\gamma_{AB})^2\right\}}{(\gamma_{AC}+1)^2(\gamma_{BC}+1)^2(\gamma_{AB}+1)^2} \quad , \tag{42}$$

and

$$1-\cos(n)=\frac{\left\{(\gamma_{AB}^2-1)(\gamma_{BC}^2-1)-(\gamma_{AC}-\gamma_{BC}\gamma_{AB})^2\right\}}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)} \quad . \tag{43}$$

Hence,

$$\cos(n)=1-\frac{\left\{(\gamma_{AB}^2-1)(\gamma_{BC}^2-1)-(\gamma_{AC}-\gamma_{BC}\gamma_{AB})^2\right\}}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}=$$

$$=\frac{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)-(\gamma_{AB}^2-1)(\gamma_{BC}^2-1)+(\gamma_{AC}-\gamma_{BC}\gamma_{AB})^2}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}$$

$$=\frac{\gamma_{AC}(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}-\gamma_{BC}\gamma_{AB})+\gamma_{BC}(\gamma_{BC}+1)+\gamma_{AB}(\gamma_{AB}+1)}{(\gamma_{AC}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)} \quad . \tag{44}$$

Here, we perform some of the arithmetic for $\sin^2+\cos^2$ in MAXIMA:

```
om2: (gbc*gbc-1)*(gab*gab-1)-(gac-gbc*gab)*(gac-gbc*gab);
den: (gac+1)*(gbc+1)*(gab+1);
s2: (1+gac+gbc+gab)*(1+gac+gbc+gab)*om2;
c: den-om2;
p: c*c+s2-den*den;
expand(p);
quit();
```

which results in $P = 0$, hence proofs that $\sin^2 + \cos^2 = 1$.

The tangent of the rotation angle n is given by

$$\tan^2(n) = \frac{(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB})^2\left\{(\gamma_{AB}^2-1)(\gamma_{BC}^2-1)-(\gamma_{AC}-\gamma_{BC}\gamma_{AB})^2\right\}}{\left\{\gamma_{AC}(1+\gamma_{AC}+\gamma_{BC}+\gamma_{AB}-\gamma_{BC}\gamma_{AB})+\gamma_{BC}(\gamma_{BC}+1)+\gamma_{AB}(\gamma_{AB}+1)\right\}^2} \quad , \qquad (45)$$

where, from Eq. (21) one has the relation of γ_{AC} with $\vec{\beta}_{BC}$ and $\vec{\beta}_{AB}$, namely

$$\gamma_{AC} = \gamma_{BC}\gamma_{AB}\left(1+\vec{\beta}_{BC}\cdot\vec{\beta}_{AB}\right) \quad \text{and} \quad \gamma_{AC}-\gamma_{BC}\gamma_{AB} = \gamma_{BC}\gamma_{AB}\,\vec{\beta}_{BC}\cdot\vec{\beta}_{AB} \quad . \qquad (46)$$

Hence, the rotation axis and the rotation angle can be fully expressed in terms of $\vec{\beta}_{BC}$ and $\vec{\beta}_{AB}$.

For further studies of the rotation R, formulas (39) and (40) seem more transparant. After the substitutions

$$\gamma_{AC} = \gamma_{BC}\gamma_{AB}\left\{1+\beta_{BC}\beta_{AB}\cos(\theta)\right\} \quad \text{and} \quad \left|\vec{\beta}_{BC}\times\vec{\beta}_{AB}\right| = \beta_{BC}\beta_{AB}\sin(\theta) \quad , \qquad (47)$$

where θ represents the angle between the two boosts $\vec{\beta}_{BC}$ and $\vec{\beta}_{AB}$, one obtains for formulas (39) and (40) the following expressions.

$$\sin(n) = -\frac{\gamma_{BC}\gamma_{AB}(1+\gamma_{BC}\gamma_{AB}\left\{1+\beta_{BC}\beta_{AB}\cos(\theta)\right\}+\gamma_{BC}+\gamma_{AB})}{(\gamma_{BC}\gamma_{AB}\left\{1+\beta_{BC}\beta_{AB}\cos(\theta)\right\}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\beta_{BC}\beta_{AB}\sin(\theta) \quad , \qquad (48)$$

and

$$1-\cos(n) = \frac{\gamma_{BC}^2\gamma_{AB}^2}{(\gamma_{BC}\gamma_{AB}\left\{1+\beta_{BC}\beta_{AB}\cos(\theta)\right\}+1)(\gamma_{BC}+1)(\gamma_{AB}+1)}\beta_{BC}^2\beta_{AB}^2\sin^2(\theta) \quad . \qquad (49)$$

