Some Notes on Field Theory

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Chapter 1

Introduction to Quantum Field Theory

Quantum Field Theory is a general technique for dealing with systems with an infinite number of degrees of freedom. Examples are systems of many interacting particles or critical phenomena like second order phase transitions. Here we will concentrate on the scattering of particles, but the general framework can be applied to any domain in physics.

For an introduction, we simplify Nature as much as possible and hence assume that Nature exists out of only one type of particles, without spin, without charge and all with the same mass, m. Such particles are moreover their own antiparticles.

The objects of our interest are n-points Green's functions, $G(x_1, x_2, \ldots, x_n)$, which represent n-particle processes where $(n-k)$ particles enter the interaction area before scattering and k particles leave the interaction area after scattering.

On the subject of Quantum Field Theory exists a vast amount of literature. Here we will just mention some books, but the list is very incomplete.

Many of the ideas behind the theory have been developed by R.P. Feynman and can be found in his book entitled "Quantum Electrodynamics" [9].
A classic course on the subject is contained in "Relativistic Quantum Fields" by J.D. Bjorken and S.D. Drell [10].
Also the books entitled "Quantum Field Theory" by C. Itzykson and J-B Zuber [11] and "Gauge theory of elementary particle physics" by Ta-Pei Cheng and Ling-Fong Li [12], which contain a lot of ideas worked out in detail, have become classic works in the mean time.
More modern, and also with a great deal of detail, is the book of George Sterman entitled "An introduction to Quantum Field Theory" [14].
But theories develop, some of the stuff becomes obsolete and other new areas enter the game, and therefore new strategies are followed for courses written in a modern language and intended for those who want to work in the frontier areas of physics. Good examples are the lectures of Pierre Ramond entitled "Field Theory (a modern primer)" [15] and of R.J. Rivers "Path integral methods in quantum field theory" [16].
Path integral techniques form the basis of almost all modern literature on field theory. The classic book "Quantum Mechanics and Path Integrals" is written by R.P. Feynman and A.R. Hibbs [17].

The dimensional regularization methods, which were important for the proof that non-Abelian Gauge Theories are renormalizable, are developed by Gerard 't Hooft and Tiny Veltman, and can be found in their Cern publication [18] or in their publication in Nuclear Physics [19]. Any modern lecture contains a chapter on the issue.
Not exactly on the subject of introducing quantum field theory, but still with everything necessary to study the subject, is the book of Sidney Coleman entitled "Aspects of Symmetry" [20], which is strongly recommended for further reading.

1.1 Huygens' principle versus Schrödinger equation

In the 17th century Christiaan Huygens (The Hague, 1629-1695) formulated the foundations of modern wave mechanics and the theory of light. The description of the propagation of waves in matter is nowadays known by Huygens' principle.

According to Huygens' principle one may calculate the wave amplitude of an oscillatory phenomenon at each point in space at a certain instant t when one disposes of the following two informations: (1) the wave amplitudes at all points in space at an earlier instant t' and (2) the way in which the wave propagates through space. The first information we denote by $\psi\left(\vec{x}\,', t'\right)$, whereas the second information is supposed to be contained in the Green's function $G\left(\vec{x}, t; \vec{x}\,', t'\right)$. With those definitions one may express Huygens' principle by the following relation

$$\psi\left(\vec{x}, t\right) \; = \; i \int d^3x' \; G\left(\vec{x}, t; \vec{x}\,', t'\right) \; \psi\left(\vec{x}\,', t'\right) \quad \text{for} \quad t > t' \quad . \tag{1.1}$$

In order to quantify the condition $t > t'$ in formula (1.1), one may introduce the step function, $\theta\left(t - t'\right)$, which vanishes for negative argument and equals 1 for positive argument, *i.e.*

$$\theta\left(t - t'\right) \; = \; \begin{cases} 0 \quad \text{for} \qquad t < t' \\ \\ 1 \quad \text{for} \qquad t > t' \end{cases} \quad . \tag{1.2}$$

We obtain then from formula (1.1) the relation

$$\theta\left(t - t'\right) \; \psi\left(\vec{x}, t\right) \; = \; i \int d^3x' \; G\left(\vec{x}, t; \vec{x}\,', t'\right) \; \psi\left(\vec{x}\,', t'\right) \tag{1.3}$$

for Huygens' principle.

In this section we study the relation of formula (1.3) with the Schrödinger equation. For that purpose, we first express the step function (1.2) by an integral representation, given by

$$-(2\pi i) \; \theta\left(\tau\right) \; = \; \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} d\omega \; \frac{e^{-i\omega\tau}}{\omega + i\varepsilon} \quad . \tag{1.4}$$

The integral can be carried out as follows. For $\tau < 0$ one closes the contour in the complex ω-plane by a semicircle in the upper half plane which does not contain any singularity. Consequently, the complex contour integral vanishes and we obtain as a result the upper equation of formula (1.2). For $\tau > 0$ one closes the contour in the complex ω-plane by a semicircle in the lower half plane which does contain the singularity at $-i\varepsilon$. The residue of the resulting complex contour integral equals 1 in the limit of $\varepsilon \to 0$. Hence we obtain $-(2\pi i)\theta\left(\tau\right) = -(2\pi i)$, which results in the lower equation of formula (1.2).

From the integral representation it is moreover easy to verify that

$$\frac{d}{dt} \, \theta\left(t - t'\right) \; = \; \int_{-\infty}^{+\infty} d\omega \; \frac{e^{-i\omega\left(t - t'\right)}}{2\pi} \; = \; \delta\left(t - t'\right) \quad . \tag{1.5}$$

So, by applying $\partial/\partial t$ to equation (1.3), we find the following relation

$$\delta\left(t - t'\right) \; \psi\left(\vec{x}, t\right) \; + \; \theta\left(t - t'\right) \; \frac{\partial}{\partial t}\psi\left(\vec{x}, t\right) \; = \; i \int d^3x' \; \frac{\partial}{\partial t}G\left(\vec{x}, t; \vec{x}\,', t'\right) \; \psi\left(\vec{x}\,', t'\right) \quad . \tag{1.6}$$

Now, we come to the Schrödinger equation which we will consider here, given by

$$\left(i\frac{\partial}{\partial t} \; - \; H_0\left(\vec{x}\,\right)\right) \; \psi\left(\vec{x},t\right) \; = \; V\left(\vec{x},t\right) \; \psi\left(\vec{x},t\right) \quad , \tag{1.7}$$

where $H_0\left(\vec{x}\,\right)$ might represent the operator $-\nabla^2/2m$, but could be more complicated, and where V represents the potential which has to be specified for each different problem under study.

Associated with equation (1.7) we define the Green's function for free propagation, or *free propagator*, $G_0\left(x, x'\,\right)$, given by

$$\left(i\frac{\partial}{\partial t} \; - \; H_0\left(\vec{x}\,\right)\right) \; G_0\left(x,x'\,\right) \; = \; \delta^{(4)}\left(x-x'\,\right) \quad , \tag{1.8}$$

where we introduced $x = (\vec{x}, t)$.

Equation (1.8) can be solved as we will assume here. Later on, we will encounter some examples.

The relation between Huygens' priciple (1.3) and the Schrödinger equation (1.7) can now be formulated as follows

$$G\left(x,x'\,\right) \; = \; G_0\left(x,x'\,\right) \; + \; \int d^4x'' \, G_0\left(x,x''\,\right) \; V\left(x''\,\right) \; G\left(x''\,,x'\,\right) \quad , \tag{1.9}$$

which is an integral equation and can be solved by iteration, a procedure which we will study first. In the remaining part of this section we will outline a proof of relation (1.9).

When one substitutes $G\left(x,x'\,\right)$ as defined on the lefthand side of formula (1.9) into the expression of the righthand side, then one obtains

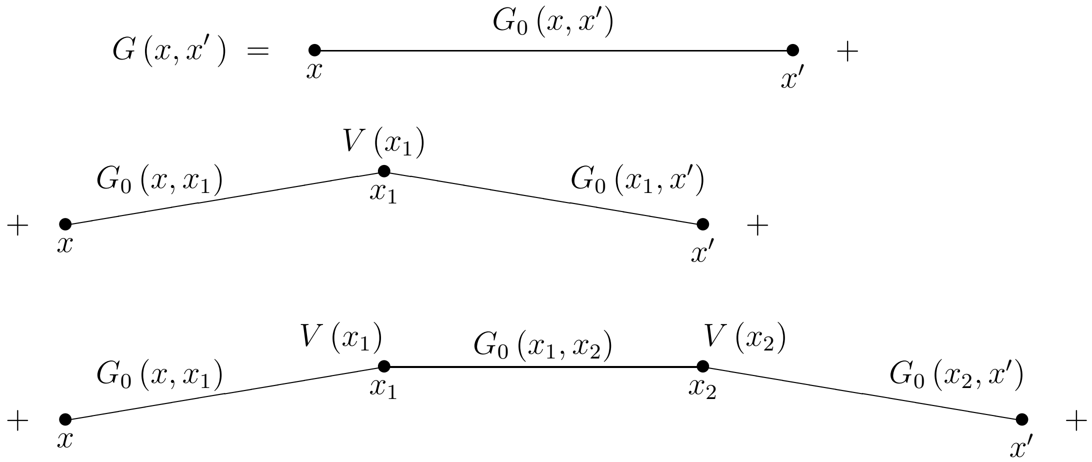
$$\begin{aligned} G\left(x,x'\,\right) \; &= \; G_0\left(x,x'\,\right) \; + \; \int d^4x_1 \, G_0\left(x,x_1\,\right) \; V\left(x_1\,\right) \; \Big\{ \; G_0\left(x_1\,,x'\,\right) \; + \\ &\qquad + \; \int d^4x_2 \, G_0\left(x_1\,,x_2\,\right) \; V\left(x_2\,\right) \; G\left(x_2\,,x'\,\right) \Big\} \\ &= \; G_0\left(x,x'\,\right) \; + \; \int d^4x_1 \, G_0\left(x,x_1\,\right) \; V\left(x_1\,\right) \; G_0\left(x_1\,,x'\,\right) \; + \\ &\qquad + \; \int d^4x_1 \; \int d^4x_2 \, G_0\left(x,x_1\,\right) \; V\left(x_1\,\right) \; G_0\left(x_1\,,x_2\,\right) \; V\left(x_2\,\right) \; G\left(x_2\,,x'\,\right) \quad . \end{aligned} \tag{1.10}$$

The substitution can be repeated. One finds

$$\begin{aligned} G\left(x,x'\,\right) \; &= \; G_0\left(x,x'\,\right) \; + \; \int d^4x_1 \, G_0\left(x,x_1\,\right) \; V\left(x_1\,\right) \; G_0\left(x_1\,,x'\,\right) \; + \\ &+ \int d^4x_1 d^4x_2 \, G_0\left(x,x_1\right) \; V\left(x_1\right) \; G_0\left(x_1,x_2\right) \; V\left(x_2\right) \; G_0\left(x_2,x'\right) \; + \\ &+ \int d^4x_1 d^4x_2 d^4x_3 \, G_0\left(x,x_1\right) V\left(x_1\right) G_0\left(x_1,x_2\right) V\left(x_2\right) G_0\left(x_2,x_3\right) V\left(x_3\right) G_0\left(x_3,x'\right) \; + \\ &+ \; \ldots \quad . \end{aligned} \tag{1.11}$$

Each term in the sum (1.11) can be evaluated, once the free propagator is known. Hence, when the sum converges one can determine the full propagator $G\,(x, x'\,)$. This is the case for weak potentials V.

A way to memorize formula (1.11) is by means of the following graphical representation for each of the terms.



$+ \;\ldots \quad .$ (1.12)

The first graph at the righthand side of formula (1.12) represents the free propagator

$$G_0\,(x, x'\,) \quad .$$

The second graph, where the free propagators connect x to x_1 and x_1 to x' and where the potential acts once, at space-time point x_1, represents the second term of formula (1.11), given by

$$\int d^4x_1\; G_0\,(x, x_1\,)\;\; V\,(x_1\,)\;\; G_0\,(x_1\,, x'\,) \quad ,$$

The third graph, where the free propagators connect x to x_1, x_1 to x_2 and x_2 to x' and where the potential acts twice, one time at space-time point x_1 and another time at space-time point x_2, represents the third term of formula (1.11), given by

$$\int d^4x_1 d^4x_2\; G_0\,(x, x_1)\;\; V\,(x_1)\;\; G_0\,(x_1, x_2)\;\; V\,(x_2)\;\; G_0\,(x_2, x') \quad .$$

And so on.

One important property, *causality*, of both, the free propagator $G_0\,(x, x'\,)$ and the full propagator $G\,(x, x'\,)$, should be mentioned here: No signal can travel faster than light. Consequently, nothing can be observed before it happens. Or in formula

$$G\,(\vec{x}, t; \vec{x}\,', t') \;=\; G_0\,(\vec{x}, t; \vec{x}\,', t') \;=\; 0 \quad \text{for} \quad t < t' \quad . \tag{1.13}$$

1.1.1 Proof of formula (1.9)

Below, we will study a proof of formula (1.9).

We show that by substituting expression (1.9) into formula (1.3) one ends up with the Schrödinger equation (1.7).

The substitution results in the following relation

$$
\begin{aligned}
&\theta\left(t-t'\right)\ \psi\left(x\right)\ = \qquad\qquad (1.14)\\
&\quad = \ i \int d^3x' \ \left\{ G_0\left(x, x'\right) \ + \ \int d^4x''\ G_0\left(x, x''\right)\ V\left(x''\right)\ G\left(x'', x'\right) \right\}\ \psi\left(x'\right) \quad .
\end{aligned}
$$

Next, we let the operator

$$
i\frac{\partial}{\partial t} \ - \ H_0\left(\vec{x}\right)
$$

work at both sides of equation (1.14). From the lefthand side of (1.14), also using the result (1.5), one finds

$$
i\delta\left(t-t'\right)\ \psi\left(x\right) \ + \ \theta\left(t-t'\right)\ \left\{ i\frac{\partial}{\partial t} \ - \ H_0\left(\vec{x}\right) \right\}\ \psi\left(x\right) \quad . \qquad (1.15)
$$

Whereas, from the righthand side, also using the result (1.8), we obtain

$$
\begin{aligned}
& i \int d^3x' \ \left\{ i\frac{\partial}{\partial t} \ - \ H_0\left(\vec{x}\right) \right\}\ G_0\left(x, x'\right)\ \psi\left(x'\right) \ + \\
& + i \int d^3x' \ \int d^4x'' \ \left\{ i\frac{\partial}{\partial t} \ - \ H_0\left(\vec{x}\right) \right\} G_0\left(x, x''\right)\ V\left(x''\right)\ G\left(x'', x'\right)\ \psi\left(x'\right) \ = \\
= \ & i \int d^3x' \ \delta^{(4)}\left(x-x'\right)\ \psi\left(x'\right) \ + \\
& + i \int d^3x' \ \int d^4x'' \ \delta^{(4)}\left(x-x''\right)\ V\left(x''\right)\ G\left(x'', x'\right)\ \psi\left(x'\right) \\
= \ & i\delta\left(t-t'\right)\ \psi\left(x\right) \ + \ i \int d^3x' \ V\left(x\right)\ G\left(x, x'\right)\ \psi\left(x'\right) \\
= \ & i\delta\left(t-t'\right)\ \psi\left(x\right) \ + \ V\left(x\right)\ \theta\left(t-t'\right)\ \psi\left(x\right) \quad . \qquad (1.16)
\end{aligned}
$$

In the last step of equation (1.16) we used once more equation (1.3). Combining results (1.15) and (1.16) one finds the Schrödinger equation (1.7).

1.2 Free Klein Gordon particles

Non-interacting particles without spin or charge are described by the Klein-Gordon equation, which satisfies the wave equation

$$\frac{\partial^2}{\partial t^2}\,\psi(x,t) \;=\; \left(\frac{\partial^2}{\partial x^2} \,-\, m^2\right)\,\psi(x,t) \quad . \tag{1.17}$$

Here we define

$$\partial^\mu\partial_\mu \;=\; \frac{\partial^2}{\partial t^2} \,-\, \frac{\partial^2}{\partial x^2} \,-\, \frac{\partial^2}{\partial y^2} \,-\, \frac{\partial^2}{\partial z^2} \quad ,$$

in order to write the Klein-Gordon equation in the usual form

$$\left(\partial^\mu\partial_\mu \,+\, m^2\right)\,\psi(x) \;=\; 0 \quad , \tag{1.18}$$

where $\psi(x)$ stands for $\psi\left(\vec{x},t\right)$.

Notice, that we assume here that gravitational effects can be completely ignored and consequently that our particles move in a Minkowskian background for which we adopted the metric $(+---)$.

As easily can be verified, a general solution to the free Klein-Gordon equation (1.18) is given by the following wave packet

$$\psi(x) \;=\; \int \frac{d^3k}{(2\pi)^3 2E}\,\left\{\alpha\left(\vec{k}\,\right)\,e^{-ikx} \,+\, \alpha^*\left(\vec{k}\,\right)\,e^{ikx}\right\} \quad , \tag{1.19}$$

provided that k, which stands for $\left(E,\vec{k}\,\right)$, satisfies the mass-shell relation

$$E^2 \;=\; \left(\vec{k}\,\right)^2 \,+\, m^2 \quad . \tag{1.20}$$

1.3 Green's function for free Klein-Gordon particles

The Green's function, G_0, for a free Klein-Gordon particle, which has the correct boundary conditions, is a solution of the differential equation given by

$$\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x_{\mu}} + m^2\right) G_0\left(x, x'\right) = \delta^{(4)}\left(x - x'\right) \quad . \tag{1.21}$$

One may construct the correct solution by defining the Fourier transform, $\tilde{G}_0$, of G_0, by

$$G_0\left(x, x'\right) = \int \frac{d^4p}{(2\pi)^4}\, e^{ipx} \int \frac{d^4p'}{(2\pi)^4}\, e^{ip'x'}\, \tilde{G}_0\left(p, p'\right) \quad .$$

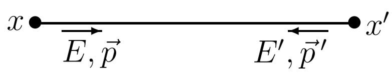
For this Fourier transform one finds, by applying the Klein-Gordon differential equation (1.21), the relation

$$\int \frac{d^4p}{(2\pi)^4} \left(-p^2 + m^2\right) e^{ipx} \int \frac{d^4p'}{(2\pi)^4}\, e^{ip'x'}\, \tilde{G}_0\left(p, p'\right) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\left(x - x'\right)} \quad ,$$

which is solved by

$$\left(-p^2 + m^2\right) \tilde{G}_0\left(p, p'\right) = (2\pi)^4\, \delta^{(4)}\left(p + p'\right) \quad . \tag{1.22}$$

Graphically one may represent this solution by



which graph can be interpreted as follows: Four momentum propagates from event x to event x'. This is represented by four momentum p which flows away from x and four momentum p' which flows away from x'. Now, four momentum conservation demands that p' equals $-p$. This is expressed by the delta function in formula (1.22).

One defines the **Feynman propagator**, S_F, by

$$S_F(p, m^2) = \frac{i}{p^2 - m^2} \quad \text{and} \quad \tilde{G}_0\left(p, p'\right) = i(2\pi)^4\, \delta^{(4)}\left(p + p'\right)\, S_F(p, m^2) \quad . \tag{1.23}$$

As we will see in the following, it is usually very convenient to do all calculations with the Feynman propagators and only at the end to bother about four momentum conservation.

1.4 Second Quantization Procedure

Our goal is to describe many interacting particles, not just one-particle states. To that aim we define a Hilbert space of many-particle states, also called *Fock space*.

The most elementary state of this space is called the *vacuum*, symbolized by $|0\rangle$. It is assumed to be the state with no particles at all or just simply the ground state of the system of states one considers.

Next in the hierarchy come the one-particle states, for our world, just existing of Klein-Gordon particles, denoted by $\left|\vec{k}\right\rangle$. It is supposed to describe a particle with momentum $\vec{k}$. The operator, which creates out of the vacuum a one-particle state, is denoted by $a^\dagger\left(\vec{k}\right)$. Consequently, we may write

$$\left|\vec{k}\right\rangle = a^\dagger\left(\vec{k}\right)\ |0\rangle \quad . \tag{1.24}$$

Two-particle states, which describe the situation in which in our world only two particles are present, one with momentum $\vec{k}_1$ and the other with momentum $\vec{k}_2$, are supposed to be given by

$$\left|\vec{k}_1, \vec{k}_2\right\rangle = a^\dagger\left(\vec{k}_1\right)\ a^\dagger\left(\vec{k}_2\right)\ |0\rangle \quad . \tag{1.25}$$

Now, we suppose that the order in which the particles are created, which is not a time-order but just an operation order, does not influence in any way the resulting two-particle state. Hence, we find as a property of the creation operators defined in formula (1.24) that they commute, *i.e.*

$$a^\dagger\left(\vec{k}_1\right)\ a^\dagger\left(\vec{k}_2\right) = a^\dagger\left(\vec{k}_2\right)\ a^\dagger\left(\vec{k}_1\right) \quad . \tag{1.26}$$

We also define annihilation operators, $a\left(\vec{k}\right)$, with the following properties

$$\begin{aligned}
a\left(\vec{k}\right)\ |0\rangle &= 0 \quad , \\
a\left(\vec{k}_1\right)\ a\left(\vec{k}_2\right) &= a\left(\vec{k}_2\right)\ a\left(\vec{k}_1\right) \quad , \quad \text{and} \\
\left[a\left(\vec{k}_1\right)\ ,\ a^\dagger\left(\vec{k}_2\right)\right] &= (2\pi)^3\ 2E_1\ \delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right) \quad .
\end{aligned} \tag{1.27}$$

Notice that the commutation relations for the creation and annihilation operators are the continuum generalizations of the commutators for n harmonic oscillators, which also vanish except for $\left[a_i\ ,\ a_j^\dagger\right] = \delta_{ij}$.

The next step in the second quantization procedure is the replacement of the free Klein-Gordon wave packet, which is defined in formula (1.19), by a free Klein-Gordon *quantum field*, *i.e.*

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E} \left\{ a\left(\vec{k}\right)\ e^{-ikx} + a^\dagger\left(\vec{k}\right)\ e^{ikx} \right\} \quad , \tag{1.28}$$

which is an operator which acts in the many-particle state Hilbert space.

The reason why this procedure is called second quantization stems from the fact that we can also define a conjugate momentum

$$\pi(x) \;=\; \frac{\partial}{\partial t}\,\phi(x) \quad , \tag{1.29}$$

for which one has the following *equal time* commutation relations

$$\begin{aligned} \left[\phi\left(\vec{x},t\right)\;,\;\phi\left(\vec{x}\,',t\right)\right] \;&=\; \left[\pi\left(\vec{x},t\right)\;,\;\pi\left(\vec{x}\,',t\right)\right] \;=\; 0 \\ \left[\pi\left(\vec{x},t\right)\;,\;\phi\left(\vec{x}\,',t\right)\right] \;&=\; -i\delta^{(3)}\left(\vec{x}-\vec{x}\,'\right) \quad . \end{aligned} \tag{1.30}$$

1.4.1 Proof of formula (1.30)

First, we write the explicit expression for the conjugate momentum $\pi\left(\vec{x},t\right)$ of $\phi\left(\vec{x},t\right)$, namely

$$\begin{aligned} \pi\left(\vec{x},t\right) &= \frac{\partial}{\partial t}\phi\left(\vec{x},t\right) \\ &= \int\frac{d^3k}{(2\pi)^3 2E}\,iE\left\{-\,a\left(\vec{k}\,\right)\,e^{i\left(\vec{k}\cdot\vec{x}-Et\right)} \;+\; a^{\dagger}\left(\vec{k}\,\right)\,e^{-i\left(\vec{k}\cdot\vec{x}-Et\right)}\right\} \\ &= i\int\frac{d^3k}{2(2\pi)^3}\,\left\{-\,a\left(\vec{k}\,\right)\,e^{i\left(\vec{k}\cdot\vec{x}-Et\right)} \;+\; a^{\dagger}\left(\vec{k}\,\right)\,e^{-i\left(\vec{k}\cdot\vec{x}-Et\right)}\right\} \quad . \end{aligned} \tag{1.31}$$

Then, we substitute formulas (1.28) for the field and (1.31) for its conjugate momentum in the expression for the equal-time commutators (1.30). This gives:

$$\begin{aligned} &\left[\phi\left(\vec{x},t\right)\;,\;\phi\left(\vec{x}\,',t\right)\right] = \int\frac{d^3k}{(2\pi)^3 2E}\int\frac{d^3k'}{(2\pi)^3 2E'} \\ &\left\{\left[a\left(\vec{k}\,\right)\;,\;a\left(\vec{k}\,'\right)\right]e^{i\left(\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{x}\,'-(E+E')\,t\right)}\;+\right. \\ &+\;\left[a\left(\vec{k}\,\right)\;,\;a^{\dagger}\left(\vec{k}\,'\right)\right]e^{i\left(\vec{k}\cdot\vec{x}-\vec{k}\,'\cdot\vec{x}\,'-(E-E')\,t\right)}\;+ \\ &+\;\left[a^{\dagger}\left(\vec{k}\,\right)\;,\;a\left(\vec{k}\,'\right)\right]e^{i\left(-\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{x}\,'-(-E+E')\,t\right)}\;+ \\ &\left.+\;\left[a^{\dagger}\left(\vec{k}\,\right)\;,\;a^{\dagger}\left(\vec{k}\,'\right)\right]e^{i\left(-\vec{k}\cdot\vec{x}-\vec{k}\,'\cdot\vec{x}\,'+(E+E')\,t\right)}\right\} \end{aligned}$$

Next, we insert expressions (1.26) and (1.27) to find

$$\left[\phi\left(\vec{x},t\right)\;,\;\phi\left(\vec{x}\,',t\right)\right] = \int\frac{d^3k}{(2\pi)^3 2E}\int\frac{d^3k'}{(2\pi)^3 2E'}$$

$$\left\{(2\pi)^3 2E\delta^{(3)}\left(\vec{k}-\vec{k}\,'\right)e^{i\left(\vec{k}\cdot\vec{x}-\vec{k}\,'\cdot\vec{x}\,'-\left(E-E'\right)t\right)}+\right.$$

$$\left.-\,(2\pi)^3 2E\delta^{(3)}\left(\vec{k}-\vec{k}\,'\right)e^{i\left(-\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{x}\,'-\left(-E+E'\right)t\right)}\right\}$$

Upon integration over $\vec{k}\,'$, we obtain $\vec{k}\,'=\vec{k}$ and

$$E'=\sqrt{\left(\vec{k}\,'\right)^2+m^2}=\sqrt{\left(\vec{k}\right)^2+m^2}=E$$

hence

$$\left[\phi\left(\vec{x},t\right)\ ,\ \phi\left(\vec{x}\,',t\right)\right]=\int\frac{d^3k}{(2\pi)^3 2E}\left\{e^{i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)}\ -\ e^{-i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)}\right\}\quad .$$

In the second term one may perform the substition $\vec{k}\leftrightarrow-\vec{k}$ for the integration variable, in order to obtain two equal terms with opposite sign and thus

$$\left[\phi\left(\vec{x},t\right)\ ,\ \phi\left(\vec{x}\,',t\right)\right]=0\quad .$$

The proof for the equal-time commutator of two conjugate momentum fields is very similar.

For the equal-time commutator of the field and its conjugate momentum we obtain

$$\left[\pi\left(\vec{x},t\right)\ ,\ \phi\left(\vec{x}\,',t\right)\right]=\int\frac{id^3k}{2(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3 2E'}$$

$$\left\{-\left[a\left(\vec{k}\right)\ ,\ a^\dagger\left(\vec{k}\,'\right)\right]e^{i\left(\vec{k}\cdot\vec{x}-\vec{k}\,'\cdot\vec{x}\,'-\left(E-E'\right)t\right)}+\right.$$

$$\left.+\ \left[a^\dagger\left(\vec{k}\right)\ ,\ a\left(\vec{k}\,'\right)\right]e^{i\left(-\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{x}\,'-\left(-E+E'\right)t\right)}\right\}$$

$$=\int\frac{id^3k}{2(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3 2E'}\left\{-(2\pi)^3 2E\delta^{(3)}\left(\vec{k}-\vec{k}\,'\right)e^{i\left(\vec{k}\cdot\vec{x}-\vec{k}\,'\cdot\vec{x}\,'-\left(E-E'\right)t\right)}+\right.$$

$$\left.-\,(2\pi)^3 2E\delta^{(3)}\left(\vec{k}-\vec{k}\,'\right)e^{i\left(-\vec{k}\cdot\vec{x}+\vec{k}\,'\cdot\vec{x}\,'-\left(-E+E'\right)t\right)}\right\}$$

$$=\int\frac{id^3k}{2(2\pi)^3}\left\{-\ e^{i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)}\ -\ e^{-i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)}\right\}=-i\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\left(\vec{x}-\vec{x}\,'\right)}$$

$$=-i\delta^{(3)}\left(\vec{x}-\vec{x}\,'\right)\quad .$$

1.5 Self-interacting Klein-Gordon field

In general, one starts a quantum field theory by defining a *Lagrangian density*, $\mathcal{L}$, which is a functional of a quantum field, φ, and its derivatives

$$\mathcal{L}\left(\varphi\left(\vec{x},t\right),\partial_\mu\,\varphi\left(\vec{x},t\right)\right) \quad . \tag{1.32}$$

The object $\partial_\mu\varphi$ in formula (1.32) stands for the four partial derivatives given by

$$\partial_0\varphi \;=\; \frac{\partial}{\partial t}\varphi \;\;,\;\; \partial_1\varphi \;=\; \frac{\partial}{\partial x}\varphi \;\;,\;\; \partial_2\varphi \;=\; \frac{\partial}{\partial y}\varphi \;\;,\;\; \text{and} \;\; \partial_3\varphi \;=\; \frac{\partial}{\partial z}\varphi \;\; .$$

The total Lagrangian, L, for the system under consideration is given by the volume integral of the Lagrangian density over all space

$$L \;=\; \int d^3x\,\mathcal{L}\left(\varphi\left(\vec{x},t\right),\partial_\mu\,\varphi\left(\vec{x},t\right)\right) \quad .$$

All dynamics of the system is contained in the Lagrangian density.

The field equations for the quantum field can be derived from the Lagrangian density by the use of the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\varphi} \;=\; \partial_\mu\,\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)} \quad , \tag{1.33}$$

where

$$\partial_\mu\,\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)} \;=\; \partial_0\,\frac{\partial\mathcal{L}}{\partial\left(\partial_0\varphi\right)} \;-\; \partial_1\,\frac{\partial\mathcal{L}}{\partial\left(\partial_1\varphi\right)} \;-\; \partial_2\,\frac{\partial\mathcal{L}}{\partial\left(\partial_2\varphi\right)} \;-\; \partial_3\,\frac{\partial\mathcal{L}}{\partial\left(\partial_3\varphi\right)} \quad .$$

Now, the Lagrangian density for the self-interacting scalar field, or Klein-Gordon field, which we will consider here, is given by

$$\mathcal{L}\left(\varphi,\partial_\mu\,\varphi\right) \;=\; \frac{1}{2}\left(\partial_\mu\,\varphi\right)^2 \;-\; \frac{1}{2}m^2\varphi^2 \;-\; \frac{\lambda}{4!}\varphi^4 \quad , \tag{1.34}$$

where

$$\left(\partial_\mu\,\varphi\right)^2 \;=\; \left(\partial_0\,\varphi\right)^2 \;-\; \left(\partial_1\,\varphi\right)^2 \;-\; \left(\partial_2\,\varphi\right)^2 \;-\; \left(\partial_3\,\varphi\right)^2 \quad .$$

The theory, which follows from the above Lagrangian density (1.34), is in the literature known as φ^4 theory.

Applying the Euler-Lagrange equations (1.33) to the Lagrangian density (1.34), yields the following quantum field equation

$$\left(\partial^\mu\partial_\mu \;+\; m^2\right)\,\varphi(x) \;=\; -\frac{\lambda}{3!}\,\varphi^3(x) \quad . \tag{1.35}$$

When we compare the field equation (1.35) to the wave equation (1.18) for a free Klein-Gordon particle we may conclude that, for vanishing λ, equation (1.35) may be interpreted as the field equation for a free Klein-Gordon field. The term on the righthand side of equation (1.35), which stems from the term $-\lambda\varphi^4/4!$ in the Lagrangian density (1.34), may be interpreted as the *source term* which describes the deviation of the theory for self-interacting particles from the free theory because of the presence of interaction between the particles. For this reason we split the Lagrangian density in two parts, the free Lagrangian density $\mathcal{L}_0$ and the interaction part $\mathcal{L}_{\text{int}}$, defined by

$$\mathcal{L}_0 \;=\; \frac{1}{2}\left(\partial_\mu\,\varphi\right)^2 \;-\; \frac{1}{2}m^2\varphi^2 \quad \text{and} \quad \mathcal{L}_{\text{int}} \;=\; -\frac{\lambda}{4!}\varphi^4 \quad . \tag{1.36}$$

The first term in $\mathcal{L}_0$, which generates the term $\partial^\mu\partial_\mu\;\varphi$ in the field equation and is therefore related to the momentum squared of a free Klein-Gordon particle, is called the *kinetic term*; the second term in $\mathcal{L}_0$ the *mass term*.

As been observed above, in the absence of the source term the field equation (1.35) describes a free scalar quantum field, ϕ, for which the expression (1.28) is a general solution.

As mentioned before, the objects of our interest are the n-point Green's functions, which we are now capable of defining

$$G\left(x_1,\ldots,x_n\right) \;=\; \frac{\left\langle 0 \middle| T\left\{\phi\left(x_1\right)\cdots\phi\left(x_n\right)\; e^{i\int d^4y\;\mathcal{L}_{\text{int}}\left(\phi(y)\right)}\right\} \middle| 0\right\rangle}{\left\langle 0 \middle| T\left\{e^{i\int d^4y\;\mathcal{L}_{\text{int}}\left(\phi(y)\right)}\right\} \middle| 0\right\rangle} \quad , \tag{1.37}$$

where T stands for *time-ordering*, which means that in all expressions the fields must be permuted in such a way that the time components of their arguments are decreasing.

1.6 Time-ordered product of two fields

In this section we determine in all detail the vacuum expectation value of the time ordered product of two boson fields, also called *propagator*, and which is defined by

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \right\} \right| 0 \rangle \quad . \tag{1.38}$$

When we express the time-ordering in terms of the θ-function, defined in (1.2), which vanishes for negative argument and equals 1 for positive argument, then we obtain the following two terms

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \right\} \right| 0 \rangle \;=\; \langle 0 \left| \phi\left(x_1\right) \phi\left(x_2\right) \right| 0 \rangle \theta\left(t_1 - t_2\right) \; + \langle 0 \left| \phi\left(x_2\right) \phi\left(x_1\right) \right| 0 \rangle \theta\left(t_2 - t_1\right) \quad . \tag{1.39}$$

i.e. each term being characterized by one of the two permutations of the numbers one and two.

From expression (1.39) we learn that the first thing to be calculated, are the simple vacuum expectation values of two fields

$$\langle 0 \left| \phi\left(x_1\right) \phi\left(x_2\right) \right| 0 \rangle \quad \text{and} \quad \langle 0 \left| \phi\left(x_2\right) \phi\left(x_1\right) \right| 0 \rangle \quad . \tag{1.40}$$

The full expressions for those objects, after the substitution of formula (1.28) for the fields, are also quite long, but things become more managable by the use of the definitions

$$a(x) \;=\; \int \frac{d^3k}{(2\pi)^3 2E}\, e^{-ikx}\, a\left(\vec{k}\,\right) \quad \text{and} \quad \phi(x) \;=\; a(x) + a^{\dagger}(x) \quad . \tag{1.41}$$

Substituting those definitions into the first term of formula (1.40), one obtains for the vacuum expectation value of two fields

$$\left\langle 0 \left| \left\{ a\left(x_1\right) + a^{\dagger}\left(x_1\right) \right\} \left\{ a\left(x_2\right) + a^{\dagger}\left(x_2\right) \right\} \right| 0 \right\rangle \quad , \tag{1.42}$$

which upon multiplication leaves us with the following four terms

$$\langle 0 \left| a\left(x_1\right) a\left(x_2\right) \right| 0 \rangle + \left\langle 0 \left| a^{\dagger}\left(x_1\right) a\left(x_2\right) \right| 0 \right\rangle + \left\langle 0 \left| a\left(x_1\right) a^{\dagger}\left(x_2\right) \right| 0 \right\rangle + \left\langle 0 \left| a^{\dagger}\left(x_1\right) a^{\dagger}\left(x_2\right) \right| 0 \right\rangle \quad . \tag{1.43}$$

Three of the four terms in the expansion (1.43) vanish, as for example one has from the definition (1.27) for the annihilation operators that

$$a(x)|0\rangle \;=\; \int \frac{d^3k}{(2\pi)^3 2E}\, e^{-ikx}\, a\left(\vec{k}\,\right)|0\rangle \;=\; 0 \quad , \tag{1.44}$$

and hence, for a creation operator

$$\langle 0|a^{\dagger}(x) \;=\; \left\{ a(x)|0\rangle \right\}^{\dagger} \;=\; 0 \quad . \tag{1.45}$$

As a consequence of those properties for the operators defined in formula (1.41), we are then left with only one nonzero contribution to first of the two vacuum expectation values (1.40) of two fields, *i.e.*

$$\langle 0 \left| \phi\left(x_1\right) \phi\left(x_2\right) \right| 0 \rangle \;=\; \left\langle 0 \left| a\left(x_1\right) a^{\dagger}\left(x_2\right) \right| 0 \right\rangle \quad , \tag{1.46}$$

which, upon insertion of the full expression (1.41) for the operators $a(x)$ and $a^\dagger(x)$, reads

$$\int \frac{d^3k_1}{(2\pi)^3 2E_1} \int \frac{d^3k_2}{(2\pi)^3 2E_2}\, e^{-ik_1x_1 + ik_2x_2} \left\langle 0 \left| a\left(\vec{k}_1\right) a^\dagger\left(\vec{k}_2\right) \right| 0 \right\rangle$$

and hence contains the vacuum expectation value

$$\left\langle 0 \left| a\left(\vec{k}_1\right) a^\dagger\left(\vec{k}_2\right) \right| 0 \right\rangle \quad .$$

The latter expression can easily be handled by the use of the commutation relations (1.27) and the properties (1.27) for the annihilation operators, which leads to

$$\begin{aligned}
\left\langle 0 \left| a\left(\vec{k}_1\right) a^\dagger\left(\vec{k}_2\right) \right| 0 \right\rangle &= \left\langle 0 \left| \left\{ \left[a\left(\vec{k}_1\right) \, , \, a^\dagger\left(\vec{k}_2\right) \right] \, + \, a^\dagger\left(\vec{k}_2\right) a\left(\vec{k}_1\right) \right\} \right| 0 \right\rangle \\
&= \langle 0 | 0 \rangle (2\pi)^3 2E_1 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right) \, + \, \left\langle 0 \left| a^\dagger\left(\vec{k}_2\right) a\left(\vec{k}_1\right) \right| 0 \right\rangle \\
&= (2\pi)^3 2E_1 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right) \quad ,
\end{aligned}$$

and which turns expression (1.46) into

$$\langle 0 | \phi(x_1)\, \phi(x_2) | 0 \rangle \; = \; \int \frac{d^3k_1}{(2\pi)^3 2E_1} \int \frac{d^3k_2}{(2\pi)^3 2E_2}\, e^{-ik_1x_1 + ik_2x_2}\, (2\pi)^3 2E_1 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right) \quad .$$

Because of the Dirac delta function, one may perform the $\vec{k}_2$-integration and then rename the dummy $\vec{k}_1$ integration variable for $\vec{k}$. This gives the vacuum expectation value of formula (1.46) its final form

$$\langle 0 | \phi(x_1)\, \phi(x_2) | 0 \rangle \; = \; \int \frac{d^3k}{(2\pi)^3 2E}\, e^{-ik\,(x_1 - x_2)} \tag{1.47}$$

The second term of formula (1.40) equals the first term with the numbers one and two exchanged. So, we obtain for the vacuum expectation value (1.39) of the time ordered product of two boson fields the expression

$$\langle 0 | T\{\phi(x_1)\, \phi(x_2)\} | 0 \rangle \; = \; \left\{ \begin{array}{lll} \int \frac{d^3k}{(2\pi)^3 2E}\, e^{-ik\,(x_1 - x_2)} & \text{for} & t_1 \rangle t_2 \\ & & \\ \int \frac{d^3k}{(2\pi)^3 2E}\, e^{-ik\,(x_2 - x_1)} & \text{for} & t_1 \langle t_2 \end{array} \right. \quad . \tag{1.48}$$

Now, in the exponents of (1.48) comes kx, which in our metric equals $Et - \vec{k}\cdot\vec{x}$. Hence, in the above expression we must take $E\,(t_1 - t_2)$ for $t_1 \rangle t_2$ and $E\,(t_2 - t_1)$ for $t_1 \langle t_2$, which is equivalent to taking $E\,|t_1 - t_2|$ irrespective of the order of t_1 and t_2, *i.e.*

$$\langle 0 | T\{\phi(x_1)\, \phi(x_2)\} | 0 \rangle \; = \; \left\{ \begin{array}{lll} \int \frac{d^3k}{(2\pi)^3 2E}\, e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2) - iE\,|t_1 - t_2|} & \text{for} & t_1 \rangle t_2 \\ & & \\ \int \frac{d^3k}{(2\pi)^3 2E}\, e^{i\vec{k}\cdot(\vec{x}_2 - \vec{x}_1) - iE\,|t_1 - t_2|} & \text{for} & t_1 \langle t_2 \end{array} \right. \quad . \tag{1.49}$$

Furthermore, by changing the integration variable $\vec{k}$ to $-\vec{k}$ in the lower of the two expressions in formula (1.49), results

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \right\} \right| 0 \rangle \;=\; \int \frac{d^3k}{(2\pi)^3 2E}\; e^{i\vec{k}\cdot\left(\vec{x}_1 - \vec{x}_2\right) - iE\left|t_1 - t_2\right|} \quad . \tag{1.50}$$

With complex function theory one can easily show the following identity

$$i\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi}\; \frac{e^{-ik_0 t}}{\left(k_0\right)^2 - \left(\vec{k}\,\right)^2 - m^2} \;=\; \frac{e^{-i\sqrt{\left(\vec{k}\,\right)^2 + m^2}\,|t|}}{2\sqrt{\left(\vec{k}\,\right)^2 + m^2}} \quad , \tag{1.51}$$

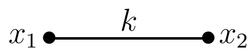
which, upon substitution in formula (1.50),
also remembering that E actually stands for $\sqrt{\left(\vec{k}\,\right)^2 + m^2}$, gives

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \right\} \right| 0 \rangle \;=\; i\int \frac{d^4k}{(2\pi)^4}\; \frac{e^{-ik\left(x_1 - x_2\right)}}{k^2 - m^2} \quad , \tag{1.52}$$

where k stands for $\left(k_0, \vec{k}\,\right)$ and d^4k for $dk_0 d^3k$.

Notice, that, since k_0 is an integration variable, k^2, which equals $\left(k_0\right)^2 - \left(\vec{k}\,\right)^2$, is not identical to m^2, *i.e.* is **off-mass-shell**.

A graphical representation for the propagator (1.52) is as shown below.



One might moreover recognize in the final expression (1.52) for the vacuum expectation value of the time ordered product of two boson fields the Feynman propagator which is given in formula (1.23).

1.6.1 Proof of formula (1.51)

For the proof of formula (1.51), which we cast here in the form

$$i\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi}\; \frac{e^{-ik_0 t}}{\left(k_0\right)^2 - M^2} \;=\; \frac{e^{-iM\,|t|}}{2M} \quad , \tag{1.53}$$

we introduce a small positive real number ϵ, such that the righthand side of equation (1.53) gives

$$\frac{e^{-iM\,|t| - \epsilon\,|t|}}{2M + \mathcal{O}(\epsilon)} \quad , \tag{1.54}$$

which vanishes in the limits $t \to \pm\infty$.

At the end of the calculations we take $\epsilon \downarrow 0$.