Notice from Eqs. (48) and (49) that when the boosts $\vec{\beta}_{BC}$ and $\vec{\beta}_{AB}$ are parallel, $\sin(n) = 0$ and $\cos(n) = 1$, hence, no rotation occurs.

Furthermore, when the boosts $\vec{\beta}_{BC}$ and $\vec{\beta}_{AB}$ are perpendicular, $\sin(n) = 1$ and $\cos(n) = 0$, one finds

$$\cos(n) = \frac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} \quad \text{and} \quad \sin(n) = -\frac{\gamma_{AB}\gamma_{BC}\beta_{AB}\beta_{BC}}{1+\gamma_{AB}\gamma_{BC}} \quad .$$

Literature

1. Those who like pictorial representations of boosts, may enjoy the work of Riad Chamseddine, entitled *"Vectorial form of the successive Lorentz transformations. Application: Thomas rotation"* and published in Foundations of Physics, Vol. 42, 488511 (2012).

2. With a notation slightly different from the one used above, W.L. Kennedy has written a similar derivation for the rotation axis and angle in *"Thomas rotation: a Lorentz matrix approach"*, which is published in the European Journal of Physics, Vol. 23, 235247 (2002).

3. At the internet pages for lecture notes of the Physics Department of Berkeley, one may find an interesting discussion on the *"Thomas precession"* at http://bohr.physics.berkeley.edu/classes/221/0708/notes/thomprec.ps

4. An introductory article, entitled *"The Thomas rotation"*, is written by John P. Costella, Bruce H. J. McKellar, Andrew A. Rawlinsonc and Gerard J. Stephenson, Jr. and published in The American Journal of Physics, Vol. 69, 837-847 (2001).

5. A nice article on the subject, entitled *"Thomas rotation and the parametrization of the Lorentz transformation group"*, is written by Abraham A. Ungar and published in Foundations of Physics Letters, Vol. 1, pp 57-89 (1988).

6. Interesting applications are found in *"The Thomas precession"*, written by George P. Fisher and published in The American Journal of Physics, Vol. 40, 1772-1781 (1972).

See also:

L.H. Thomas, Nature, 117, 514 (1926); Phil. Mag. 3, 1 (1927).

E.P. Wigner, Ann. Math. 40, 149 (1939).

Rotations in three dimensions

The three rotations around the principal axes of the orthogonal coordinate system (x, y, z) are given by:

$$R(\hat{x},\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad R(\hat{y},\vartheta) = \begin{pmatrix} \cos(\vartheta) & 0 & \sin(\vartheta) \\ 0 & 1 & 0 \\ -\sin(\vartheta) & 0 & \cos(\vartheta) \end{pmatrix},$$

$$\text{and} \quad R(\hat{z},\varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

Those matrices are unimodular (*i.e.* have unit determinant) and orthogonal.

As an example that rotations in general do not commute, let us take a rotation of 90° around the x-axis and a rotation of 90° around the y-axis. It is, using the above definitions (50), easy to show that:

$$R(\hat{x}, 90^\circ)R(\hat{y}, 90^\circ) \neq R(\hat{y}, 90^\circ)R(\hat{x}, 90^\circ) \quad . \tag{51}$$

An arbitrary rotation can be characterized in various different ways. One way is as follows: Let $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ represent the orthonormal basis vectors of the coordinate system and let $\vec{u}_1 = \hat{e}_1$, $\vec{u}_2 = \hat{e}_2$ and $\vec{u}_3 = \hat{e}_3$ be three vectors in three dimensions which before the rotation R are at the positions of the three basis vectors. The images of the three vectors are after an active rotation R given by:

$$\vec{u}_i^{\,\prime} = R\,\vec{u}_i \quad \text{for} \quad i = 1, 2, 3.$$

The rotation matrix for R at the above defined basis $\hat{e}_i$ $(i = 1, 2, 3)$, consists then of the components of those image vectors, *i.e.*