By comparison of formulae (1.53) and (1.54), we conclude that we must choose the substitution

$$M \;\longrightarrow\; M - i\epsilon \quad . \tag{1.55}$$

The integral which consequently has to be calculated is then

$$i \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0 t}}{(k_0 - M + i\epsilon)\,(k_0 + M - i\epsilon)} \quad . \tag{1.56}$$

In the literature one often finds the form

$$i \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0 t}}{(k_0)^2 - M^2 + i\epsilon} \quad . \tag{1.57}$$

This can be achieved from expression (1.56) by the substitution

$$2M\epsilon \longrightarrow \epsilon \quad , \tag{1.58}$$

moreover ignoring the term quadratic in ϵ.

The integrand of formula (1.56) has two singularities, or poles, in the complex k_0 plane at $-M + i\epsilon$ and at $M - i\epsilon$, as indicated in figure (1.1).

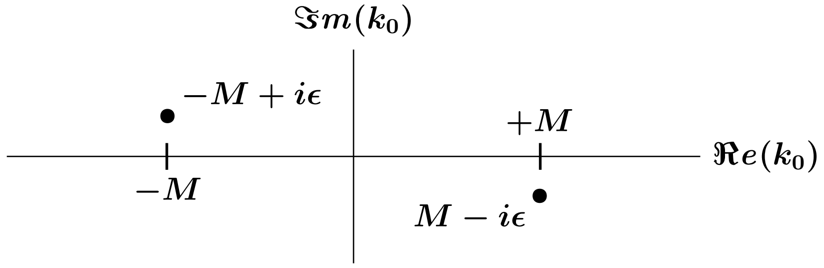


Figure 1.1: The two poles of the integrand of formula (1.56) in the complex k_0 plane at $-M + i\epsilon$ and at $M - i\epsilon$.

Next, we concentrate on the numerator of the integrand of expression (1.56), *i.e.*

$$e^{-ik_0 t} \quad . \tag{1.59}$$

When k_0 is complex, then it has a real part and an imaginary part

$$k_0 \; = \; \Re e(k_0) \; + \; i\Im m(k_0) \quad . \tag{1.60}$$

Hence, formula (1.59) turns into

$$e^{-i\Re e(k_0)t \; + \; \Im m(k_0)t} \quad . \tag{1.61}$$

In the following, we study what happens to the expression (1.61) when we take $|k_0| \to \infty$. Except for the cases where k_0 is real, hence $\Im m(k_0) = 0$, we find

$$\Im m(k_0) \longrightarrow +\infty \quad \text{for} \quad |k_0| \to \infty \quad \text{in the upper half complex } k_0 \text{ plane} \quad , \tag{1.62}$$

and

$$\Im m(k_0) \longrightarrow -\infty \quad \text{for} \quad |k_0| \to \infty \quad \text{in the lower half complex } k_0 \text{ plane} \quad . \tag{1.63}$$

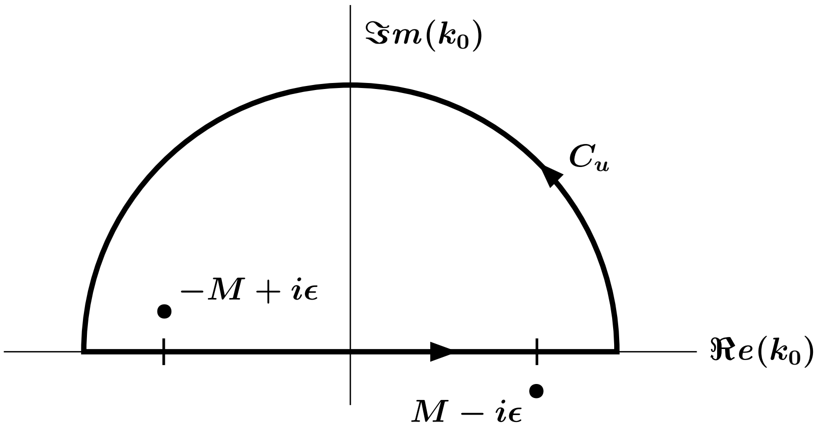
Consequently, for $\boldsymbol{t < 0}$, we obtain

$$e^{-i\Re e(k_0)t \; + \; \Im m(k_0)t} \longrightarrow 0 \quad \text{for} \quad |k_0| \to \infty \quad \text{in the upper half complex } k_0 \text{ plane} \quad , \tag{1.64}$$

whereas, for $\boldsymbol{t > 0}$, we obtain

$$e^{-i\Re e(k_0)t \; + \; \Im m(k_0)t} \longrightarrow 0 \quad \text{for} \quad |k_0| \to \infty \quad \text{in the lower half complex } k_0 \text{ plane} \quad , \tag{1.65}$$

Let us consider for $\boldsymbol{t < 0}$ the following contour in the complex k_0 plane.



When we let the radius of the half circle of contour C_u go to infinity, then we have

$$i\oint_{C_u \to \infty} \frac{dk_0}{2\pi} \, \frac{e^{-ik_0t}}{(k_0)^2 - M^2 + 2iM\epsilon} = \tag{1.66}$$

$$= \; i\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \, \frac{e^{-ik_0t}}{(k_0)^2 - M^2 + 2iM\epsilon} \; + \; i\int_{\text{half circle}} \frac{dk_0}{2\pi} \, \frac{e^{-ik_0t}}{(k_0)^2 - M^2 + 2iM\epsilon} \quad .$$

However, the integrand of the integral over the half circle vanishes when its radius approaches infinity, according to formula (1.64). Accordingly, for infinite radius one has

$$i\oint_{C_u \to \infty} \frac{dk_0}{2\pi} \, \frac{e^{-ik_0t}}{(k_0)^2 - M^2 + 2iM\epsilon} = i\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \, \frac{e^{-ik_0t}}{(k_0)^2 - M^2 + 2iM\epsilon} \quad . \tag{1.67}$$

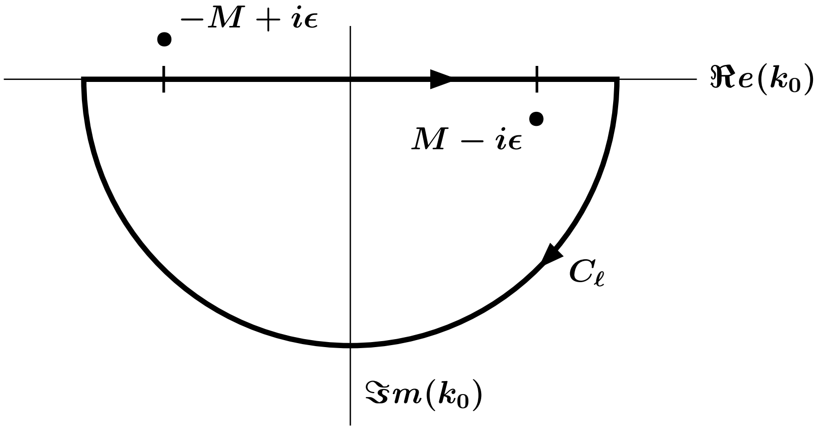
Now, according to complex function theory, the counterclockwise integral of a closed contour in the complex plane equals $2\pi i$ times the sum of the residues on the poles contained in the closed contour. Here, we have one pole at $k_0 = -M + i\epsilon$, where the residue may easily be determined using formula (1.56), to give

$$\frac{i}{2\pi} \, \frac{e^{iMt \, + \, \epsilon t}}{(-2M + 2i\epsilon)} \quad . \tag{1.68}$$

Hence, for $\boldsymbol{t < 0}$ we obtain

$$i\int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \, \frac{e^{-ik_0t}}{(k_0)^2 - M^2 + 2iM\epsilon} = \frac{e^{iMt \, + \, \epsilon t}}{2M - 2i\epsilon} = \frac{e^{-iM\,|t| \, - \, \epsilon\,|t|}}{2M - 2i\epsilon} \quad . \tag{1.69}$$

For $\boldsymbol{t > 0}$ we consider the following contour in the complex k_0 plane.



The integrand of the integral over the half circle vanishes when its radius approaches infinity, according to formula (1.65). Accordingly, for infinite radius one has here

$$i \oint_{C_\ell \to \infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0 t}}{(k_0)^2 - M^2 + 2iM\epsilon} = i \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0 t}}{(k_0)^2 - M^2 + 2iM\epsilon} \quad . \tag{1.70}$$

Furthermore, according to complex function theory, the clockwise integral of a closed contour in the complex plane equals $-2\pi i$ times the sum of the residues on the poles contained in the closed contour. Here, we have one pole at $k_0 = M - i\epsilon$, where the residue may easily be determined using formula (1.56), to give

$$\frac{i}{2\pi} \frac{e^{-iMt - \epsilon t}}{(2M - 2i\epsilon)} \quad . \tag{1.71}$$

Hence, for $\boldsymbol{t > 0}$ we obtain

$$i \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0 t}}{(k_0)^2 - M^2 + 2iM\epsilon} = \frac{e^{-iMt - \epsilon t}}{2M - 2i\epsilon} = \frac{e^{-iM|t| - \epsilon|t|}}{2M - 2i\epsilon} \quad . \tag{1.72}$$

By comparison of formulae (1.69) and (1.72), we find for any sign of t the result

$$i \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0 t}}{(k_0)^2 - M^2 + 2iM\epsilon} = \frac{e^{-iM|t| - \epsilon|t|}}{2M - 2i\epsilon} \quad . \tag{1.73}$$

Taking the limit $\epsilon \downarrow 0$, one finds formula (1.53).

1.7 Time-ordered product of four fields

In this section we determine in all detail the vacuum expectation value of the time ordered product of four boson fields, which is defined by

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right\} \right| 0 \rangle \quad . \tag{1.74}$$

When we express the time-ordering in terms of the θ-function, then we obtain the following twenty-four terms

$$\begin{aligned} \langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right\} \right| 0 \rangle \; = & \\ = \; \langle 0 \left| \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right| 0 \rangle \theta\left(t_1 - t_2\right) \theta\left(t_2 - t_3\right) \theta\left(t_3 - t_4\right) \; + & \\ + \; \langle 0 \left| \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_4\right) \phi\left(x_3\right) \right| 0 \rangle \theta\left(t_1 - t_2\right) \theta\left(t_2 - t_4\right) \theta\left(t_4 - t_3\right) \; + & \\ + \; \langle 0 \left| \phi\left(x_1\right) \phi\left(x_3\right) \phi\left(x_2\right) \phi\left(x_4\right) \right| 0 \rangle \theta\left(t_1 - t_3\right) \theta\left(t_3 - t_2\right) \theta\left(t_2 - t_4\right) \; + \; \cdots \quad , & \end{aligned} \tag{1.75}$$

i.e. each term being characterized by one of the twenty-four permutations of the numbers one to four.

From expression (1.75) we learn that the first thing to be calculated, is the simple vacuum expectation value of four fields. There are twenty-four of them, which are all just permutations of the first, given by

$$\langle 0 \left| \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right| 0 \rangle \quad . \tag{1.76}$$

The full expression for this object is also quite long, but things become more managable by the use of the definitions given in formula (1.41). Substituting those definitions into the expression of formula (1.76) for the simple vacuum expectation value of four fields, one obtains

$$\left\langle 0 \left| \left\{ a\left(x_1\right) + a^\dagger\left(x_1\right) \right\} \left\{ a\left(x_2\right) + a^\dagger\left(x_2\right) \right\} \left\{ a\left(x_3\right) + a^\dagger\left(x_3\right) \right\} \left\{ a\left(x_4\right) + a^\dagger\left(x_4\right) \right\} \right| 0 \right\rangle \quad . \tag{1.77}$$

Here we perform the various multiplications, to end up with sixteen terms given by

$$\begin{aligned} & \langle 0 \left| a\left(x_1\right) a\left(x_2\right) a\left(x_3\right) a\left(x_4\right) \right| 0 \rangle \; + \; \left\langle 0 \left| a^\dagger\left(x_1\right) a\left(x_2\right) a\left(x_3\right) a\left(x_4\right) \right| 0 \right\rangle \; + \\ & + \; \left\langle 0 \left| a\left(x_1\right) a^\dagger\left(x_2\right) a\left(x_3\right) a\left(x_4\right) \right| 0 \right\rangle \; + \; \cdots \quad . \end{aligned} \tag{1.78}$$

Several of the terms in the expansion (1.78) vanish because of the properties (1.44) and (1.45) for the operators defined in formula (1.41).

More complicated cases, like

$$\left\langle 0 \left| a\left(x_1\right) a\left(x_2\right) a\left(x_3\right) a^\dagger\left(x_4\right) \right| 0 \right\rangle \quad ,$$

which, according to the definitions (1.41), equals

$$\int \frac{d^3k_1}{(2\pi)^3 2E_1} \int \frac{d^3k_2}{(2\pi)^3 2E_2} \int \frac{d^3k_3}{(2\pi)^3 2E_3} \int \frac{d^3k_4}{(2\pi)^3 2E_4} \times$$

$$\times\, e^{-ik_1x_1 - ik_2x_2 - ik_3x_3 + ik_4x_4} \left\langle 0 \left| a\left(\vec{k}_1\right) a\left(\vec{k}_2\right) a\left(\vec{k}_3\right) a^\dagger\left(\vec{k}_4\right) \right| 0 \right\rangle \quad (1.79)$$

and hence contains the vacuum expectation value

$$\left\langle 0 \left| a\left(\vec{k}_1\right) a\left(\vec{k}_2\right) a\left(\vec{k}_3\right) a^\dagger\left(\vec{k}_4\right) \right| 0 \right\rangle \quad ,$$

can be handled by the use of the commutation relations (1.27), which leads to

$$\begin{aligned}
&\left\langle 0 \left| a\left(\vec{k}_1\right) a\left(\vec{k}_2\right) a\left(\vec{k}_3\right) a^\dagger\left(\vec{k}_4\right) \right| 0 \right\rangle \ = \\
&= \left\langle 0 \left| a\left(\vec{k}_1\right) a\left(\vec{k}_2\right) \left\{ \left[a\left(\vec{k}_3\right) \ , \ a^\dagger\left(\vec{k}_4\right) \right] \ + \ a^\dagger\left(\vec{k}_4\right) a\left(\vec{k}_3\right) \right\} \right| 0 \right\rangle \\
&= \left\langle 0 \left| a\left(\vec{k}_1\right) a\left(\vec{k}_2\right) \right| 0 \right\rangle (2\pi)^3 2E_3 \delta^{(3)}\left(\vec{k}_3 - \vec{k}_4\right) \ + \ \left\langle 0 \left| a\left(\vec{k}_1\right) a\left(\vec{k}_2\right) a^\dagger\left(\vec{k}_4\right) a\left(\vec{k}_3\right) \right| 0 \right\rangle \\
&= \ 0 \quad .
\end{aligned}$$

Inspection of all sixteen terms of (1.78) gives as a result that fourteen of those vanish. We are then left with only two nonzero contributions

$$\begin{aligned}
&\left\langle 0 \left| \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right| 0 \right\rangle \ = \qquad (1.80) \\
&\quad = \left\langle 0 \left| a\left(x_1\right) a\left(x_2\right) a^\dagger\left(x_3\right) a^\dagger\left(x_4\right) \right| 0 \right\rangle \ + \ \left\langle 0 \left| a\left(x_1\right) a^\dagger\left(x_2\right) a\left(x_3\right) a^\dagger\left(x_4\right) \right| 0 \right\rangle \quad .
\end{aligned}$$

This can easily be seen, since, first, a vacuum expectation value for an operator which does not have an equal number of creation and annihilation operators, like the one given in formula (1.79), allways ends up with an annihilation operator acting on $|0\rangle$ or a creation operator acting on $\langle 0|$, by the use of commutation relations (1.27) whenever necessary. Moreover, a vacuum expectation value also vanishes when a creation operator stands on the lefthand side or when an annihilation operator stands on the righthand side. Consequently, for x_1 we must have an annihilation operator and for x_4 a creation operator. This then implies that for x_2 and x_3 we must have one annihilation and one creation operator. There are only two possibilities, which are shown in formula (1.80).

The first term of (1.80), which, in a way similar to formula (1.79), contains the vacuum expectation value

$$\left\langle 0 \left| a\left(\vec{k}_1\right) a\left(\vec{k}_2\right) a^\dagger\left(\vec{k}_3\right) a^\dagger\left(\vec{k}_4\right) \right| 0 \right\rangle \quad ,$$

can be handled by the use of the commutation relations (1.27). First, we commute $a\left(\vec{k}_2\right)$ and $a^\dagger\left(\vec{k}_3\right)$, which leads to

$$\left\langle 0 \left| a\left(\vec{k}_1\right) a^\dagger\left(\vec{k}_4\right) \right| 0 \right\rangle (2\pi)^3 2E_2 \delta^{(3)}\left(\vec{k}_2 - \vec{k}_3\right) \ + \ \left\langle 0 \left| a\left(\vec{k}_1\right) a^\dagger\left(\vec{k}_3\right) a\left(\vec{k}_2\right) a^\dagger\left(\vec{k}_4\right) \right| 0 \right\rangle \quad .$$

Then, we commute in the first of the above two terms $a\left(\vec{k}_1\right)$ and $a^\dagger\left(\vec{k}_4\right)$, whereas in the second of the above two terms we commute as well $a\left(\vec{k}_1\right)$ with $a^\dagger\left(\vec{k}_3\right)$, as $a\left(\vec{k}_2\right)$ with $a^\dagger\left(\vec{k}_4\right)$. The result of those operations is given by

$$4(2\pi)^6 E_1 E_2 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_4\right)\ \delta^{(3)}\left(\vec{k}_2 - \vec{k}_3\right)\ +\ 4(2\pi)^6 E_1 E_2 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_3\right)\ \delta^{(3)}\left(\vec{k}_2 - \vec{k}_4\right)\quad . \tag{1.81}$$

In the second term of (1.80), which, in a way similar to formula (1.79), contains the vacuum expectation value

$$\left\langle 0\left| a\left(\vec{k}_1\right) a^\dagger\left(\vec{k}_2\right) a\left(\vec{k}_3\right) a^\dagger\left(\vec{k}_4\right)\right| 0\right\rangle\quad ,$$

we commute as well $a\left(\vec{k}_1\right)$ with $a^\dagger\left(\vec{k}_2\right)$, as $a\left(\vec{k}_3\right)$ with $a^\dagger\left(\vec{k}_4\right)$. The result of those operations is given by

$$4(2\pi)^6 E_1 E_3 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right)\ \delta^{(3)}\left(\vec{k}_3 - \vec{k}_4\right)\quad . \tag{1.82}$$

When we sum the two expressions (1.81) and (1.82) and also include the integrations and the corresponding exponentials, we find for the vacuum expectation value of formula (1.80) the result

$$\begin{aligned}
&\langle 0\left|\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\right| 0\rangle\ = \\
&= \int \frac{d^3k_1}{(2\pi)^3 2E_1}\int \frac{d^3k_2}{(2\pi)^3 2E_2}\int \frac{d^3k_3}{(2\pi)^3 2E_3}\int \frac{d^3k_4}{(2\pi)^3 2E_4}\ \times \\
&\times\Big\{\ e^{-ik_1x_1 - ik_2x_2 + ik_3x_3 + ik_4x_4}\ \Big[4(2\pi)^6 E_1 E_2 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_4\right)\ \delta^{(3)}\left(\vec{k}_2 - \vec{k}_3\right)\ + \\
&+4(2\pi)^6 E_1 E_2 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_3\right)\ \delta^{(3)}\left(\vec{k}_2 - \vec{k}_4\right)\Big]\ + \\
&+\ e^{-ik_1x_1 + ik_2x_2 - ik_3x_3 + ik_4x_4}\ 4(2\pi)^6 E_1 E_3 \delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right)\ \delta^{(3)}\left(\vec{k}_3 - \vec{k}_4\right)\Big\}\quad .
\end{aligned}$$

Because of the Dirac delta functions, one may perform two of the four $\vec{k}$-integrations in each of the three above terms. In the first two terms we perform the $\vec{k}_3$ and the $\vec{k}_4$ integrations. In the third term we perform the $\vec{k}_2$ and the $\vec{k}_4$ integrations, and then rename the $\vec{k}_3$ integration variable for $\vec{k}_2$. This gives the vacuum expectation value of formula (1.80) its final form

$$\begin{aligned}
&\langle 0\left|\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\right| 0\rangle\ = \\
&= \int \frac{d^3k_1}{(2\pi)^3 2E_1}\int \frac{d^3k_2}{(2\pi)^3 2E_2}\Big\{\ e^{-ik_1\left(x_1 - x_4\right) - ik_2\left(x_2 - x_3\right)}\ + \\
&+\ e^{-ik_1\left(x_1 - x_3\right) - ik_2\left(x_2 - x_4\right)}\ +\ e^{-ik_1\left(x_1 - x_2\right) - ik_2\left(x_3 - x_4\right)}\Big\}
\end{aligned} \tag{1.83}$$

Not a very terrible result, but remember that the vacuum expectation value (1.75) of the time ordered product of four boson fields contains twenty-four of such terms, which now has to be multiplied by three. So, we have ended up with seventy-two terms, hence some bookkeeping is in order.

For convenience we define

$$A\left(x_1 - x_2\right) \;=\; \int \frac{d^3k}{(2\pi)^3 2E}\; e^{-ik\left(x_1 - x_2\right)} \quad . \tag{1.84}$$

Using this definition and the result (1.83), we obtain for the vacuum expectation value (1.75) of the time ordered product of four boson fields the expression

$$\begin{aligned}
&\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\right\}\right|0\right\rangle \;= \\
&= \;\left\{A\left(x_1 - x_4\right)A\left(x_2 - x_3\right) \;+\; A\left(x_1 - x_3\right)A\left(x_2 - x_4\right) \;+\right. \\
&\left.+\; A\left(x_1 - x_2\right)A\left(x_3 - x_4\right)\right\}\theta\left(t_1 - t_2\right)\theta\left(t_2 - t_3\right)\theta\left(t_3 - t_4\right) \;+ \\
&+\;\; \text{(all possible permutations of 1,2,3 and 4)} \quad .
\end{aligned} \tag{1.85}$$

Indeed $24 \times 3 = 72$ terms! However, as we will see in the following, their number can be reduced to three. By inspection of all twenty-four permutations of (1.85), we find that there are several terms which contain the same combination of A's. Notice, from their definition (1.84), that the order of the A's in a product of A's does not matter, but that the order of the coordinate variables x inside one A do matter. In the following table, where we denote $t_1 \rangle t_2 \rangle t_3 \rangle t_4$ by 1234 and similar for the other time-orderings, we have collected all twenty-four possible time-orderings which contribute to (1.85) and the A's to which they are multiplied.

time-ordering	Amplitudes involved		
1234	$\underline{A(1-2)\times A(3-4)}$	$A(1-3)\times A(2-4)$	$A(1-4)\times A(2-3)$
1243	$A(1-2)\times A(4-3)$	$A(1-4)\times A(2-3)$	$A(1-3)\times A(2-4)$
1324	$A(1-3)\times A(2-4)$	$\underline{A(1-2)\times A(3-4)}$	$A(1-4)\times A(3-2)$
1342	$A(1-3)\times A(4-2)$	$A(1-4)\times A(3-2)$	$\underline{A(1-2)\times A(3-4)}$
1432	$A(1-4)\times A(3-2)$	$A(1-3)\times A(4-2)$	$A(1-2)\times A(4-3)$
1423	$A(1-4)\times A(2-3)$	$A(1-2)\times A(4-3)$	$A(1-3)\times A(4-2)$
2134	$A(2-1)\times A(3-4)$	$A(2-3)\times A(1-4)$	$A(2-4)\times A(1-3)$
2143	$A(2-1)\times A(4-3)$	$A(2-4)\times A(1-3)$	$A(2-3)\times A(1-4)$
2314	$A(2-3)\times A(1-4)$	$A(2-1)\times A(3-4)$	$A(2-4)\times A(3-1)$
2341	$A(2-3)\times A(4-1)$	$A(2-4)\times A(3-1)$	$A(2-1)\times A(3-4)$
2431	$A(2-4)\times A(3-1)$	$A(2-3)\times A(4-1)$	$A(2-1)\times A(4-3)$
2413	$A(2-4)\times A(1-3)$	$A(2-1)\times A(4-3)$	$A(2-3)\times A(4-1)$
3214	$A(3-2)\times A(1-4)$	$A(3-1)\times A(2-4)$	$A(3-4)\times A(2-1)$
3241	$A(3-2)\times A(4-1)$	$A(3-4)\times A(2-1)$	$A(3-1)\times A(2-4)$
3124	$A(3-1)\times A(2-4)$	$A(3-2)\times A(1-4)$	$\underline{A(3-4)\times A(1-2)}$
3142	$A(3-1)\times A(4-2)$	$\underline{A(3-4)\times A(1-2)}$	$A(3-2)\times A(1-4)$
3412	$\underline{A(3-4)\times A(1-2)}$	$A(3-1)\times A(4-2)$	$A(3-2)\times A(4-1)$
3421	$A(3-4)\times A(2-1)$	$A(3-2)\times A(4-1)$	$A(3-1)\times A(4-2)$
4231	$A(4-2)\times A(3-1)$	$A(4-3)\times A(2-1)$	$A(4-1)\times A(2-3)$
4213	$A(4-2)\times A(1-3)$	$A(4-1)\times A(2-3)$	$A(4-3)\times A(2-1)$
4321	$A(4-3)\times A(2-1)$	$A(4-2)\times A(3-1)$	$A(4-1)\times A(3-2)$
4312	$A(4-3)\times A(1-2)$	$A(4-1)\times A(3-2)$	$A(4-2)\times A(3-1)$
4132	$A(4-1)\times A(3-2)$	$A(4-3)\times A(1-2)$	$A(4-2)\times A(1-3)$
4123	$A(4-1)\times A(2-3)$	$A(4-2)\times A(1-3)$	$A(4-3)\times A(1-2)$

For example, when we collect all terms, underlined in the previous table, which contain the product of $A\left(x_1-x_2\right)$ and $A\left(x_3-x_4\right)$, then we find six such terms, *i.e.*

$$
\begin{aligned}
& A\left(x_1-x_2\right) A\left(x_3-x_4\right)\left\{\theta\left(t_1-t_2\right) \theta\left(t_2-t_3\right) \theta\left(t_3-t_4\right)\ +\right. \\
& +\ \theta\left(t_1-t_3\right) \theta\left(t_3-t_2\right) \theta\left(t_2-t_4\right)\ +\ \theta\left(t_1-t_3\right) \theta\left(t_3-t_4\right) \theta\left(t_4-t_2\right)\ + \\
& +\ \theta\left(t_3-t_1\right) \theta\left(t_1-t_2\right) \theta\left(t_2-t_4\right)\ +\ \theta\left(t_3-t_1\right) \theta\left(t_1-t_4\right) \theta\left(t_4-t_2\right)\ + \\
& \left.+\ \theta\left(t_3-t_4\right) \theta\left(t_4-t_1\right) \theta\left(t_1-t_2\right)\right\} \quad .
\end{aligned} \tag{1.86}
$$

In total the vacuum expectation value (1.85) of the time ordered product of four boson fields contains twelve such expressions, similar to the one in formula (1.86), which gives again $12 \times 6 = 72$ terms. They are collected in the table below.

distinct terms	time-ordering contributions					
$A(1-2) \times A(3-4)$	1234	1324	1342	3124	3142	3412
$A(1-3) \times A(2-4)$	1234	1243	1324	2134	2143	2413
$A(1-4) \times A(2-3)$	1234	1243	1423	2134	2143	2314
$A(1-2) \times A(4-3)$	1243	1432	1423	4312	4132	4123
$A(1-4) \times A(3-2)$	1324	1342	1432	3214	3124	3142
$A(1-3) \times A(4-2)$	1342	1432	1423	4213	4132	4123
$A(2-1) \times A(3-4)$	2134	2314	2341	3214	3241	3421
$A(2-1) \times A(4-3)$	2143	2431	2413	4231	4213	4321
$A(2-4) \times A(3-1)$	2314	2341	2431	3214	3241	3124
$A(2-3) \times A(4-1)$	2341	2431	2413	4231	4213	4123
$A(3-2) \times A(4-1)$	3241	3412	3421	4321	4312	4132
$A(3-1) \times A(4-2)$	3142	3412	3421	4231	4321	4312

Now, when we inspect carefully the six combinations of θ-functions in formula (1.86), then we find that they together just built up the time interval given by $t_1\rangle t_2$ and $t_3\rangle t_4$. For instance, when we denote $t_1\rangle t_2\rangle t_3\rangle t_4$ by 1234 and similar for the other time-orderings, then the time interval $t_1\rangle t_2$ and $t_3\rangle t_4$ just consists of 1234, 1324, 1342, 3124, 3142, and 3412, which time-orderings represent precisely the six combinations of θ-functions in formula (1.86). Any other permutation of 1, 2, 3 and 4 lies outside the referred time interval. Consequently, formula (1.86) can be substituted by

$$A\left(x_1-x_2\right) A\left(x_3-x_4\right)\ \theta\left(t_1-t_2\right)\ \theta\left(t_3-t_4\right) \quad . \tag{1.87}$$

The seventy-two terms contained in formula (1.85) have reduced to twelve terms of the form (1.87).

A similar expression which comes from the product of $A\left(x_2-x_1\right)$ and $A\left(x_3-x_4\right)$, and which is built up of the time-ordered terms 2134, 2314, 2341, 3214, 3241, and 3421, is given by

$$A\left(x_2-x_1\right) A\left(x_3-x_4\right)\ \theta\left(t_2-t_1\right)\ \theta\left(t_3-t_4\right) \quad . \tag{1.88}$$

The sum of (1.87) and (1.88) gives

$$\left\{A\left(x_1-x_2\right) \theta\left(t_1-t_2\right)\ +\ A\left(x_2-x_1\right) \theta\left(t_2-t_1\right)\right\}\ A\left(x_3-x_4\right) \theta\left(t_3-t_4\right) \quad ,$$

which equals

$$A\left(x_1-x_2\right) A\left(x_3-x_4\right) \theta\left(t_3-t_4\right) \quad \text{when} \quad t_1\rangle t_2 \ ,$$

or

$$A\left(x_2-x_1\right) A\left(x_3-x_4\right) \theta\left(t_3-t_4\right) \quad \text{when} \quad t_1\langle t_2 \ .$$

Now, using the same procedure which lead from formula (1.48) to formula (1.52), we find for the sum of (1.87) and (1.88) the following

$$\int \frac{d^3k}{(2\pi)^3 2E}\, e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}\, e^{-iE\,|t_1 - t_2|}\, A\,(x_3 - x_4)\,\theta\,(t_3 - t_4)$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\,(x_1 - x_2)}}{k^2 - m^2}\, A\,(x_3 - x_4)\,\theta\,(t_3 - t_4) \quad . \tag{1.89}$$

Continuing the above procedure, all seventy-two terms of (1.85) can be summed setwise in six sets of twelve terms. For example, another such set of twelve terms sums up to

$$i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\,(x_1 - x_2)}}{k^2 - m^2}\, A\,(x_4 - x_3)\,\theta\,(t_4 - t_3) \quad . \tag{1.90}$$

Now, at this stage, it might be clear that, along the same reasoning which lead to formula (1.89), we obtain for the sum of (1.89) and (1.90) the result

$$i^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{-ik_1\,(x_1 - x_2)}}{(k_1)^2 - m^2} \int \frac{d^4k_2}{(2\pi)^4} \frac{e^{-ik_2\,(x_3 - x_4)}}{(k_2)^2 - m^2} \quad . \tag{1.91}$$

A set of twenty-four terms of (1.85) neatly summed up in a compact expression. The other two sets of twenty-four terms yield:

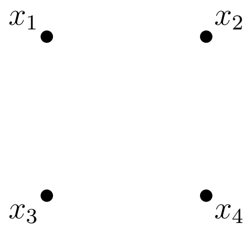
$$i^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{-ik_1\,(x_1 - x_3)}}{(k_1)^2 - m^2} \int \frac{d^4k_2}{(2\pi)^4} \frac{e^{-ik_2\,(x_2 - x_4)}}{(k_2)^2 - m^2} \quad , \tag{1.92}$$

and

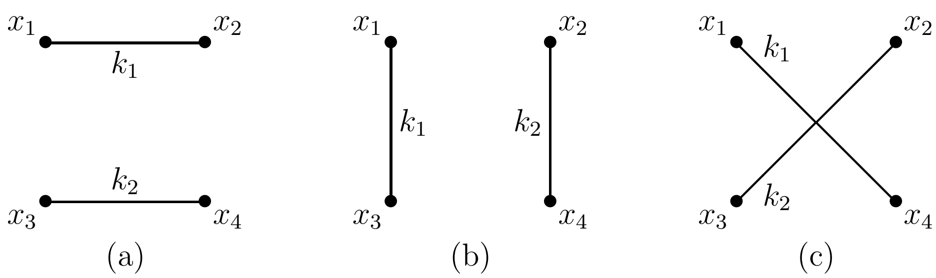
$$i^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{-ik_1\,(x_1 - x_4)}}{(k_1)^2 - m^2} \int \frac{d^4k_2}{(2\pi)^4} \frac{e^{-ik_2\,(x_2 - x_3)}}{(k_2)^2 - m^2} \quad . \tag{1.93}$$

The whole vacuum expectation value of the time ordered product of four boson fields is just given by the sum of the three expressions, (1.91), (1.92), and (1.93). Each of those expressions is just the product of two Feynman propagators as given in formula (1.52).

A graphical representation for the three expressions, (1.91), (1.92), and (1.93), can be constructed as follows: The coordinates x_1, x_2, x_3, and x_4 are represented by four dots, as shown below.



Each possible pairwise connection of those dots represents one of the three above expressions according to the combination of coordinates in the exponents. The three possible pairwise connections are given below.



Graph (a) corresponds to expression (1.91), graph (b) to expression (1.92), and graph (c) to expression (1.93).

It might be clear that for more fields the procedure becomes more tedious. For this reason we will make use of the *Feynman rules*, which take care (for us) of all time-orderings and leave us with the final integrals.

1.8 Feynman rules (part I)

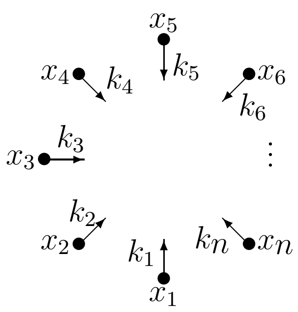
In order to determine an analytic expression for the vacuum expection value of a time-ordered product of n fields, given by

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \cdots \phi\left(x_n\right) \right\} \right| 0 \rangle \quad , \tag{1.94}$$

one proceeds as follows. Each field ϕ in (1.94) brings, following the definition (1.28) for the quantum fields as well as the procedure which lead from formula (1.48) to formula (1.52), a Fourier transform integration of the form

$$\int \frac{d^4 k_i}{(2\pi)^4}\, e^{-ik_i x_i} \quad \text{for} \quad i = 1, \ldots, n \quad . \tag{1.95}$$

The n events are graphically represented by n dots, as shown in the figure below



From each dot flows momentum away, as also indicated in the same picture. Those momenta correspond to the Fourier transform integration variables and are closely related to the creation and annihilation operators, $a^\dagger\left(\vec{k}\right)$ and $a\left(\vec{k}\right)$. Consequently, each momentum flow relates to one field defined at the corresponding event. Hence, when for an event y more fields are involved, as many momenta flow from the related dot as there come fields with argument y in the expression for the vacuum expectation value.

Overall momentum conservation gives moreover a factor

$$(2\pi)^4\, \delta^{(4)}\left(k_1 + k_2 + \cdots + k_n\right) \quad . \tag{1.96}$$

So far, the procedure is the same for each contribution, *i.e.*

$$\begin{aligned} \langle 0 \left| T \left\{ \phi\left(x_1\right) \cdots \phi\left(x_n\right) \right\} \right| 0 \rangle \;=\; & \int \frac{d^4 k_1}{(2\pi)^4} e^{-ik_1 x_1} \int \frac{d^4 k_2}{(2\pi)^4} e^{-ik_2 x_2} \cdots \int \frac{d^4 k_n}{(2\pi)^4} e^{-ik_n x_n} \times \\ & \times (2\pi)^4\, \delta^{(4)}\left(k_1 + k_2 + \cdots + k_n\right) \left\{ \text{ something } \right\} \end{aligned} \tag{1.97}$$

The *something* contains all possible contributions, which are found by contracting pairwise, in all possible combinations, the momenta. When you do it with a pencil, then you obtain the *Feynman graphs*. In the analytic expression one writes for each contraction a Feynman propagator, as defined in formula (1.23), and moreover a Dirac delta function to assure momentum conservation (multiplied with $(2\pi)^4$ of course), except for one pair which follows already from the overall plus all the other Dirac delta functions.

We give below three examples, the already known vacuum expectation values of the time-ordered products of two and four fields and a new vacuum expectation value which also involves some combinatorics.

I The vacuum expectation value of the time-ordered product of two fields.

From expression (1.38), following the above outlined procedure, we find for the vacuum expectation value of the time-ordered product of two fields the following graphic representation

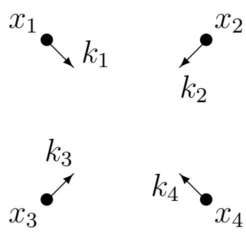
Consequently, one has only one possible contraction, which leads to the analytic expression given by

$$\int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1} \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2}\, (2\pi)^4\, \delta^{(4)}\,(k_1+k_2)\, \frac{i}{(k_1)^2-m^2} \quad , \tag{1.98}$$

for which it is a simple task (just perform the k_2-integration) to convince oneself that this equals the previous expression (1.52).

II The vacuum expectation value of the time-ordered product of four fields.

From expression (1.74), following the above outlined procedure, we find for the vacuum expectation value of the time-ordered product of four fields the following graphic representation



Consequently, the general form of the analytic expression reads

$$\int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1} \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2} \int \frac{d^4k_3}{(2\pi)^4}\, e^{-ik_3x_3} \int \frac{d^4k_4}{(2\pi)^4}\, e^{-ik_4x_4} \times$$

$$\times\, (2\pi)^4\, \delta^{(4)}\,(k_1+k_2+k_3+k_4)\, \{\text{contractions}\} \quad . \tag{1.99}$$

There are three different possible ways to contract the four momenta in this case, as we already know from section (1.6).

Contracting k_1 with k_2 and k_3 with k_4 gives

$$(2\pi)^4 \, \delta^{(4)} \left(k_1 + k_2\right) \, \frac{i}{\left(k_1\right)^2 - m^2} \, \times \, \frac{i}{\left(k_3\right)^2 - m^2} \quad .$$

Notice that only one of the two contractions involves a Dirac delta function for the momentum conservation, the other pair is then automatically conserved because of the Dirac delta function in formule (1.99) for the overall momentum conservation.

The other two contributions to (1.99), with comparable expressions to the one above, come from the other two possible ways to contract the momenta. In total, we find then

$$\begin{aligned}
&\left\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right\} \right| 0 \right\rangle \; = \qquad\qquad (1.100) \\
&\int \frac{d^4k_1}{(2\pi)^4} \, e^{-ik_1x_1} \int \frac{d^4k_2}{(2\pi)^4} \, e^{-ik_2x_2} \int \frac{d^4k_3}{(2\pi)^4} \, e^{-ik_3x_3} \int \frac{d^4k_4}{(2\pi)^4} \, e^{-ik_4x_4} \; \times \\
&\quad \times \, (2\pi)^4 \, \delta^{(4)} \left(k_1 + k_2 + k_3 + k_4\right) \; \times \\
&\times \, \left\{ (2\pi)^4 \, \delta^{(4)} \left(k_1 + k_2\right) \, \frac{i}{\left(k_1\right)^2 - m^2} \, \times \, \frac{i}{\left(k_3\right)^2 - m^2} \, + \right. \\
&\quad + \, (2\pi)^4 \, \delta^{(4)} \left(k_1 + k_3\right) \, \frac{i}{\left(k_1\right)^2 - m^2} \, \times \, \frac{i}{\left(k_2\right)^2 - m^2} \, + \\
&\quad \left. + \, (2\pi)^4 \, \delta^{(4)} \left(k_1 + k_4\right) \, \frac{i}{\left(k_1\right)^2 - m^2} \, \times \, \frac{i}{\left(k_3\right)^2 - m^2} \right\} \quad .
\end{aligned}$$

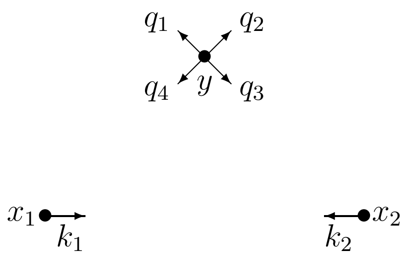
After performing two of the four k-integrations one obtains the same result as given by the sum of the three expressions, (1.91), (1.92), and (1.93).

III The vacuum expectation value of the time-ordered product of six fields, out of which four are at the same event.

The vacuum expectation value of the time-ordered product of six fields, out of which four are at the same event, is given by

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi(y)\phi(y)\phi(y)\phi(y) \right\} \right| 0 \rangle \quad , \tag{1.101}$$

whereas its general structure is represented by the following graph



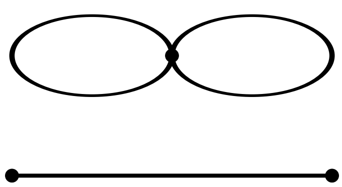
and, moreover, its corresponding analytic expression takes the form

$$\begin{aligned} & \int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1} \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2} \; \times \\ & \times \int \frac{d^4q_1}{(2\pi)^4}\, e^{-iq_1y} \int \frac{d^4q_2}{(2\pi)^4}\, e^{-iq_2y} \int \frac{d^4q_3}{(2\pi)^4}\, e^{-iq_3y} \int \frac{d^4q_4}{(2\pi)^4}\, e^{-iq_4y} \; \times \\ & \times (2\pi)^4\, \delta^{(4)}\left(k_1 + k_2 + q_1 + q_2 + q_3 + q_4\right)\; \{\text{something}\} \quad . \end{aligned} \tag{1.102}$$

The *something* contains fifteen contributions: For, one of the six momenta can be contracted with each of the five other momenta. One of the four remaining momenta can be contracted with one out of three momenta. Whereas, the finally remaining two momenta can only be contracted amongst each other. This gives indeed

$$5 \;\times\; 3 \;\times\; 1 \;=\; 15 \tag{1.103}$$

possibilities. There are two types of contractions which can be distinguished. The first type, which we will refer to as *type A* contributions, is the result of contracting k_1 with k_2 and the q's amongst each other. The generic graph is depicted below.



type A

There are three such contributions, which result all three in the same analytic expression, because one of the q's can be contracted with each of the remaining three q's and moreover integration variables are dummy. We obtain then for type A the expression

$$3\ (2\pi)^4\ \delta^{(4)}\left(q_1+q_4\right)\ \frac{i}{\left(q_1\right)^2-m^2}\ \times\ (2\pi)^4\ \delta^{(4)}\left(q_2+q_3\right)\ \frac{i}{\left(q_2\right)^2-m^2}\ \times\ \frac{i}{\left(k_1\right)^2-m^2}\quad .$$

So, the type A contractions lead to the contribution

$$\begin{aligned}
&3\int\frac{d^4k_1}{(2\pi)^4}\ e^{-ik_1x_1}\int\frac{d^4k_2}{(2\pi)^4}\ e^{-ik_2x_2}\ \times \qquad\qquad (1.104)\\
&\times\int\frac{d^4q_1}{(2\pi)^4}\ e^{-iq_1y}\int\frac{d^4q_2}{(2\pi)^4}\ e^{-iq_2y}\int\frac{d^4q_3}{(2\pi)^4}\ e^{-iq_3y}\int\frac{d^4q_4}{(2\pi)^4}\ e^{-iq_4y}\ \times\\
&\times(2\pi)^4\ \delta^{(4)}\left(k_1+k_2+q_1+q_2+q_3+q_4\right)\ \times\\
&\times(2\pi)^4\ \delta^{(4)}\left(q_1+q_4\right)\ \frac{i}{\left(q_1\right)^2-m^2}\ \times\ (2\pi)^4\ \delta^{(4)}\left(q_2+q_3\right)\ \frac{i}{\left(q_2\right)^2-m^2}\ \times\ \frac{i}{\left(k_1\right)^2-m^2}\quad .
\end{aligned}$$

When we perform the q_3 and q_4 integrations, then, because of the Dirac delta functions, we end up with

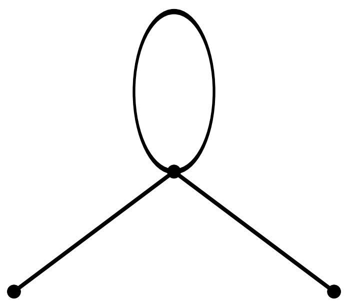
$$\begin{aligned}
&3\int\frac{d^4k_1}{(2\pi)^4}\ e^{-ik_1x_1}\int\frac{d^4k_2}{(2\pi)^4}\ e^{-ik_2x_2}\ (2\pi)^4\ \delta^{(4)}\left(k_1+k_2\right)\ \frac{i}{\left(k_1\right)^2-m^2}\ \times\\
&\times\int\frac{d^4q_1}{(2\pi)^4}\ \frac{i}{\left(q_1\right)^2-m^2}\ \times\int\frac{d^4q_2}{(2\pi)^4}\ \frac{i}{\left(q_2\right)^2-m^2}\quad . \qquad\qquad (1.105)
\end{aligned}$$

The latter two integrals are so-called *loop integrations* since each can be associated with one of the two loops in the graph for the type A contributions. When in formula (1.100) one substitutes x_1, x_2, x_3, and x_4, by y one obtains exactly three times the product of those loop integrations. Moreover, by comparison to formula (1.98), one finds that the first part of the above expression (1.105) equals the vacuum expectation value of the time-ordered product of two fields. Consequently, one may write for the type A contributions the following identity

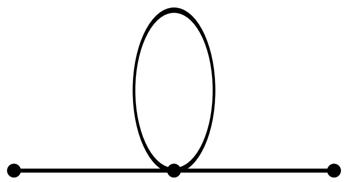
$$\begin{aligned}
&\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi(y)\phi(y)\phi(y)\phi(y)\right\}\right|0\right\rangle\ \text{(type A contributions)}\ =\\
&=\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\right\}\right|0\right\rangle\ \times\ \left\langle 0\left|T\left\{\phi(y)\phi(y)\phi(y)\phi(y)\right\}\right|0\right\rangle\quad . \qquad\qquad (1.106)
\end{aligned}$$

Contributions, which are represented by graphs similar to the graph for the type A contribution, *i.e.* graphs which have disconnected parts, are called *vacuum bubbles.* They do not play any role in real physics as we will see furtheron.