$$R = \begin{pmatrix} (\vec{u}_1^{\,\prime})_1 & (\vec{u}_2^{\,\prime})_1 & (\vec{u}_3^{\,\prime})_1 \\ (\vec{u}_1^{\,\prime})_2 & (\vec{u}_2^{\,\prime})_2 & (\vec{u}_3^{\,\prime})_2 \\ (\vec{u}_1^{\,\prime})_3 & (\vec{u}_2^{\,\prime})_3 & (\vec{u}_3^{\,\prime})_3 \end{pmatrix} \quad . \tag{52}$$

A second way to characterize a rotation is by means of its rotation axis, which is the one-dimensional subspace of the three dimensional space which remains invariant under the rotation, and by its rotation angle.

In both cases are three free parameters involved: The direction of the rotation axis in the second case, needs two parameters and the rotation angle gives the third. In the case of the rotation R of formula (52) we have nine different matrix elements. Now, if $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ form a righthanded set of unit vectors, *i.e.* $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, then consequently form the rotated vectors $\vec{u}_1^{\,\prime}$, $\vec{u}_2^{\,\prime}$ and $\vec{u}_3^{\,\prime}$ also a righthanded orthonormal system, *i.e.* $\vec{u}_3^{\,\prime} = \vec{u}_1^{\,\prime} \times \vec{u}_2^{\,\prime}$. This leaves us with the six components of $\vec{u}_1^{\,\prime}$ and $\vec{u}_2^{\,\prime}$ as parameters. But there are three more conditions, $|\vec{u}_1^{\,\prime}| = |\vec{u}_2^{\,\prime}| = 1$ and $\vec{u}_1^{\,\prime} \cdot \vec{u}_2^{\,\prime} = 0$. So, only three of the nine components of R are free.

The matrix R of formula (52) is unimodular and orthogonal. In order to proof those properties of R, we first introduce, for now and for later use, the Levi-Civita tensor ϵ_{ijk}, given by:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for} \quad ijk = \ 123,\ 312 \ \text{and} \ \ 231. \\ -1 & \text{for} \quad ijk = \ 132,\ 213 \ \text{and} \ \ 321. \\ \ \ 0 & \text{for} \quad \text{all other combinations.} \end{cases} \tag{53}$$

This tensor has the following properties:

(i) For symmetric permutations of the indices:

$$\epsilon_{jki} = \epsilon_{kij} = \epsilon_{ijk}. \tag{54}$$

(ii) For antisymmetric permutations of indices:

$$\epsilon_{ikj} = \epsilon_{jik} = \epsilon_{kji} = -\epsilon_{ijk}. \tag{55}$$

Now, using the definition of the Levi-Civita tensor and the fact that the rotated vectors $\vec{u}'_1$, $\vec{u}'_2$ and $\vec{u}'_3$ form a righthanded orthonormal system, we find for the determinant of the rotation matrix R of formula (52) the following:

$$\begin{aligned} \det(R) &= \epsilon_{ijk} R_{i1} R_{j2} R_{k3} = \epsilon_{jik} (\vec{u}'_1)_i (\vec{u}'_2)_j (\vec{u}'_3)_k \\ &= (\vec{u}'_1 \times \vec{u}'_2)_k (\vec{u}'_3)_k = \vec{u}'_3 \cdot \vec{u}'_3 = 1 \quad . \end{aligned}$$

And for the transposed of the rotation matrix R we obtain moreover:

$$\left(R^T R\right)_{ij} = \left(R^T\right)_{ik} R_{kj} = R_{ki} R_{kj} = (\vec{u}'_i)_k (\vec{u}'_j)_k = \delta_{ij} \quad .$$

Consequently, the rotation matrix R is unimodular and orthogonal.

The Euler angles.