The second type of contributions to the *something* of formula (1.102), which we will refer to as *type B* contributions, stem from the contractions of k_1 and k_2 each with one of the q's. The generic graph is depicted below



In the literature this graph is usually drawn as shown hereafter



type B

There are twelve such contributions, which result all twelve in the same analytic expression, because one of the k's can be contracted with each of the four q's, the other k with any of the three remaining q's and moreover integration variables are dummy, which gives

$$4 \times 3 = 12 \tag{1.107}$$

contributions. We obtain then for type B the expression

$$12\,(2\pi)^4\,\delta^{(4)}\,(k_1+q_1)\;\frac{i}{(k_1)^2-m^2}\;\times\;(2\pi)^4\,\delta^{(4)}\,(k_2+q_2)\;\frac{i}{(k_2)^2-m^2}\;\times\;\frac{i}{(q_3)^2-m^2}\quad .$$

So, the type B contractions lead to the contribution

$$\begin{aligned}
&12\int\frac{d^4k_1}{(2\pi)^4}\,e^{-ik_1x_1}\int\frac{d^4k_2}{(2\pi)^4}\,e^{-ik_2x_2}\;\times \qquad (1.108)\\
&\times\int\frac{d^4q_1}{(2\pi)^4}\,e^{-iq_1y}\int\frac{d^4q_2}{(2\pi)^4}\,e^{-iq_2y}\int\frac{d^4q_3}{(2\pi)^4}\,e^{-iq_3y}\int\frac{d^4q_4}{(2\pi)^4}\,e^{-iq_4y}\;\times\\
&\times(2\pi)^4\,\delta^{(4)}\,(k_1+k_2+q_1+q_2+q_3+q_4)\;\times\\
&\times(2\pi)^4\,\delta^{(4)}\,(k_1+q_1)\;\frac{i}{(k_1)^2-m^2}\;\times\;(2\pi)^4\,\delta^{(4)}\,(k_2+q_2)\;\frac{i}{(k_2)^2-m^2}\;\times\;\frac{i}{(q_3)^2-m^2}
\end{aligned}$$

When we perform the q_1 and q_2 integrations, then, because of the Dirac delta functions, we end up with

$$12 \int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1\,(x_1-y)}\, \frac{i}{(k_1)^2-m^2} \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2\,(x_2-y)}\, \frac{i}{(k_2)^2-m^2}\ \times$$

$$\times \int \frac{d^4q_3}{(2\pi)^4}\, e^{-iq_3y} \int \frac{d^4q_4}{(2\pi)^4}\, e^{-iq_4y}\, (2\pi)^4\, \delta^{(4)}\,(q_3+q_4)\, \frac{i}{(q_3)^2-m^2} \quad . \tag{1.109}$$

Next, we may perform the q_4 integration, to end up with

$$12 \int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1\,(x_1-y)}\, \frac{i}{(k_1)^2-m^2} \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2\,(x_2-y)}\, \frac{i}{(k_2)^2-m^2}\ \times$$

$$\times \int \frac{d^4q}{(2\pi)^4}\, \frac{i}{q^2-m^2} \quad . \tag{1.110}$$

For the latter part of this expression we recognize again a loop integral, corresponding to the loop in the type B graph.

Chapter 2

Two-points Green's function

Following formula (1.37), also substituting expression (1.36) for the interaction Lagrangian density, the two-points Green's function is in ϕ^4 theory defined by

$$G\left(x_1, x_2\right) \;=\; \frac{\left\langle 0 \left| T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\;\exp\left[i\int d^4y\;\left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]\right\}\right| 0\right\rangle}{\left\langle 0 \left| T\left\{\exp\left[i\int d^4y\;\left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]\right\}\right| 0\right\rangle} \quad , \tag{2.1}$$

When we expand the exponent in the numerator of (2.1), then we obtain for the numerator the following series of time-ordered vacuum expectation values

$$\begin{aligned}
&\left\langle 0 \left| T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\;\exp\left[i\int d^4y\;\left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]\right\}\right| 0\right\rangle \;= \qquad (2.2)\\
&= \left\langle 0 \left| T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\;\left\{1 \;+\; i\int d^4y\;\left(-\frac{\lambda}{4!}\right)\phi^4(y) \;+\right.\right.\right.\right.\\
&\left.\left.\left.\left. +\;\frac{1}{2!}\left[i\int d^4y\;\left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]^2 \;+\; \frac{1}{3!}\left[i\int d^4y\;\left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]^3 \;+\;\cdots\right\}\right\}\right| 0\right\rangle\\
&= \left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\right\}\right|0\right\rangle \;+\; \left(-i\frac{\lambda}{4!}\right)\int d^4y\;\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi^4(y)\right\}\right|0\right\rangle \;+\\
&+\;\frac{1}{2!}\left(-i\frac{\lambda}{4!}\right)^2\int d^4y_1\;\int d^4y_2\;\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi^4\left(y_1\right)\phi^4\left(y_2\right)\right\}\right|0\right\rangle \;+\\
&+\;\frac{1}{3!}\left(-i\frac{\lambda}{4!}\right)^3\int d^4y_1\;\int d^4y_2\;\int d^4y_3\;\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi^4\left(y_1\right)\phi^4\left(y_2\right)\phi^4\left(y_3\right)\right\}\right|0\right\rangle \;+\\
&+\;\cdots \quad ,
\end{aligned}$$

which may be considered as an expansion in the coupling constant λ.

For the first term of the expansion (2.2) we recognize the vacuum expectation value of the time-ordered product of two fields, for which we have the analytic expressions (1.38) or (1.98). The second term, linear in λ, contains the vacuum expectation value (1.101), which we have determined previously to be equal to the sum of the expressions

(1.105), referred to as the type A contribution, and (1.110), which we called the type B contribution. So, up to the first order in λ we find for the numerator of (2.1) the result

$$\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\ \exp\left[i\int d^4y\ \left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]\right\}\right|0\right\rangle\ = \tag{2.3}$$

$$=\ \left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\right\}\right|0\right\rangle\ +\ \left(-i\frac{\lambda}{4!}\right)\int d^4y\ \{\text{type A}\ +\ \text{type B}\}\ +\ \cdots\quad .$$

Now, for the type A contribution we have the identity given in formula (1.106). Consequently, we may also write the numerator of (2.1) like

$$\left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\ \exp\left[i\int d^4y\ \left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]\right\}\right|0\right\rangle\ = \tag{2.4}$$

$$=\ \left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\right\}\right|0\right\rangle\ \left\{1\ +\ \left(-i\frac{\lambda}{4!}\right)\int d^4y\ \left\langle 0\left|T\left\{\phi^4(y)\right\}\right|0\right\rangle\right\}\ +$$

$$+\ \left(-i\frac{\lambda}{4!}\right)\int d^4y\ \{\text{type B}\}\ +\ \cdots\quad .$$

The denominator of (2.1), expanded to first order in λ, reads

$$\left\langle 0\left|T\left\{\exp\left[i\int d^4y\ \left(-\frac{\lambda}{4!}\right)\phi^4(y)\right]\right\}\right|0\right\rangle\ =\ 1\ +\ \left(-i\frac{\lambda}{4!}\right)\int d^4y\ \left\langle 0\left|T\left\{\phi^4(y)\right\}\right|0\right\rangle\ +\ \cdots\quad . \tag{2.5}$$

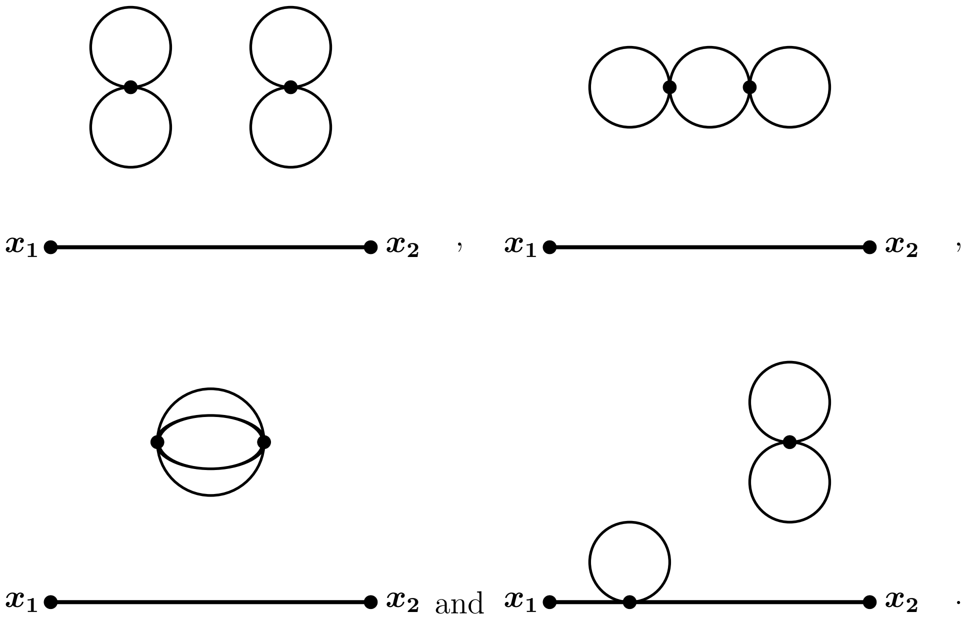
So, by dividing out the denominator (2.5) of the two-points Green's function (2.1) from the expression (2.4) for its numerator, we obtain to first order in λ the result

$$G\left(x_1,x_2\right)\ =\ \left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\right\}\right|0\right\rangle\ +\ \left(-i\frac{\lambda}{4!}\right)\int d^4y\ \{\text{type B}\}\ +\ \cdots\quad . \tag{2.6}$$

The type A contribution has disappeared from the final expression for the two-points Green's function, which result can be generalized, as we will discuss in the next section.

2.1 Vacuum bubbles

The events y, which stem from the interaction part of the Lagrangian density, are in general called the *internal points* of a contribution to the n-points Green's function. The other events x, which come as arguments of the n-points Green's function, are referred as the *external points*. Now, when in a Feynman graph for one or more internal points do not exist any propagators, directly or indirectly, which connect them to the external points, then the bubble-like structure(s) around those internal points are called vacuum bubbles. The type A contribution to the vacuum expectation value given in formula (1.101), contains such vacuum bubble. Other examples, for which the graphs are shown below, come from the second order in λ term of the expansion (2.2) for the 2-points Green's function.



The sum of the contributions represented by the first three of the here shown graphs is, similarly to the factorization (1.106) for the type A contribution, given by

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \right\} \right| 0 \rangle \left\langle 0 \left| T \left\{ \tfrac{1}{2!} \left[i \int d^4y \, \left(-\frac{\lambda}{4!} \right) \phi^4(y) \right]^2 \right\} \right| 0 \right\rangle \quad , \tag{2.7}$$

which can be considered to represent the second order in λ vacuum bubble extension of the vacuum expectation value of the time-ordered product of two fields. One can easily imagine how the higher order extensions look like. In fact, one can proof that the sum of all possible vacuum bubble extensions of the vacuum expectation value of the time-ordered product of two fields is just given by

$$\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \right\} \right| 0 \rangle \left\langle 0 \left| T \left\{ \exp \left[i \int d^4y \, \left(-\frac{\lambda}{4!} \right) \phi^4(y) \right] \right\} \right| 0 \right\rangle \quad . \tag{2.8}$$

The latter of the four above second order in λ vacuum bubble graphs reads analytically

$$\{\text{type B}\} \left\langle 0 \left| T \left\{ i \int d^4y \, \left(-\frac{\lambda}{4!} \right) \phi^4(y) \right\} \right| 0 \right\rangle \quad , \tag{2.9}$$

which forms the first order in λ vacuum bubble extension of the type B contribution, given in formula (1.110) and discussed in the text preceding that formula, to the two-points Green's function.

One can, moreover, proof in general that the whole numerator of (2.1) is given by

$$\left\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \; \exp\left[i \int d^4y \; \left(-\frac{\lambda}{4!} \right) \phi^4(y) \right] \right\} \right| 0 \right\rangle \; = \tag{2.10}$$

$$= \; \{\text{all contributions without vacuum bubbles}\} \times \left\langle 0 \left| T \left\{ e^{\displaystyle i \int d^4y \; \left(-\frac{\lambda}{4!} \right) \phi^4(y)} \right\} \right| 0 \right\rangle ,$$

and hence the two-points Green's function by

$$G\left(x_1, x_2\right) \; = \; \text{sum over all contributions without vacuum bubbles} \quad . \tag{2.11}$$

Vacuum bubble terms do not contribute to any n-points Green's function and do not even have to be considered.

2.2 Two-points Green's function (continuation)

So, from formula (2.11) we may conclude that to first order in λ the two-points Green's function reads

$$G(x_1, x_2) \;=\; \langle 0\,|T\{\phi(x_1)\,\phi(x_2)\}|\,0\rangle \;+\; \left(-i\frac{\lambda}{4!}\right)\int d^4y \text{ type B } + \;\cdots \quad . \tag{2.12}$$

When we substitute for type B the expression of formula (1.110) and moreover perform the y-integration, then we arrive for the type B term of formula (2.12) at

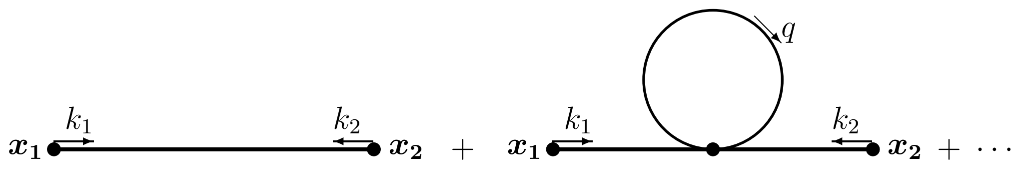
$$\begin{aligned}
&\left(-i\frac{\lambda}{4!}\right)\int d^4y \text{ type B } = \qquad\qquad (2.13)\\
&= \left(-i\frac{\lambda}{4!}\right)\int d^4y\; 12 \int\frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1}\,\frac{i}{(k_1)^2-m^2}\int\frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2}\,\frac{i}{(k_2)^2-m^2}\;\times\\
&\times\, e^{i\,(k_1+k_2)\,y}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\\
&= 12\left(-i\frac{\lambda}{4!}\right)\int\frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1}\,\frac{i}{(k_1)^2-m^2}\int\frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2}\,\frac{i}{(k_1)^2-m^2}\;\times\\
&\times\,(2\pi)^4\delta^{(4)}(k_1+k_2)\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\quad .
\end{aligned}$$

Notice that we changed k_2 in one of the propagators for k_1, which can be done because of the Dirac delta function.

Substituting in formula (2.12) both, the above result (2.13) for the type B contribution and the previous result (1.98) for the vacuum expectation value of the time-ordered product of two fields, one obtains for the two-points Green's function to first order in λ the following

$$\begin{aligned}
G(x_1, x_2) \;&=\; \int\frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1}\int\frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2}\,(2\pi)^4\delta^{(4)}(k_1+k_2)\;\times \qquad (2.14)\\
&\times\left\{\frac{i}{k_1^2-m^2}+\frac{i}{k_1^2-m^2}\left[12\left(-i\frac{\lambda}{4!}\right)\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\right]\frac{i}{k_1^2-m^2}\;+\;\cdots\right\}\quad .
\end{aligned}$$

A graphical representation of formula (2.14) reads



2.3 Feynman rules (part II)

From formula (2.14) and its graphical representation one can read off further Feynman rules for ϕ^4 theory.

For each external point x, from which flows away momentum k, we have a Fourier integration of the form

$$\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} \quad . \tag{2.15}$$

For overall momentum conservation we have a factor

$$(4\pi)^4 \delta^{(4)}(\text{sum of the external momenta}) \quad . \tag{2.16}$$

For each propagator in which flows momentum p we have a factor

$$\frac{i}{p^2 - m^2} \quad . \tag{2.17}$$

For each internal point, usually referred to as *vertex*, one has a factor related to the expansion parameter, or *coupling constant*, given by

$$-i\frac{\lambda}{4!} \quad . \tag{2.18}$$

From the series in formula (2.2) we learn moreover that a graph with s vertices brings a factor $(s!)^{-1}$ from the expansion of the exponent. Consequently, for a Feynman graph with s vertices, also taking into account the factors (2.18), one has to include an overall factor

$$\frac{1}{s!}\left(-i\frac{\lambda}{4!}\right)^s \quad . \tag{2.19}$$

Then there is a *combinatorial factor*, which, for example, for the second term of (2.14), or the type B contribution, equals twelve as shown in formula (1.107).

And finally, for each internal loop, with loop momentum q, we find an integration of the form

$$\int \frac{d^4q}{(2\pi)^4} \quad . \tag{2.20}$$

Following these Feynman rules, one determines all possible contributions to any n-point Green's function for ϕ^4 theory.

2.4 The second order in λ contribution to $G(x_1, x_2)$

In order to get some training in applying the Feynman rules which are discussed in section (2.3), and to, moreover, discover new properties for the series expansion of an n-points Green's function, we determine here in all detail the second order, in the coupling constant, contributions to the two-points Green's function.

A second order Feynman graph has two internal points and hence, following the Feynman rule (2.19), yields an overal factor

$$\frac{1}{2!}\left(-i\frac{\lambda}{4!}\right)^2$$

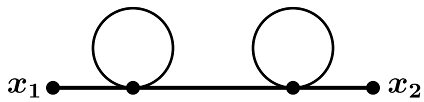
Furthermore, since we are dealing with a two-points Green's function, we have two Fourier integrations of the form (2.15). Then, according to equation (2.11) we only need to find the Feynman graphs without vacuum bubbles, for which the external points cannot be contracted amongst each other. Consequently, each possible contribution has two external propagators, or *legs*, for which the factors are given in Feynman rule (2.17). When we include moreover the factor (2.16), which guarantees momentum conservation for the external momenta, then we find the following generic form for the second order contribution to the two-points Green's function.

$$\frac{1}{2!}\left(-i\frac{\lambda}{4!}\right)^2 \int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1}\, \frac{i}{k_1^2-m^2} \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2}\, \frac{i}{k_1^2-m^2}\, (2\pi)^4\, \delta^{(4)}(k_1+k_2)$$
$$\times\ \{\text{something}\} \quad . \tag{2.21}$$

Since in the expression for *something* we do not have to bother any more about the external legs, this is also called the *amputed* Green's function.

As mentioned before, according to equation (2.11) we only need to find the Feynman graphs without vacuum bubbles, in order to determine the *something* of formula (2.21). Below, we discuss the three graphically distinct possibilities.

1. The first second order contribution which comes to our mind has a graphical representation which consists just of two times the type B Feynman graph for formula (1.108), *i.e.*



From the above graph we can read the combinatorial factor, which indicates how many different contractions are possible. First, we can contract each of the two external points to any of the two internal points, which gives two possibilities. One external momentum can be contracted to any of the four momenta from an internal point, which gives four times four possiblities. Then, there are three momenta left at each vertex, which gives three times three possibilities to contract any pair of them. So, in total we obtain

$$2\ \times\ 4\ \times\ 4\ \times\ 3\ \times\ 3\ =\ 288 \tag{2.22}$$

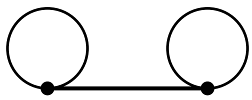
different ways for performing the contractions and still end up with the same graph, which means that this graph represents 288 different, but analytically the same, contributions.

Besides the external propagators, which are already taken care of in expression (2.21), there are three more propagators, the two loops and the propagator which results from the contraction of the momenta of two different internal points. Momentum conservation demands that the momentum, which flows in the propagator which connects the two internal points, equals the external momenta, *i.e.* k_1. For the two loop momenta we select q_1 and q_2.