So, rotations in three dimensions are characterized by three parameters. Here we consider the rotation which rotates a point $\vec{a}$, defined by:

$$\vec{a} = \left(\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta)\right), \tag{56}$$

to the position $\vec{b}$, defined by

$$\vec{b} = \left(\sin(\vartheta')\cos(\varphi'), \sin(\vartheta')\sin(\varphi'), \cos(\vartheta')\right). \tag{57}$$

Notice that there exists various different rotations which perform this operation. Here, we just select one. Using the definitions (50), (56) and (57), it is not very difficult to show that:

$$R(\hat{y}, -\vartheta)R(\hat{z}, -\varphi)\vec{a} = \hat{z},\ R(\hat{z}, \varphi')R(\hat{y}, \vartheta')\hat{z} = \vec{b} \text{ and } R(\hat{y}, \vartheta')R(\hat{y}, -\vartheta) = R(\hat{y}, \vartheta' - \vartheta).$$

As a consequence of this results, we may conclude that a possible rotation which transforms $\vec{a}$ (56) into $\vec{b}$ (57), is given by:

$$R(\varphi', \vartheta' - \vartheta, \varphi) = R(\hat{z}, \varphi')R(\hat{y}, \vartheta' - \vartheta)R(\hat{z}, -\varphi). \tag{58}$$

This parametrization of an arbitrary rotation in three dimensions is due to Euler. The three independent angles φ', $\vartheta' - \vartheta$ and φ are called the *Euler angles*.

The generators.

A second parametrization involves the *generators* of rotations in three dimensions, as are called the following three matrices which result from the three basic rotations defined in (50):

$$A_1 = \left.\frac{d}{d\alpha}R(\hat{x},\alpha)\right|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \left.\frac{d}{d\vartheta}R(\hat{y},\vartheta)\right|_{\vartheta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad A_3 = \left.\frac{d}{d\varphi}R(\hat{z},\varphi)\right|_{\varphi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{59}$$

In terms of the Levi-Civita tensor, defined in formula (53), we can express the matrix representation (59) for the generators of $SO(3)$, by:

$$(A_i)_{jk} = -\epsilon_{ijk}. \tag{60}$$

The introduction of the Levi-Civita tensor is very useful for the various derivations in the following, since it allows a compact way of formulating matrix multiplications, as we will see. However, one more property of this tensor should be given here, *i.e.* the contraction of one index in the product of two Levi-Civita tensors:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{ilm} &= \epsilon_{1jk}\epsilon_{1lm} + \epsilon_{2jk}\epsilon_{2lm} + \epsilon_{3jk}\epsilon_{3lm} \\ &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \end{aligned} \tag{61}$$

Equiped with this knowledge, let us determine the commutator of two generators (59), using the above properties (54), (55) and (61). First we concentrate on one matrix element (60) of the commutator:

$$\begin{aligned} \{[A_i, A_j]\}_{kl} &= (A_iA_j)_{kl} - (A_jA_i)_{kl} = (A_i)_{km}(A_j)_{ml} - (A_j)_{km}(A_i)_{ml} \\ &= \epsilon_{ikm}\epsilon_{jml} - \epsilon_{jkm}\epsilon_{iml} = \epsilon_{mik}\epsilon_{mlj} - \epsilon_{mjk}\epsilon_{mli} \\ &= \delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl} - (\delta_{jl}\delta_{ki} - \delta_{ji}\delta_{kl}) = \delta_{il}\delta_{kj} - \delta_{jl}\delta_{ki} \\ &= \epsilon_{mij}\epsilon_{mlk} = -\epsilon_{ijm}\epsilon_{mkl} = \epsilon_{ijm}(A_m)_{kl} = (\epsilon_{ijm}A_m)_{kl}. \end{aligned}$$

So, for the commutator of the generators (59) we find:

$$[A_i, A_j] = \epsilon_{ijm}A_m. \tag{62}$$

The rotation axis and angle.

In order to determine a second parametrization of a rotation in three dimensions, we define an arbitrary vector $\vec{n}$ by:

$$\vec{n} = (n_1, n_2, n_3), \tag{63}$$

as well as its "innerproduct" with the three generators (59), given by the expression:

$$\vec{n} \cdot \vec{A} = n_i A_i = n_1 A_1 + n_2 A_2 + n_3 A_3. \tag{64}$$

In the following we need the higher order powers of this "innerproduct". Actually, it is sufficient to determine the third power of (64), *i.e.*:

$$(\vec{n} \cdot \vec{A})^3 = (n_i A_i)(n_j A_j)(n_k A_k) = n_i n_j n_k A_i A_j A_k.$$