We find then the following contribution to the *something* of formula (2.21)

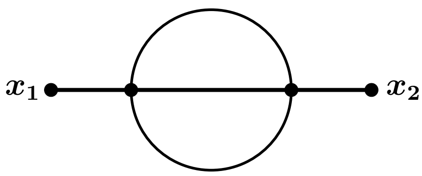
$$288 \left[\int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2} \right] \frac{i}{k_1^2 - m^2} \left[\int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_2^2 - m^2} \right] \quad . \tag{2.23}$$

Since, in fact, for this contribution there are no external legs involved, its correct graphical representation is given by



However, for the combinatorics it is easier to also consider the external legs.

2. The next second order contribution has the following Feynman graph.



As in the previous case, there are two different ways to connect the external points to the internal points. Also the contraction of one external momentum to any of the four vertex momenta can be done in four different ways and the same for the other external momentum. Each vertex has then three remaining momenta. For the first choice to contract one momentum of one vertex with any of the three momenta of the other vertex, are three possibilities. For the second choice two ways. Whereas for the last choice only one possibility is left. So, we obtain as a result that this Feynman graph represents

$$2 \times 4 \times 4 \times 3 \times 2 \times 1 = 192 \tag{2.24}$$

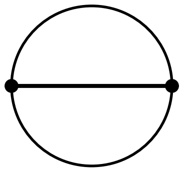
different, though analytically the same, contributions.

Besides the external propagators, which are already taken care of in expression (2.21), there are three more propagators, each connecting the two different internal points. Let us take the momentum flow in those three propagators in the direction away from x_1 towards x_2. Then, if one of those three propagators takes momentum q_1, and a second momentum q_2, the third, because of momentum conservation, must take $k_1 - q_1 - q_2$. For the two loop momenta we select q_1 and q_2.

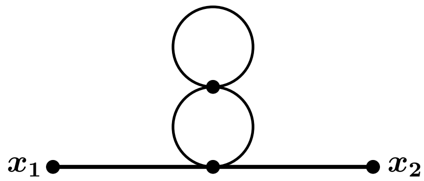
We find then the following contribution to the *something* of formula (2.21) for this case

$$192 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_1^2 - m^2} \frac{i}{q_2^2 - m^2} \frac{i}{(k_1 - q_1 - q_2)^2 - m^2} \quad . \tag{2.25}$$

Since also for this contribution there are no external legs involved, its correct graphical representation is given by



3. The third second order contribution has the following Feynman graph.



Again, there are two different ways to connect the external points to the internal points. Also the contraction of one external momentum to any of the four vertex momenta can be done in four different ways. But, then, for the other external momentum only three choices are left. The vertex which is connected to the two external points, has then two remaining momenta. For the first choice to contract one momentum of that vertex with any of the four momenta of the other vertex, are four possibilities, for the second choice three. For the last choice no more freedom is left. So, we obtain as a result that this Feynman graph represents

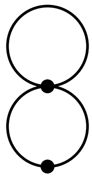
$$2 \times 4 \times 3 \times 4 \times 3 \times 1 = 288 \tag{2.26}$$

different, though analytically the same, contributions.

Besides the external propagators, which are already taken care of in expression (2.21), there are three more propagators. Two propagators in the lower loop, for which we select loop momentum q_1, and one propagator in the upper loop, for which we select loop momentum q_2. We find then the following contribution to the *something* of formula (2.21) for this case

$$288 \left[\int \frac{d^4 q_1}{(2\pi)^4} \left(\frac{i}{q_1^2 - m^2} \right)^2 \right] \left[\int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_2^2 - m^2} \right] \quad . \tag{2.27}$$

Since also for this contribution there are no external legs involved, its correct graphical representation is given by



The *something* of the second order, in λ, contribution (2.21) to the two-points Green's function, is just the sum of the three above determined expressions, (2.23), (2.25), and (2.27).

When, one wants to be sure that no contribution has been forgotten, then one may also take the vacuum bubble diagrams of section (2.1) into account. The reason is, that the total number of possible contractions can easily be determined. There are ten momenta flowing from two external points, which contribute each one momentum, and from two internal points, which contribute each four momenta. The first external momentum can be contracted with any of the other nine momenta, the second with any of the remaining momenta. One of the then six remaining momenta can be contracted in five different ways. One of the then four remaining momenta can be contracted in three different ways. For the last two momenta no more freedom exists. We find then

$$9 \times 7 \times 5 \times 3 \times 1 = 945 \tag{2.28}$$

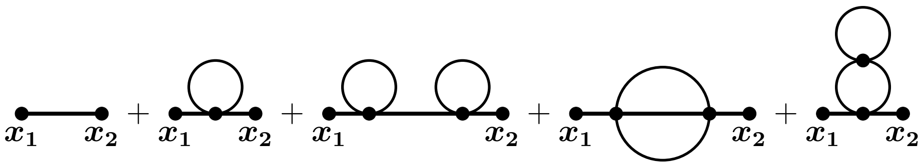
For the vacuum bubbles of section (2.1), in the order of appearance, one has the multiplicities 9, 72, 24, and 72 respectively. Summing those possible different ways of contracting the momenta, to the numbers of formulas (2.22), (2.24), and (2.26), one finds

$$9 + 72 + 24 + 72 + 288 + 192 + 288 = 945 \quad ,$$

which result agrees indeed with the total number given in formula (2.28).

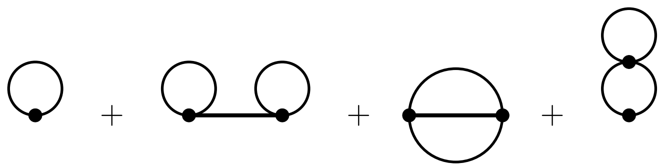
2.5 The amputed Green's function

For the two-points Green's function (2.1), using formulas (2.14), (2.21), and the second order contributions (2.23), (2.25), and (2.27), we obtain to second order in λ the result

$G(x_1, x_2) \; = $  $+ \cdots$

$$= \int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1x_1} \int \frac{d^4k_2}{(2\pi)^4}\, e^{-ik_2x_2}\, (2\pi)^4 \delta^{(4)}(k_1+k_2) \; \times \tag{2.29}$$
$$\times \left\{ \frac{i}{k_1^2-m^2} + \frac{i}{k_1^2-m^2}\, F\left(k_1, \lambda, m^2\right) \frac{i}{k_1^2-m^2} \right\} \quad ,$$

where $F(k_1, \lambda, m^2)$ is the so-called *amputed* two-points Green's function, given by

$F(k_1, \lambda, m^2) \; = $  $+ \cdots$

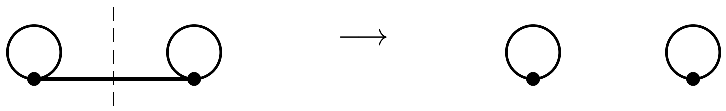
$$= -i\frac{\lambda}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{i}{q_1^2-m^2} \; + \tag{2.30}$$
$$-\frac{\lambda^2}{4} \left[\int \frac{d^4q_1}{(2\pi)^4} \frac{i}{q_1^2-m^2} \right] \frac{i}{k_1^2-m^2} \left[\int \frac{d^4q_2}{(2\pi)^4} \frac{i}{q_2^2-m^2} \right] +$$
$$-\frac{\lambda^2}{6} \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{i}{q_1^2-m^2} \frac{i}{q_2^2-m^2} \frac{i}{(k_1-q_1-q_2)^2-m^2} +$$
$$-\frac{\lambda^2}{4} \left[\int \frac{d^4q_1}{(2\pi)^4} \left(\frac{i}{q_1^2-m^2} \right)^2 \right] \left[\int \frac{d^4q_2}{(2\pi)^4} \frac{i}{q_2^2-m^2} \right] + \cdots \quad ,$$

Notice, that though the multiplicities for contributions of higher orders in λ are large, the factors for the powers of λ are moderate because of the factor 4! in the interaction Lagrangian (1.36).

2.6 1PI graphs and the self-energy

A further usefull reduction of the amount of graphs, which have to be calculated, is to consider only 1PI (one-particle-irreducible) graphs.

In order to define what is a one-particle-irreducible graph, we return to the amputed two-points Green's function, which, to second order in the coupling constant λ, is shown in formula (2.30). One of the four graphs of formula (2.30) differs from the other three graphs in the following sense: When we remove one internal line from the graph, then we obtain two disconnected graphs, *i.e.*

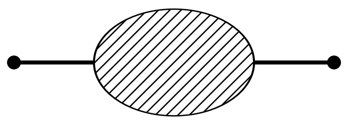


Such a graph is said to be *one-particle-reducible* and thus not 1PI. The other three graphs of formula (2.30) are 1PI.

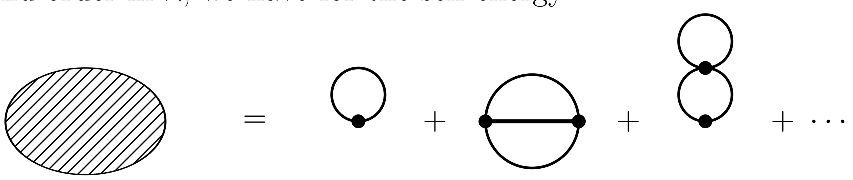
We define the sum of all 1PI contributions to the amputed two-points Green's function by

$$-i\,\Sigma\left(k_1, \lambda, m^2\right) \quad , \tag{2.31}$$

which is called the *self-energy* and which is depicted by



Up to the second order in λ, we have for the self-energy



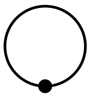
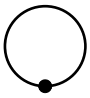
or in analytic form

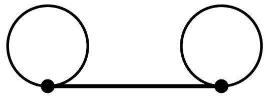
$$\begin{aligned}
\Sigma\left(k_1, \lambda, m^2\right) \;=\; & \frac{\lambda}{2}\int \frac{d^4q_1}{(2\pi)^4}\,\frac{i}{q_1^{\,2}-m^2} \;+ \qquad\qquad (2.32)\\
-i\frac{\lambda^2}{6}\int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4}\, & \frac{i}{q_1^{\,2}-m^2}\,\frac{i}{q_2^{\,2}-m^2}\,\frac{i}{(k_1-q_1-q_2)^2-m^2} \;+\\
-i\frac{\lambda^2}{4}\left[\int \frac{d^4q_1}{(2\pi)^4}\left(\frac{i}{q_1^{\,2}-m^2}\right)^2\right] & \left[\int \frac{d^4q_2}{(2\pi)^4}\,\frac{i}{q_2^{\,2}-m^2}\right] + \cdots \quad ,
\end{aligned}$$

Between the self-energy (2.31) and the amputed two-points Green's function (2.30) can be shown the following relation

$$F\left(k,\lambda,m^2\right) \;=\; \left[-i\Sigma\left(k,\lambda,m^2\right)\right] + \left[-i\Sigma\left(k,\lambda,m^2\right)\right]\,\frac{i}{k^2-m^2}\,\left[-i\Sigma\left(k,\lambda,m^2\right)\right] + \cdots \quad . \tag{2.33}$$

The second term contains for example

 $\dfrac{i}{k^2-m^2}$  $\;=\; \left[-i\frac{\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\right]\frac{i}{k^2-m^2}\left[-i\frac{\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\right]$

$=$  (2.34)

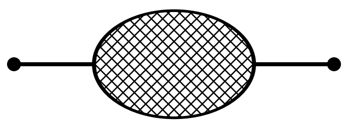
Similarly, any kind of one-particle-reducible contribution is automatically taken care of by the righthand side of (2.33).

2.7 Full propagator

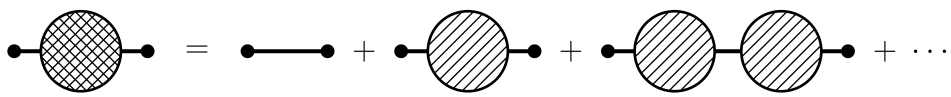
One might have noticed, for instance by inspection of formula (2.29), that for the free theory, for which $\lambda = 0$ and hence for which the interaction Lagrangian of (1.34) is absent, the two-points Green's function (2.1) equals the vacuum expectation value of the time-ordered product of two fields, given in formula (1.98). The central part of formula (1.98) is the *free propagator* S_F, which has been defined in equation (1.23). For the central part of the two-points Green's function for the complete theory, we define the *full propagator*, S'_F, *i.e.*

$$G\left(x, x'\right) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} \int \frac{d^4k'}{(2\pi)^4}\, e^{-ik'x'}\, (2\pi)^4 \delta^{(4)}\left(k + k'\right)\, S'_F\left(k, \lambda, m^2\right) \quad . \tag{2.35}$$

The full propagator is graphically represented by



From formulas (1.23), (2.30), (2.33), and (2.35) one reads off the following relation between the full propagator, the free propagator and the self-energy, given by



or using their symbolic notation, by

$$S'_F = S_F + S_F\,(-i\Sigma)\,S_F + S_F\,(-i\Sigma)\,S_F\,(-i\Sigma)\,S_F + \cdots \quad , \tag{2.36}$$

which series can be formally summed and written in a compact form

$$S'_F\left(k, \lambda, m^2\right) = \frac{i}{k^2 - m^2 - \Sigma\left(k, \lambda, m^2\right)} \quad . \tag{2.37}$$

2.8 Divergencies

In the foregoing, we have obtained a beautiful analytic expression for the two-points Green's function in formulas (2.29) and (2.30). However, we still have not come very far, since the loop integrals are divergent and hence the whole expression does not exist. But, would we discuss a theory which does not lead to any sensible result? Of course not!

There exist several regularization methods to get rid of the infinities, which accompany almost any quantum field theory (see, for example Bjorken and Drell, chapters 8 and 19, or Itzykson and Zuber, chapter 8). Here, we will study an elegant procedure, which is developed by G. 't Hooft and M. Veltman. Although we need then the concept of non-integer dimensions, this is no problem since all relevant integrals in arbitrary dimensions are tabulated. In appendix B of *Diagrammar*, or appendix A of their Nuclear Physics article, G. 't Hooft and M. Veltman give (see also section 2.8.1)

$$\int d^n p \, \frac{1}{p^2+m^2} \;=\; \frac{i\pi^{\frac{1}{2}n}}{\left(m^2\right)^{1-\frac{1}{2}n}}\, \Gamma\left(1-\frac{1}{2}n\right) \quad . \tag{2.38}$$

However, their metric differs from ours, *i.e.*

$$p^2 \;=\; \begin{cases} -E^2+\vec{p}^{\,2} & \text{(G. 't Hooft and M. Veltman)} \\ +E^2-\vec{p}^{\,2} & \text{(our definition.)} \end{cases} \tag{2.39}$$

Consequently, in our definition of the metric we obtain for formula (2.38) the following result

$$\int d^n q \, \frac{i}{q^2-m^2} \;=\; \frac{\pi^{\frac{1}{2}n}}{\left(m^2\right)^{1-\frac{1}{2}n}}\, \Gamma\left(1-\frac{1}{2}n\right) \quad . \tag{2.40}$$

Γ represents the gamma function, which is an extension to complex values of the factorial function and which has the following property

$$\Gamma(\alpha+1) \;=\; \alpha\Gamma(\alpha) \quad . \tag{2.41}$$

Notice that as a consequence of this property, $\Gamma(0)$, $\Gamma(-1)$, $\Gamma(-2)$, $\Gamma(-3)$, ..., do not exist (are infinite) and hence for 4-dimensional physics (*i.e.* $n=4$), formula (2.40) diverges.

The divergent integral which contributes to the first order in λ term of the self-energy (2.32) is exactly given by formula (2.40) for $n=4$. The limit $n=4$ does not exist, because $\Gamma(-1)$ does not exist. However, we define $\epsilon = n-4$, in order to obtain

$$\int d^4 q \, \frac{i}{q^2-m^2} \;=\; \lim_{\epsilon\to 0} \frac{\pi^{2+\frac{1}{2}\epsilon}}{\left(m^2\right)^{-1-\frac{1}{2}\epsilon}}\, \Gamma\left(-1-\frac{1}{2}\epsilon\right) \quad . \tag{2.42}$$

The gamma function we might handle by using the property (2.41), which leads to

$$\Gamma\left(-1-\frac{1}{2}\epsilon\right) \;=\; \frac{\Gamma\left(-\frac{1}{2}\epsilon\right)}{-1-\frac{1}{2}\epsilon} \quad ,$$

and moreover the Laurent series expansion (see for example M.Abramowitz and I. Stegun, Handbook of Mathematics, formula 6.1.35) for the gamma function in the neighborhood of zero, *i.e.*

$$\Gamma(\alpha) = \frac{1}{\alpha} + \text{finite part} \quad ,$$

where the *finite part* does not contain infinities in the limit $\alpha \to 0$.

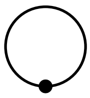
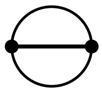
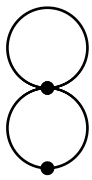
So, when we forget about the limit ($\epsilon \to 0$), but keep it in mind, then we find for expression (2.42), the result

$$\int d^4q \, \frac{i}{q^2 - m^2} = \pi^2 m^2 \left\{ \frac{2}{\epsilon} + \text{finite part} \right\} \quad . \tag{2.43}$$

Now, the expression (2.32) for the self-energy, which is an expansion in the coupling constant λ, has the following structure

$$\Sigma = \lambda \Sigma_1 + \lambda^2 \Sigma_2 + \cdots \tag{2.44}$$

where

$\lambda\Sigma_1 =$  (one loop) and $\lambda^2\Sigma_2 =$  $+$  (two loops)

Here, we are studying the regularization procedure up to one loop term(s), for which it suffices to consider from the above expansion (2.44) only

$$\begin{aligned} \Sigma &= \lambda \Sigma_1 + \cdots \\ &= \frac{\lambda}{2} \int \frac{d^4q}{(2\pi)^4} \, \frac{i}{q^2 - m^2} + \cdots \\ &= \lambda \, m^2 \left\{ \left(\frac{1}{16\pi^2} \right) \frac{1}{\epsilon} + \text{finite part} \right\} \end{aligned} \tag{2.45}$$

2.8.1 Integration in n dimensions

In an n dimensional Euclidean space we assume Cartesian coordinates $\left\{ x_i^{(n)} \; , \; i = 1, \ldots, n \right\}$ and define spherical coordinates through

$$\begin{aligned} x_1^{(n)} &= r \, \sin(\vartheta_{n-1}) \cdots \sin(\vartheta_4) \, \sin(\vartheta_3) \, \sin(\vartheta_2) \, \sin(\vartheta_1) \\ x_2^{(n)} &= r \, \sin(\vartheta_{n-1}) \cdots \sin(\vartheta_4) \, \sin(\vartheta_3) \, \sin(\vartheta_2) \, \cos(\vartheta_1) \\ x_3^{(n)} &= r \, \sin(\vartheta_{n-1}) \cdots \sin(\vartheta_4) \, \sin(\vartheta_3) \, \cos(\vartheta_2) \\ x_4^{(n)} &= r \, \sin(\vartheta_{n-1}) \cdots \sin(\vartheta_4) \, \cos(\vartheta_3) \\ &\vdots \\ x_n^{(n)} &= r \, \cos(\vartheta_{n-1}) \quad , \end{aligned} \tag{2.46}$$

where

$$0 \leq \vartheta_2 \; , \; \ldots \; , \; \vartheta_{n-1} \leq \pi \quad \text{and} \quad 0 \leq \vartheta_1 \leq 2\pi \quad . \tag{2.47}$$

The volume elements for Cartesian and spherical coordinates are related by

$$d^n x \; = \; r^{n-1} dr \; \sin^{n-2}\left(\vartheta_{n-1}\right) d\vartheta_{n-1} \; \sin^{n-3}\left(\vartheta_{n-2}\right) d\vartheta_{n-2} \; \cdots \; d\vartheta_1 \quad . \tag{2.48}$$

Relation (2.48) can be proven by means of induction. For $n = 1$ one has $x_1 = r$ and $dx = dr$, for $n = 2$ one has, employing the usual cylindrical coordinates r and φ, that $d^2x = rdr\; d\varphi$, whereas moreover, for $n = 3$, with the ordinary three-dimensional spherical coordinates r, ϑ and φ, one has $d^3x = r^2 dr\; \sin(\vartheta)\, d\vartheta\; d\varphi$, all in agreement with formula (2.48).

For $(n+1)$ dimensions we define the spherical coordinates through

$$\begin{aligned} x_i^{(n+1)} &= x_i^{(n)} \; \sin\left(\vartheta_n\right) \quad \text{for} \quad i \neq n+1 \\ x_{n+1}^{(n+1)} &= r \; \cos\left(\vartheta_n\right) \quad . \end{aligned} \tag{2.49}$$

Hence, when we define the Jacobian in n dimensions by

$$J(n) \; = \; \left| \begin{array}{cccc} \dfrac{\partial x_1^{(n)}}{\partial r} & \dfrac{\partial x_1^{(n)}}{\partial \vartheta_1} & \cdots & \dfrac{\partial x_1^{(n)}}{\partial \vartheta_{n-1}} \\ \dfrac{\partial x_2^{(n)}}{\partial r} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial x_n^{(n)}}{\partial r} & \dfrac{\partial x_n^{(n)}}{\partial \vartheta_1} & \cdots & \dfrac{\partial x_n^{(n)}}{\partial \vartheta_{n-1}} \end{array} \right| \quad , \tag{2.50}$$

then we obtain in $(n+1)$ dimensions for the Jacobian the form

$$J(n+1) \; = \; \left| \begin{array}{cccc} \multicolumn{3}{c}{\left| \; \sin^n\left(\vartheta_n\right) \; J(n) \; \right|} & \begin{array}{c} x_1^{(n)} \; \mathrm{cotg}\left(\vartheta_n\right) \\ x_2^{(n)} \; \mathrm{cotg}\left(\vartheta_n\right) \\ \vdots \\ x_n^{(n)} \; \mathrm{cotg}\left(\vartheta_n\right) \end{array} \\ \cos\left(\vartheta_n\right) \quad 0 & \cdots & 0 & -r \; \sin\left(\vartheta_n\right) \end{array} \right| \quad . \tag{2.51}$$

The first n elements of the last column of this determinant are equal to the first n elements of the first column, multiplied by a factor

$$r \; \frac{\cos\left(\vartheta_n\right)}{\sin^2\left(\vartheta_n\right)} \quad .$$

Consequently one has the relation

$$\begin{aligned} J(n+1) &= \left\{ r \; \mathrm{cotg}\left(\vartheta_n\right) \; \cos\left(\vartheta_n\right) \; + \; r \; \sin\left(\vartheta_n\right) \right\} \; \sin^n\left(\vartheta_n\right) \; J(n) \\ &= r \; \sin^{n-1}\left(\vartheta_n\right) \; J(n) \quad . \end{aligned} \tag{2.52}$$

The above result (2.52) combined with formula (2.48), gives then

$$\begin{aligned} d^{n+1}x &= J(n+1)\ dr\ d\vartheta_n\ \cdots\ d\vartheta_1 \\ &= r\ \sin^{n-1}(\vartheta_n)\ J(n)\ dr\ d\vartheta_n\ \cdots\ d\vartheta_1 \\ &= r^n dr\ \sin^{n-1}(\vartheta_n)\, d\vartheta_n\ \sin^{n-2}(\vartheta_{n-1})\, d\vartheta_{n-1}\ \cdots\ d\vartheta_1 \quad , \end{aligned} \tag{2.53}$$

which proofs formula (2.48).