We proceed by determining one matrix element of the resulting matrix. Using the above property (61) of the Levi-Civita tensor, we find:

$$\begin{aligned}
\{(\vec{n} \cdot \vec{A})^3\}_{ab} &= n_i n_j n_k \{A_i A_j A_k\}_{ab} = n_i n_j n_k (A_i)_{ac} (A_j)_{cd} (A_k)_{db} \\
&= -n_i n_j n_k \epsilon_{iac} \epsilon_{jcd} \epsilon_{kdb} = -n_i n_j n_k \{\delta_{id}\delta_{aj} - \delta_{ij}\delta_{ad}\} \epsilon_{kdb} \\
&= -n_d n_a n_k \epsilon_{kdb} + n^2 n_k \epsilon_{kab} = 0 - n^2 n_k (A_k)_{ab} = \{-n^2 \vec{n} \cdot \vec{A}\}_{ab}.
\end{aligned}$$

The zero in the forelast step of the above derivation, comes from the deliberation that using the antisymmetry property (55) of the Levi-Civita tensor, we have the following result for the contraction of two indices with a symmetric expression:

$$\epsilon_{ijk} n_j n_k = -\epsilon_{ikj} n_j n_k = -\epsilon_{ikj} n_k n_j = -\epsilon_{ijk} n_j n_k, \tag{65}$$

where in the last step we used the fact that contracted indices are dummy and can consequently be represented by any symbol.

So, we have obtained for the third power of the "innerproduct" (64) the following:

$$(\vec{n} \cdot \vec{A})^3 = -n^2 \vec{n} \cdot \vec{A}. \tag{66}$$

Using this relation repeatedly for the higher order powers of $\vec{n} \cdot \vec{A}$, we may also determine its exponential, *i.e.*

$$\begin{aligned}
\exp\{\vec{n} \cdot \vec{A}\} &= \mathbf{1} + \vec{n} \cdot \vec{A} + \frac{1}{2!}(\vec{n} \cdot \vec{A})^2 + \frac{1}{3!}(\vec{n} \cdot \vec{A})^3 + \frac{1}{4!}(\vec{n} \cdot \vec{A})^4 + \cdots \\
&= \mathbf{1} + \vec{n} \cdot \vec{A} + \frac{1}{2!}(\vec{n} \cdot \vec{A})^2 + \frac{1}{3!}(-n^2 \vec{n} \cdot \vec{A}) + \frac{1}{4!}(-n^2 (\vec{n} \cdot \vec{A})^2) + \cdots \\
&= \mathbf{1} + \{1 - \frac{n^2}{3!} + \frac{n^4}{5!} - \frac{n^6}{7!} + \cdots\}(\vec{n} \cdot \vec{A}) +
\end{aligned}$$

$$+\{\frac{1}{2!}-\frac{n^2}{4!}+\frac{n^4}{6!}-\frac{n^6}{8!}+\cdots\}(\vec{n}\cdot\vec{A})^2$$

$$= \mathbf{1}+\{n-\frac{n^3}{3!}+\frac{n^5}{5!}-\frac{n^7}{7!}+\cdots\}(\hat{n}\cdot\vec{A})+$$

$$+\{\frac{n^2}{2!}-\frac{n^4}{4!}+\frac{n^6}{6!}-\frac{n^8}{8!}+\cdots\}(\hat{n}\cdot\vec{A})^2.$$

We recognize here the Taylor expansions for the cosine and sine functions. So, substituting these goniometric functions for their expansions, we obtain the following result:

$$\exp\{\vec{n}\cdot\vec{A}\} = \mathbf{1}+\sin(n)(\hat{n}\cdot\vec{A})+(1-\cos(n))(\hat{n}\cdot\vec{A})^2. \tag{67}$$

Next, we will show that this exponential operator leaves the vector $\vec{n}$ invariant. For that purpose we proof, using formula (65), the following:

$$\{(\vec{n}\cdot\vec{A})\vec{n}\}_i = (\vec{n}\cdot\vec{A})_{ij}n_j = (n_kA_k)_{ij}n_j = n_k(A_k)_{ij}n_j = -n_k\epsilon_{kij}n_j = 0,$$

or equivalently:

$$(\vec{n}\cdot\vec{A})\vec{n} = 0. \tag{68}$$

Consequently, the exponential operator (67) acting at the vector $\vec{n}$, gives the following result:

$$\exp\{\vec{n}\cdot\vec{A}\}\vec{n} = \left[\mathbf{1}+\vec{n}\cdot\vec{A}+\cdots\right]\vec{n} = \mathbf{1}\vec{n} = \vec{n} \tag{69}$$