For integrands which are functions of x^2 only, one may perform the integrals over the angles independently. So, let us next concentrate on the following integral

$$I_m = \int_0^\pi d\vartheta\ \sin^m(\vartheta) \quad . \tag{2.54}$$

Integration by parts gives

$$I_m = \frac{m-1}{m}\, I_{m-2} \quad ,$$

which can be iterated. For even values of m one finds

$$I_{2k} = \frac{2k-1}{2k}\, I_{2k-2} = \cdots = \frac{(2k-1)(2k-3)\cdots 1}{2k(2k-2)\cdots 2}\, I_0 = \frac{\Gamma\left(k+\frac{1}{2}\right)/\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)}\, \pi \quad .$$

Whereas, for odd values of m one finds

$$I_{2k+1} = \frac{2k}{2k-1}\, I_{2k-1} = \cdots = \frac{2k(2k-2)\cdots 2}{(2k+1)(2k-1)\cdots 3}\, I_1 = \frac{\Gamma(k+1)}{\Gamma\left(k+\frac{3}{2}\right)/\Gamma\left(\frac{3}{2}\right)}\, 2 \quad .$$

Both cases can be written in the same way, by

$$I_m = \sqrt{\pi}\, \frac{\Gamma\left(\frac{1}{2}(m+1)\right)}{\Gamma\left(\frac{1}{2}m+1\right)} \quad . \tag{2.55}$$

With expression (2.55) we are ready for the integrals over the angles. One obtains

$$\int_0^\pi \sin^{n-2}(\vartheta_{n-1})\, d\vartheta_{n-1} \int_0^\pi \sin^{n-3}(\vartheta_{n-2})\, d\vartheta_{n-2} \cdots \int_0^\pi \sin(\vartheta_2)\, d\vartheta_2 \int_0^{2\pi} d\vartheta_1 =$$

$$= \sqrt{\pi}\, \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)}\, \sqrt{\pi}\, \frac{\Gamma\left(\frac{1}{2}(n-2)\right)}{\Gamma\left(\frac{1}{2}(n-1)\right)} \cdots \sqrt{\pi}\, \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}\, 2\pi = 2\, \frac{\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{1}{2}n\right)} \quad . \tag{2.56}$$

For the radial part of integration (2.38), we consider Euler's integral of the first kind, or Beta function, defined by

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y) = \int_0^\infty dt\, \frac{t^{x-1}}{(t+1)^{x+y}} \quad . \tag{2.57}$$

Hence,

$$\begin{aligned}\int d^n r \, \frac{1}{r^2+1} \;&=\; \int_0^\infty r^{n-1} dr \, \frac{1}{r^2+1} \int \text{angles} \;=\; \tfrac{1}{2} \int_0^\infty dt \, \frac{t^{\frac{1}{2}n-1}}{t+1} \int \text{angles} \\ &=\; B\left(\tfrac{1}{2}n \, , \; 1-\tfrac{1}{2}n\right) \int \text{angles} \;=\; \pi^{\frac{1}{2}n} \, \Gamma\left(1-\tfrac{1}{2}n\right) \quad . \end{aligned} \tag{2.58}$$

In formula (2.38) p^2 is defined by

$$p^2 \;=\; -\,{p_0}^2 \;+\; \vec{p}^{\,2} \quad . \tag{2.59}$$

In order to obtain an Euclidean integral, we introduce

$$p_0 \;=\; i\,p_n \quad , \tag{2.60}$$

which explains the i in formula (2.38).

2.9 Counterterms

Suppose that, instead of the Lagrangian density (1.34), we had chosen for a theory with the Lagrangian density given by

$$\mathcal{L}\left(\phi, \partial_\mu \phi\right) = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}Bm^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad , \tag{2.61}$$

where B is called a *counterterm*. Then, in the theory (2.61), we would obtain for the free propagator not the Feynman propagator (1.23), but instead

$$S_F(p) = \frac{i}{p^2 - Bm^2} \quad , \tag{2.62}$$

and, consequently, up to first order terms in λ for the self-energy, not expression (2.45), but instead

$$\Sigma = \lambda\, Bm^2 \left\{\left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon} + \text{finite part}\right\} \tag{2.63}$$

For the full propagator S'_F we would have found, not expression (2.37), but, to first order in λ, instead

$$S'_F = \frac{i}{k^2 - Bm^2 - \lambda\, Bm^2 \left\{\left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon} + \text{finite part}\right\}} \quad . \tag{2.64}$$

The counterterm B of the Lagrangian density (2.61) is supposed to also be expandable in a series of increasing orders of λ, like the self-energy (2.44), *i.e.*

$$B = 1 + \lambda b_1 + \lambda^2 b_2 + \cdots \quad . \tag{2.65}$$

Now, since we are only interested in terms up to order λ, we have for the second term in the denominator of the righthand side of formula (2.64)

$$Bm^2 = m^2 + \lambda b_1 m^2 + \cdots \quad ,$$

and for the third term

$$\lambda\, Bm^2 \left\{\left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon} + \text{finite part}\right\} = \lambda m^2 \left\{\left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon} + \text{finite part}\right\} \quad .$$

So, the full propagator up to first order in λ yields

$$S'_F = \frac{i}{k^2 - m^2 - \lambda\, m^2 \left\{b_1 + \left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon} + \text{finite part}\right\}} \quad . \tag{2.66}$$

The obvious choice

$$b_1 = -\left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon} \quad , \tag{2.67}$$

leads to a finite expression for the full propagator

$$S'_F = \frac{i}{k^2 - m^2 - \lambda\, m^2\ \{\text{finite part}\}} \quad . \tag{2.68}$$

This procedure is essentially the regularization method. One modifies the Lagrangian density, order by order in λ, in such a way that the the Green's functions become finite. The divergencies are absorbed in the definition of the counterterms. Lagrangian densities for which this procedure works, are called *renormalizable*. From electromagnetism, where one has the famous calculations for the anomalous magnetic moment of the electron and the Lamb shift of the Hydrogen atom, for which the regularization procedure gives very accurate agreement of the predictions with experiment, we know that the method works!

2.10 Subtraction contributions

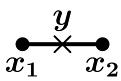
The redefinition (2.61) of the Lagrangian density of ϕ^4 theory might also be seen as a redefinition of the interaction Lagrangian density, *i.e.*

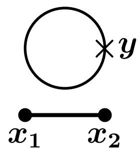
$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4 + \lambda\left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon}m^2\phi^2 + \cdots \quad , \tag{2.69}$$

leading to new contributions, *subtraction contributions*, to the two-points Green's function of the form

$$\langle 0\,|T\,\{\phi\,(x_1)\,\phi\,(x_2)\,\phi^2(y)\}|\,0\rangle \quad , \quad \text{etc.} \quad . \tag{2.70}$$

There are two different contractions possible for the vacuum expectation value shown in formula (2.70), which can be depicted by

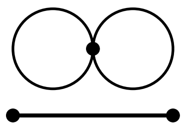


and


. (2.71)

Notice that in the point y only two fields are involved. For that reason we marked the corresponding vertex with a $\times$ to distinguish it from the four-point vertex.

The second contribution of formula (2.71) cancels the infinity in the vacuum bubble contribution (1.105), *i.e.*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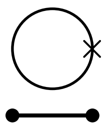
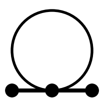

= finite , (2.72)

However, this contribution did anyhow not appear in the expression (2.11).

The first contribution of formula (2.71) cancels the infinity in the one loop contribution (1.109) to the two-points Green's function, *i.e.*


+

= finite . (2.73)

The infinities are cured by contributions which come from the new interaction Lagrangian (2.69).

Chapter 3

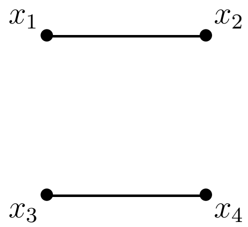
Four-points Green's function

Let us, for a moment, return to the unregularized Lagrangian (1.34) and determine the four points Green's function, which, by virtue of equation (1.37), is given by

$$
\begin{aligned}
& G\left(x_1, x_2, x_3, x_4\right) \left\langle 0 \left| T \left\{ \exp \left[i \int d^4 y \, \left(-\frac{\lambda}{4!} \right) \phi^4(y) \right] \left(\phi(y) \right) \right\} \right| 0 \right\rangle \; = \\
= \; & \left\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right\} \right| 0 \right\rangle \; + \\
& + \left\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \; i \int d^4 y \, \left(-\frac{\lambda}{4!} \right) \phi^4(y) \right\} \right| 0 \right\rangle \; + \\
& + \left\langle 0 \left| T \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \; \frac{1}{2} \left[i \int d^4 y \, \left(-\frac{\lambda}{4!} \right) \phi^4(y) \right]^2 \right\} \right| 0 \right\rangle \; + \; \cdots
\end{aligned}
\tag{3.1}
$$

As for the two points Green's function, it is easy to demonstrate that also in this case the vacuum bubble graphs do not contribute to the four points Green's function. In the following, we will not determine order by order each of the terms of the perturbation series expansion of formula (3.1). But, rather select classes of series of contributions to the four points Green's function, such that for each class it is perfectly transparant of how to proceed in order to calculate the various terms in those series. This facilitates the bookkeeping and in particular makes it more obvious to understand how the regularization procedure works. We begin by the zeroeth order terms of the series expansion (3.1) and study the classes of graphs which are associated to them. Then, we select a particular term from the first order contributions, the *vertex* contribution, and show how the remaining graphs, which together with the vertex constitute the *vertex function*, are associated to that expression and, in particular, how the regularization method works for that vertex function.

The zeroeth order term in the coupling constant λ is just the vacuum expectation value for the time ordered product of four fields, which has been studied in section (1.7) and explicitly given by the sum of formulas (1.91), (1.92), and (1.93) and represented by the corresponding graphs. Let us here concentrate on one of the three contributions, for example the one given by formula (1.91):



$$= \; i^2 \int \frac{d^4k_1}{(2\pi)^4} \, \frac{e^{-ik_1\,(x_1-x_2)}}{\left(k_1\right)^2 - m^2} \; \int \frac{d^4k_3}{(2\pi)^4} \, \frac{e^{-ik_3\,(x_3-x_4)}}{\left(k_3\right)^2 - m^2} \quad . \tag{3.2}$$

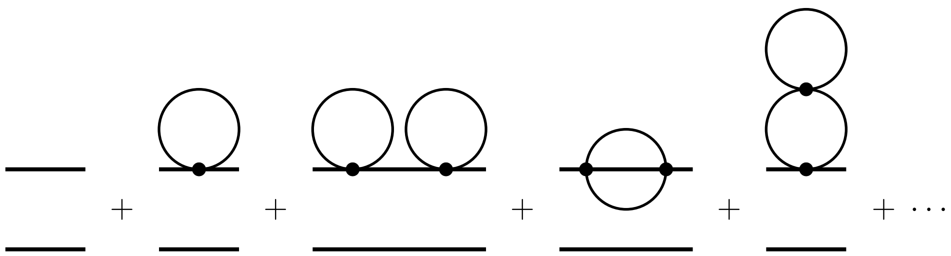
The expression for the above contribution (3.2) can also be casted in the general form, *i.e.* in the form which follows from the Feynman rules, according to

$$\left[\prod_{\ell=1}^{4} \int \frac{d^4k_\ell}{(2\pi)^4} \, e^{-ik_\ell x_\ell}\right] (2\pi)^4 \, \delta^{(4)}\left(\sum_{\ell=1}^{4} k_\ell\right) \; (2\pi)^4 \, \delta^{(4)}\left(k_1+k_2\right) \, \frac{i}{\left(k_1\right)^2 - m^2} \, \frac{i}{\left(k_3\right)^2 - m^2} \quad . \tag{3.3}$$

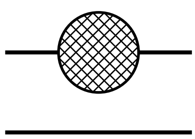
The central part of contribution (1.91) is hence given by

$$(2\pi)^4 \, \delta^{(4)}\left(k_1+k_2\right) \, \frac{i}{\left(k_1\right)^2 - m^2} \, \frac{i}{\left(k_3\right)^2 - m^2} \quad . \tag{3.4}$$

Next to the contribution (3.2), there exists a class of contributions to the four points Green's function (3.1) which can easily be summed up, *i.e.* the following series

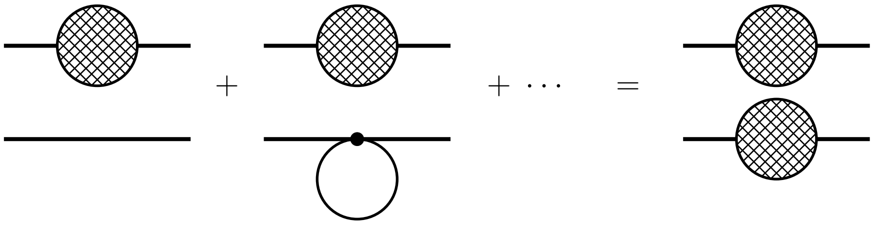


Using formula (2.35), in which the full propagator, S'_F, is defined, we can write the full sum of the above depicted contributions by



$$= \; (2\pi)^4 \, \delta^{(4)}\left(k_1+k_2\right) \; S'_F\left(k_1, \lambda, m^2\right) \, \frac{i}{\left(k_3\right)^2 - m^2} \quad . \tag{3.5}$$

One can extend this class of contributions by considering also all higher order in λ contributions to the lower propagator, *i.e.*



which represent all possible contributions one may think of for which the general structure is equivalent to (1.91), *i.e.* momentum flows from x_1 to x_2 and also from x_3 to x_4, but there are no lines which connect the upper full propagator to the lower full propagator. In formule, the central part of this total class of contributions is given by

$$(2\pi)^4\, \delta^{(4)}\left(k_1+k_2\right)\ S'_F\left(k_1,\lambda,m^2\right)\ S'_F\left(k_3,\lambda,m^2\right) \quad . \tag{3.6}$$

Following the same procedure for the contributions (1.92) and (1.93), we obtain for the four points Green's function the expression

$$\begin{aligned} G\left(x_1,x_2,x_3,x_4\right) \;=\; & \left[\prod_{\ell=1}^{4}\int\frac{d^4k_\ell}{(2\pi)^4}\,e^{-ik_\ell x_\ell}\right](2\pi)^4\,\delta^{(4)}\left(\sum_{\ell=1}^{4}k_\ell\right) \\ & \times\Big\{(2\pi)^4\,\delta^{(4)}\left(k_1+k_2\right)\ S'_F\left(k_1,\lambda,m^2\right)\ S'_F\left(k_3,\lambda,m^2\right)\ + \\ & +\ (2\pi)^4\,\delta^{(4)}\left(k_1+k_3\right)\ S'_F\left(k_1,\lambda,m^2\right)\ S'_F\left(k_2,\lambda,m^2\right)\ + \\ & +\ (2\pi)^4\,\delta^{(4)}\left(k_1+k_4\right)\ S'_F\left(k_1,\lambda,m^2\right)\ S'_F\left(k_3,\lambda,m^2\right)\ + \\ & +\ \text{other higher order terms in }\lambda\ \Big\} \quad . \end{aligned} \tag{3.7}$$

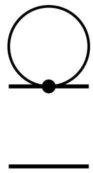
It might be clear that, once the full propagator, S'_F, is regularized, then the first three terms of the righthand side of the expansion (3.7), which actually already represent an infinity of contributions of three special types, are automatically also regularized. So, let us concentrate on the other higher order in λ contributions.

3.1 The vertex

The first order in λ term from the perturbation expansion (3.1) reads

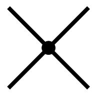
$$i\int d^4y\,\left(-\frac{\lambda}{4!}\right)\ \left\langle 0\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\ \phi^4(y)\right\}\right|0\right\rangle \quad . \tag{3.8}$$

Here, we do not have to consider the vacuum bubble contributions, which are contractions of the external momenta, k, which originate from the external events, x, amongst themselves and of the internal momenta, q, which originate at the internal event, y, amongst themselves. The contribution which is graphically represented by



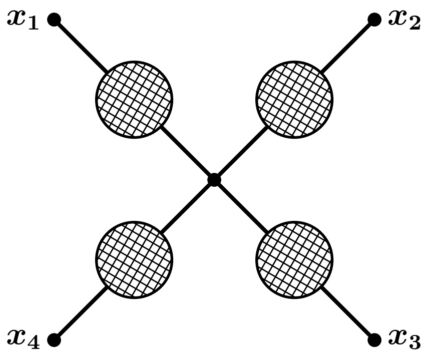
is already contained in the series (3.6) and hence does not have to be considered again.

The other one-particle irreducible graph occurs by the contraction of the four internal momenta, q, which originate from the internal event, y, each with one of the external momenta, *i.e.*



This is a new type of contribution and is called the *vertex*. Its combinatorial factor equals 4!, which compensates exactly the factor 1/4! from the interaction Lagrangian (1.36).

One can associate a whole class of contributions for (3.1), by substitution of the four external propagators by full propagators. This class of contributions is graphically represented by the following figure.



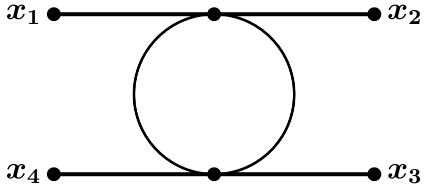
Apart from the factors 4π and the delta function, one obtains for this series the expression

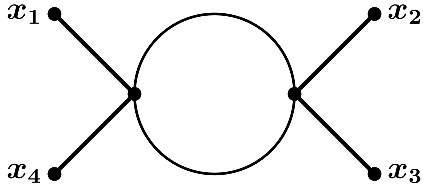
$$4! \left(-i\frac{\lambda}{4!}\right) \; {S'}_F\left(k_1, \lambda, m^2\right) \; {S'}_F\left(k_2, \lambda, m^2\right) \; {S'}_F\left(k_3, \lambda, m^2\right) \; {S'}_F\left(k_4, \lambda, m^2\right) \quad . \tag{3.9}$$

Once ${S'}_F$ is regulated, this whole class of contributions is consequently also regulated.

3.2 The second order terms

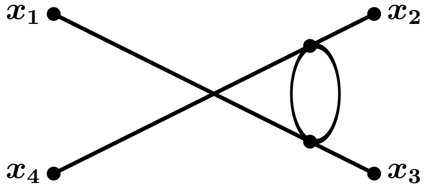
Besides contributions, which are already contained in the sums of contributions (3.6) or (3.9), one has the following one-particle irreducible graphs





,

and



(3.10)

Let us concentrate on the first of the three contributions to (3.1). The combinatorics learns us that there are $(4!)^2$ ways to make such contractions that the final result has the same analytic appearance. This factor just cancels the same, but inverted, factor from the interaction Lagrangian. So, apart from the factors 4π and the delta function, one obtains for this contribution the expression

$$(4!)^2 \frac{1}{2!} \left[\prod_{\ell=1}^{4} \frac{i}{k_\ell^2 - m^2} \right] \left(-i \frac{\lambda}{4!} \right)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2} \frac{i}{(k_1 + k_2 - q)^2 - m^2} \quad . \tag{3.11}$$

To this contribution we can also associate a whole class of contributions, by just substituting the four external propagators by full propagators. The total sum of this class of contributions gives the result

$$\left[\prod_{\ell=1}^{4} S'_F \left(k_\ell, \lambda, m^2 \right) \right] \frac{1}{2} (-i\lambda)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2} \frac{i}{(k_1 + k_2 - q)^2 - m^2} \quad . \tag{3.12}$$

The expression (3.12) could be seen as one of the first order in λ correction to the sum of contributions of formula (3.9), because for each term of the sum of contributions to the four points Green's function, which is represented by formula (3.9), a corresponding term with one more internal vertex is contained in formula (3.12). The other two first order in λ corrections to the sum of contributions of formula (3.9) are obtained, similarly, by substitution of the external propagators of the other two graphs of formula (3.10).

3.3 The amputed vertex function

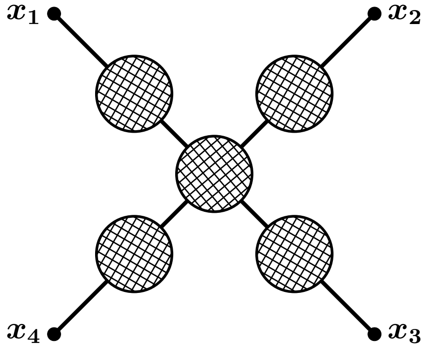
When we sum up the class extensions of the vertex, formula (3.9), the class extension of the first first order in λ correction to the vertex, formula (3.12), and the class extension of the other two graphs in formula (3.10), then we obtain the vertex function, given by

$$\left[\prod_{\ell=1}^{4} S'_F \left(k_\ell, \lambda, m^2 \right) \right] \left\{ (-i\lambda) \ + \ \frac{1}{2} (-i\lambda)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2} \frac{i}{(k_1 + k_2 - q)^2 - m^2} + \right.$$

$$+\frac{1}{2}(-i\lambda)^2\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\,\frac{i}{(k_1+k_3-q)^2-m^2}\;+$$

$$+\frac{1}{2}(-i\lambda)^2\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\,\frac{i}{(k_1+k_4-q)^2-m^2}\;+\;\cdots\Bigg\}\quad, \tag{3.13}$$

which is graphically represented by the figure below



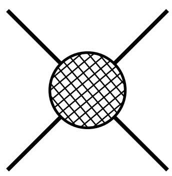
The central expression which we will consider here, is called the *amputed vertex function*, which is given by

$$\Lambda\left(k_1,k_2,k_3,k_4,\lambda,m^2\right)\;= \tag{3.14}$$

$$=\;(-i\lambda)\;+\;\frac{1}{2}(-i\lambda)^2\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2}\;\times$$

$$\times\;\left\{\frac{i}{(k_1+k_2-q)^2-m^2}\;+\;\frac{i}{(k_1+k_3-q)^2-m^2}\;+\;\frac{i}{(k_1+k_4-q)^2-m^2}\;+\;\cdots\right\}\quad,$$

and which is graphically represented by



Any contribution to the four-points Green's function belongs either to the first three terms of expression (3.7), which represent two disconnected full propagators, or to the class of contributions (3.13).