So, the exponential operator (67) leaves the vector $\vec{n}$ invariant and of course also the vectors $a\vec{n}$, where a represents an arbitrary real constant. Consequently, the axis through the vector $\vec{n}$ is invariant, which implies that it is the rotation axis when the exponential operator represents a rotation, *i.e.* when this operator represents an unimodular, orthogonal transformation. Now, the matrix $\vec{n}\cdot\vec{A}$ of formula (64) is explicitly given by:

$$\vec{n}\cdot\vec{A} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \quad , \tag{70}$$

which clearly is a traceless and anti-symmetric matrix. So we are lead to the conclusion that $\exp\{\vec{n}\cdot\vec{A}\}$ is orthogonal and unimodular and thus represents a rotation.

In order to study the angle of rotation of the transformation (67), we introduce a pair of vectors $\vec{v}$ and $\vec{w}$ in the plane perpendicular to the rotation axis $\vec{n}$:

$$\vec{v} = \begin{pmatrix} n_2-n_3 \\ n_3-n_1 \\ n_1-n_2 \end{pmatrix} \quad \text{and} \quad \vec{w} = \hat{n}\times\vec{v} = (n_1+n_2+n_3)\hat{n} - n\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad , \tag{71}$$

where n is defined by $n=\sqrt{n_1^2+n_2^2+n_3^2}$.

The vectors $\vec{v}$, $\vec{w}$ and $\vec{n}$ form an orthogonal set in three dimensions. Moreover, are the moduli of $\vec{v}$ and $\vec{w}$ equal.

Using formula (70), one finds that under the matrix $\hat{n}\cdot\vec{A}$ the vectors $\vec{v}$ and $\vec{w}$ transform according to:

$$(\hat{n} \cdot \vec{A})\vec{v} = \vec{w} \quad \text{and} \quad (\hat{n} \cdot \vec{A})\vec{w} = -\vec{v} \quad .$$

So, for the rotation $\exp(\vec{n} \cdot \vec{A})$ of formula (67) one obtains for the vectors $\vec{v}$ and $\vec{w}$ the following transformations:

$$\begin{aligned} \vec{v}\,' = \exp(\vec{n} \cdot \vec{A})\vec{v} &= \vec{v} + \sin(n)\vec{w} + (1 - \cos(n))(-\vec{v}) = \vec{v}\cos(n) + \vec{w}\sin(n), \quad \text{and} \\ \vec{w}\,' = \exp(\vec{n} \cdot \vec{A})\vec{w} &= \vec{w} + \sin(n)(-\vec{v}) + (1 - \cos(n))(-\vec{w}) = -\vec{v}\sin(n) + \vec{w}\cos(n). \end{aligned}$$

The vectors $\vec{v}$ and $\vec{w}$ are rotated over an angle n in to the resulting vectors $\vec{v}\,'$ and $\vec{w}\,'$. This rotation is moreover in the positive sense with respect to the direction $\vec{n}$ of the rotation axis, because of the choice (71) for $\vec{w}$.

Notice that the case $n_1 = n_2 = n_3$, which is not covered by the choice (71), has to be studied separately. This is left as an exercise for the reader.

Concludingly, we may state that we found a second parametrization of a rotation around the origin in three dimensions, *i.e.*:

$$R(n_1, n_2, n_3) = \exp\{\vec{n} \cdot \vec{A}\}, \tag{72}$$

where the rotation angle is determined by:

$$n = \sqrt{n_1^2 + n_2^2 + n_3^2},$$

and where the rotation axis is indicated by the direction of $\vec{n}$.

The vector $\vec{n}$ can take any direction and its modulus can take any value. Consequently, $\exp(\vec{n} \cdot \vec{A})$ may represent any rotation and so all possible unimodular, orthogonal 3×3 matrices can be obtained by formula (67) once the appropriate vectors $\vec{n}$ are selected.