3.4 Regularization of the vertex function

The expression (3.14) for the amputed vertex function contains infinities, since

$$\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{(q^2-m^2)\left((k-q)^2-m^2\right)} \;=\; -i\left\{\left(\frac{1}{16\pi^2}\right)\frac{1}{\epsilon}\;+\;\text{finite parts}\right\}\quad . \tag{3.15}$$

And, since moreover, formula (3.14) contains three such integrals, we obtain, using the above equation (3.15), for the amputed vertex function the expression

$$\Lambda\left(k_1,k_2,k_3,k_4,\lambda,m^2\right) \;=\; -i\lambda - i\lambda^2\left\{\left(\frac{3}{16\pi^2}\right)\frac{1}{\epsilon}\;+\;\text{finite parts}\right\}\quad . \tag{3.16}$$

The regularization procedure works in a similar way as for the two-points Green's function, in which case we defined a counterterm B to redefine the Lagrangian density (formula (2.61)), such that the new Lagrangian density gives finite results for the full propagator (formula (2.68)). Here we define the counterterm A, for which we assume the expansion, given by

$$A \;=\; 1\;+\;a_1\lambda\;+\;a_2\lambda^2\;+\;\cdots\quad , \tag{3.17}$$

and redefine the Lagrangian density (2.61) to also include this counterterm, *i.e.*

$$\mathcal{L}\left(\phi,\partial_\mu\,\phi\right) \;=\; \frac{1}{2}\left(\partial_\mu\,\phi\right)^2 \;-\; \frac{1}{2}Bm^2\phi^2 \;-\; A\frac{\lambda}{4!}\phi^4\quad . \tag{3.18}$$

The amputed vertex function, for which we have the divergent expression (3.16), takes for the Lagrangian density (3.18) the form

$$\begin{aligned}\Lambda \;&=\; -iA\lambda - i(A\lambda)^2\left\{\left(\frac{3}{16\pi^2}\right)\frac{1}{\epsilon}\;+\;\text{finite parts}\right\}\\ &=\; -i\lambda\left[A\;+\;A^2\lambda\left\{\left(\frac{3}{16\pi^2}\right)\frac{1}{\epsilon}\;+\;\text{finite parts}\right\}\right]\quad .\end{aligned} \tag{3.19}$$

When the expression inside the square brackets of formula (3.19) is expanded to first order in λ, then we obtain

$$\Lambda \;=\; -i\lambda\left[1\;+\;a_1\lambda\;+\;\lambda\left\{\left(\frac{3}{16\pi^2}\right)\frac{1}{\epsilon}\;+\;\text{finite parts}\right\}\right]\quad , \tag{3.20}$$

which upon the obvious choice

$$a_1 \;=\; -\left(\frac{3}{16\pi^2}\right)\frac{1}{\epsilon}$$

regulates the amputed vertex function to the "first" order perturbation in λ

$$\Lambda \;=\; -i\lambda\left[1\;+\;\lambda\left\{\text{finite parts}\right\}\right]\quad , \tag{3.21}$$

Notice that the vertex function, which is intimately related to the interaction part of the Lagrangian density, is proportional with the coupling constant λ. The first order in λ corrections yield thus terms quadratic in λ.

Chapter 4

Molding time evolution into a path integral

The concept of the path integral is rather easy to percieve. But, the precise mathematical formulation of it, is a very complicated matter. Here, we will present a fast introduction, with just the minimum of complexity, avoiding most of the mathematical rigor.

The basic idea is that the time evolution of a system can be determined by inspecting all possible ways in which the system can evolve. Each possibility is called a *path*, like in classical mechanics. To each path is associated a probability, or more accurately, an amplitude. The central expression will turn out to be a sum, or integral, over all possible paths, of the amplitudes. In quantum mechanics this *path integral* has the following appearance:

$$\int_{x(t_a)}^{x(t_b)} [dx(t)] \; e^{i\, S_{cl}(x(t))/\hbar} \quad . \tag{4.1}$$

It assumes that the system evolves from a certain state (position in classical mechanics) $x(t_a)$ at the initial time t_a to a state $x(t_b)$ at a later time t_b. The symbol $\int [dx(t)]$ indicates that the path integral is a sum over all possible paths, whereas the integration limits, $x(t_a)$ and $x(t_b)$, indicate that only those paths contribute to the integration, which at initial time t_a are in the state $x(t_a)$ and at the later time t_b in the state $x(t_b)$. The *amplitude* for each path is in expression (4.1) given by the exponent of the classical action $S_{cl}(x(t))$ for the path multiplied by $i/\hbar$. Consequently, the amplitude is just a phase factor.

Adding phase factors is a subject which is studied in optics, where we learned about constructive and destructive interference. Here, let us study expression (4.1) for subsets of possible paths which differ very little from each other. When for such a subset of nearby paths the phases, which are measured in units $\hbar$ go through large changes, then the interference is destructive. Consequently, such subset will not contribute much (basically nothing) to the path integral. However, near an extremum of $S_{cl}(x(t))$, the phases of subsets of paths will change very little. In that case one finds constructive interference. We may thus conclude that the main contribution to the path integral (4.1) stems from that subset of all possible paths which is near an extremum of the classical action.

The main issue, however, is not the calculation of the path integral itself, but merely the measurable quantities which can be extracted from it. In the following we will study how path integrals may be constructed.

4.0.1 Time evolution in Quantum Mechanics

We assume that the dynamics of the system is described by a time-independent Hamiltonian H, for which one may integrate the wave equation

$$i\,\frac{\partial}{\partial t}|\psi(t)\rangle \;=\; H|\psi(t)\rangle \quad , \tag{4.2}$$

to yield

$$|\psi(t)\rangle \;=\; e^{-iHt}|\psi(0)\rangle \qquad (t>0) \quad . \tag{4.3}$$

We also assume that the Hamiltonian H can be expressed in terms of the coordinate operator q and the momentum operator p. The eigenstates of q are denoted $|x(t)\rangle$ and of p by $|k(t)\rangle$. Wavefunctions are denoted by

$$\psi(x,t) \;=\; \langle x(t)\,|\psi(t)\rangle \quad , \tag{4.4}$$

whereas in momentum space one has

$$\varphi(k,t) \;=\; \langle k(t)\,|\psi(t)\rangle \quad . \tag{4.5}$$

We normalize the position and momentum eigenstates such that

$$\langle k(t)\,|x(t')\,\rangle \;=\; \frac{1}{\sqrt{2\pi}}\,e^{-ik(t)x(t')} \quad . \tag{4.6}$$

By applying relation (4.3), one obtains from equation (4.4) for the time evolution of the wave function the following

$$\begin{aligned}
\psi(x,t) \;&=\; \left\langle x(t)\left|e^{-iH(t-t')}\right|\psi(t')\right\rangle \qquad (t>t') \\
&=\; \int dx(t')\left\langle x(t)\left|e^{-iH(t-t')}\right|x(t')\right\rangle\langle x(t')\,|\psi(t')\rangle \\
&=\; \int dx(t')\left\langle x(t)\left|e^{-iH(t-t')}\right|x(t')\right\rangle\,\psi(x,t') \quad ,
\end{aligned} \tag{4.7}$$

The expression

$$\left\langle x(t)\left|e^{-iH(t-t')}\right|x(t')\right\rangle$$

is called the *full propagator*, or *Feynman kernel* and contains the full information on how the system develops in time from instant t' to a later instant t. Relation (4.7) says that full knowledge of the wave function at instant t' and full knowledge of the way the wave function propagates through space in the time interval from t' to t, permits us to fully reconstruct the wave function at instant t. This is called *the Huyghens principle.*

Here we will concentrate on a time interval which starts at instant t_a and ends at a later instant t_b, in order to determine an expression for the full propagator in terms of classical quantities. An intuitive way to arrive at such expression is shown by Feynman and Hibbs. Here, we will closely follow their approach.

First, the time interval (t_a,t_b) is subdivided in N equal intervals, according to

$$t_0=t_a,\;\; t_1=t_a+\Delta t,\;\; t_2=t_a+2\Delta t,\;\; \dots,\;\; t_{N-1}=t_a+(N-1)\Delta t,\;\; t_N=t_a+N\Delta t=t_b \tag{4.8}$$

which gives for the full propagator

$$\left\langle x(t_b) \left| e^{-iH(t_b - t_a)} \right| x(t_a) \right\rangle = \tag{4.9}$$

$$= \left\langle x(t_b) \left| e^{-iH(t_N - t_{N-1})} e^{-iH(t_{N-1} - t_{N-2})} \dots e^{-iH(t_2 - t_1)} e^{-iH(t_1 - t_0)} \right| x(t_a) \right\rangle$$

$$= \left\langle x(t_b) \left| e^{-iH\Delta t} \, e^{-iH\Delta t} \, \dots \, e^{-iH\Delta t} \, e^{-iH\Delta t} \right| x(t_a) \right\rangle$$

Next, we insert the identity operator, using completeness of the position eigenstates

$$\left\langle x(t_b) \left| e^{-iH(t_b - t_a)} \right| x(t_a) \right\rangle = \tag{4.10}$$

$$= \langle x(t_b)| \, e^{-iH\Delta t} \int dx(t_{N-1}) \, |x(t_{N-1})\rangle\langle x(t_{N-1})| \, e^{-iH\Delta t}$$

$$\int dx(t_{N-2}) \, |x(t_{N-2})\rangle\langle x(t_{N-2})| \, \dots$$

$$\dots \int dx(t_2) \, |x(t_2)\rangle\langle x(t_2)| \, e^{-iH\Delta t} \int dx(t_1) \, |x(t_1)\rangle\langle x(t_1)| \, e^{-iH\Delta t} \, |x(t_a)\rangle$$

$$= \int dx(t_{N-1}) \int dx(t_{N-2}) \, \dots \int dx(t_2) \int dx(t_1) \left\langle x(t_b) \left| e^{-iH\Delta t} \right| x(t_{N-1}) \right\rangle$$

$$\left\langle x(t_{N-1}) \left| e^{-iH\Delta t} \right| x(t_{N-2}) \right\rangle \, \dots \, \left\langle x(t_2) \left| e^{-iH\Delta t} \right| x(t_1) \right\rangle \left\langle x(t_1) \left| e^{-iH\Delta t} \right| x(t_a) \right\rangle$$

In the limit $N \to \infty$, the resulting expression becomes the path integral form for the full propagator over all possible paths from $x(t_a)$ at instant t_a to $x(t_b)$ at instant t_b, since by taking all possible values for $x(t_i)$ $(i = 1, \, \dots \, , N-1)$ by integration, one constructs all possible paths.

However, it should be remarked that it is not at all clear yet whether such limit exists. Here, we do not further specify under which circumstances it is possible to take the limit. Instead, we assume that for the cases of interest to us, it can be done. Many textbooks discuss particular examples of Hamiltonians which allow a suitable definition of the path integral. But, even then, it is not completely straightforward.

Before proceeding, we will first evaluate one of the terms in the product which forms the integrand of formula (4.10).

One term of the product in 4.10

In this intermezzo we study just one of the terms in the product of the integrand in formula (4.10).

We start by once more inserting unity, this time by using completeness of the momentum eigenstates, *i.e.*

$$\left\langle x(t_{n+1}) \left| e^{-iH\Delta t} \right| x(t_n) \right\rangle = \tag{4.11}$$

$$= \langle x(t_{n+1})| \int dk(t_n) \, |k(t_n)\rangle\langle k(t_n)| \, e^{-iH\Delta t} \, |x(t_n)\rangle$$

$$= \int dk(t_n)\, \langle x(t_{n+1})\, | k(t_n)\rangle \Big\langle k(t_n) \Big| \, e^{-iH\Delta t} \, \Big| x(t_n) \Big\rangle$$

As stated before, we assume that the Hamiltonian H is built of position and momentum operators, respectively q and p. Here, we assume furthermore that H is in the *normal* form, *i.e.* that momentum operators come to the left of all position operators. This is a reasonable assumption, since most expressions can be brought into such form. For example:

$$(pq)^2 = pqpq = p([q,p] + pq)q = p(i + pq)q = ipq + p^2q^2 \quad .$$

In the case H has the normal form, also using relation (4.6), we obtain

$$\langle k\, | H(p,q) |\, x\rangle \; = \; H_{cl}(k,x) \langle k\, | x\rangle \; = \; H_{cl}(k,x) \, \frac{1}{\sqrt{2\pi}} \, e^{-ikx} \quad . \tag{4.12}$$

The classical Hamiltonian H_{cl} is a function of real numbers $x(t)$ and $k(t)$, not of operators.

Now, it is generally not true that formula (4.12) can be extended to the exponent of formula (4.11). But, for small enough Δt, one may approximate

$$\begin{aligned}
\Big\langle k(t_n) \Big| \, e^{-iH\Delta t} \, \Big| x(t_n) \Big\rangle \; &\approx \; \langle k(t_n)\, | (1 - iH\Delta t) |\, x(t_n)\rangle \\
&= \; \Big(1 - iH_{cl}(k(t_n), x(t_n))\Delta t\Big) \, \langle k(t_n)\, | x(t_n)\rangle \\
&\approx \; e^{-iH_{cl}(k(t_n), x(t_n))\Delta t} \, \langle k(t_n)\, | x(t_n)\rangle \quad .
\end{aligned} \tag{4.13}$$

A more elaborate approach can be found in chapter 3 of George Sterman's book.

Hence, also using formula (4.6) and its complex conjugate, for expression (4.11) we find

$$\begin{aligned}
&\Big\langle x(t_{n+1}) \Big| \, e^{-iH\Delta t} \, \Big| x(t_n) \Big\rangle \; = \qquad\qquad (4.14) \\
&= \; \frac{1}{2\pi} \int dk(t_n) \, e^{-ik(t_n)x(t_{n+1})} \, e^{-iH_{cl}(k(t_n), x(t_n))\Delta t} \, e^{ik(t_n)x(t_n)} \\
&= \; \frac{1}{2\pi} \int dk(t_n) \, e^{i\Delta t \left[k(t_n) \, \frac{x(t_n + \Delta t) - x(t_n)}{\Delta t} \; - \; H_{cl}(k(t_n), x(t_{n+1})) \right]} \quad ,
\end{aligned}$$

where we also used $x(t_{n+1}) = x(t_n + \Delta t)$.

Anticipating moreover the limit for $\Delta t \downarrow 0$, we may write

$$\begin{aligned}
&\Big\langle x(t_{n+1}) \Big| \, e^{-iH\Delta t} \, \Big| x(t_n) \Big\rangle \; = \qquad\qquad (4.15) \\
&= \; \frac{1}{2\pi} \int dk(t_n) \, e^{i\Delta t \left[k(t_n) \, \dot{x}(t_n) \; - \; H_{cl}(k(t_n), x(t_n)) \right]} \quad .
\end{aligned}$$

Furthermore, for Hamiltonians of the type

$$H(p,q) \; = \; \frac{p^2}{2m} \; + \; V(q)$$

one obtains an integral of the Gaussian form,

$$\int_{-\infty}^{\infty} dk\; e^{-\alpha k^2 + \beta k} \;=\; \sqrt{\left(\frac{\pi}{\alpha}\right)}\; e^{\beta^2/4\alpha} \quad . \tag{4.16}$$

Consequently, we can perform the integral in formula (4.15), to find

$$\begin{aligned}
&\left\langle x(t_{n+1}) \left| e^{-iH\Delta t} \right| x(t_n) \right\rangle \;= \qquad (4.17)\\
&= \sqrt{\left(\frac{m}{2\pi i\Delta t}\right)}\; e^{i\Delta t\; \left[\frac{m}{2}\,\dot{x}^2(t_n) \;-\; V(x(t_n))\right]}\\
&\sqrt{\left(\frac{m}{2\pi i\Delta t}\right)}\; e^{i\Delta t\; L_{cl}(x(t_n),\dot{x}(t_n))} \quad ,
\end{aligned}$$

where L_{cl} is the classical Lagrangian for the system.

Back to formula 4.10

Substitution of the result (4.17) in formula (4.10), leads to

$$\begin{aligned}
&\left\langle x(t_b) \left| e^{-iH(t_b - t_a)} \right| x(t_a) \right\rangle \;= \qquad (4.18)\\
&= \left(\frac{m}{2\pi i\Delta t}\right)^{N/2} \int dx(t_{N-1}) \;\ldots\; \int dx(t_1)\\
&e^{i\Delta t\; \left[L_{cl}(x(t_1),\dot{x}(t_1)) + \ldots + L_{cl}(x(t_{N-1}),\dot{x}(t_{N-1}))\right]} \quad .
\end{aligned}$$

Formula (4.18) is the *configuration space* path integral. It is less general than the *phase space* path integral, where both the integrations over x and k are kept.

In general, the limit $N \to \infty$ might not exist. For some cases it can be shown that a suitable limit can be taken. One obtains then.

$$\begin{aligned}
&\left\langle x(t_b) \left| e^{-iH(t_b - t_a)} \right| x(t_a) \right\rangle \;= \qquad (4.19)\\
&= \int_{x(t_a)}^{x(t_b)} [dx(t)]\; e^{i\int_{t_a}^{t_b} dt\; L_{cl}(x(t),\dot{x}(t))} \;=\; \int_{x(t_a)}^{x(t_b)} [dx(t)]\; e^{i\; S_{cl}(x(t))} \quad ,
\end{aligned}$$

where S_{cl} represents the *classical action* for the system.

The symbol $\int\, [dx(t)]$ stands for the integral over all possible paths. The normalization factors have been absorbed into it.

Chapter 5

A path integral for fields

For fields ϕ one may define a path integral by summing over all possible field configurations $\int [d\phi]$. The Lagrangian $\mathcal{L}$ for fields is given in terms of a density which must be integrated over space. In total one obtains then an integral over space and time $x = (t, \vec{x})$, which formalism agrees nicely with a relativistic theory for particles. We define

$$W \;=\; \int [d\phi]\; e^{i\int d^4x\;\mathcal{L}(\phi)} \quad , \tag{5.1}$$

for the path integral of a particle described by the field ϕ whose dynamics is contained in the Lagrangian density $\mathcal{L}$.

5.1 Green's functions

Quantities of interest can be obtained from expression (5.1) by the introduction of an auxiliar field $J(x)$ according to

$$W[J] \;=\; \int [d\phi]\; e^{i\int d^4x\;\left\{\mathcal{L}(\phi(x))\;+\;J(x)\phi(x)\right\}} \quad , \tag{5.2}$$

when one studies functional derivatives with respect to the auxiliar field, *i.e.* through

$$\int d^4x_1\; j(x_1)\,\frac{\delta W[J]}{\delta J(x_1)} \;=\; \lim_{\epsilon\downarrow 0}\;\frac{W[J+\epsilon j]-W[J]}{\epsilon} \quad . \tag{5.3}$$

For the righthand side of the above expression (5.3) we determine

$$\frac{1}{\epsilon}\;\left(W[J+\epsilon j]-W[J]\right) \;= \tag{5.4}$$

$$= \quad \frac{1}{\epsilon}\int [d\phi]\;\left(e^{i\int d^4x_1\;\epsilon j(x_1)\phi(x_1)}\;-\;1\right)\; e^{i\int d^4x\;\left\{\mathcal{L}(\phi(x))\;+\;J(x)\phi(x)\right\}}$$

$$= \quad \frac{1}{\epsilon}\int [d\phi]\;\left(i\epsilon\int d^4x_1\; j(x_1)\phi(x_1)\;+\;\mathcal{O}(\epsilon^2)\right)\; e^{i\int d^4x\;\left\{\mathcal{L}(\phi(x))\;+\;J(x)\phi(x)\right\}}$$

$$= \quad \int [d\phi]\;\left(i\int d^4x_1\; j(x_1)\phi(x_1)\;+\;\mathcal{O}(\epsilon)\right)\; e^{i\int d^4x\;\left\{\mathcal{L}(\phi(x))\;+\;J(x)\phi(x)\right\}}$$

$$(\epsilon\downarrow 0)\quad i\int d^4x_1\; j(x_1)\;\int [d\phi]\;\phi(x_1)\; e^{i\int d^4x\;\left\{\mathcal{L}(\phi(x))\;+\;J(x)\phi(x)\right\}} \quad .$$

Hence, we conclude

$$\frac{\delta W[J]}{\delta J(x_1)} = \int [d\phi]\; i\phi(x_1)\; e^{i\int d^4x\; \{\mathcal{L}(\phi(x)) \;+\; J(x)\phi(x)\}} \quad , \tag{5.5}$$

nothing else than bringing $i\phi(x_1)$ in front of the exponential.

5.1.1 The free field propagator

For a free scalar field which describes a spinless particle of mass μ, with no further degrees of freedom, we define the Lagrangian density

$$\mathcal{L}_0(\phi) = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}\mu^2\phi^2 \quad . \tag{5.6}$$

The classical equations of motion can be obtained by considering the action for the Lagrangian density (5.6), *i.e.*

$$S_0[\mathcal{L}_0] = \int_{t_1}^{t_2} dt \int d^3x\; \mathcal{L}_0(\phi(\vec{x},t), \partial_\mu\phi(\vec{x},t), t) \quad , \tag{5.7}$$

where $\partial_\mu\phi(\vec{x},t)$ stands for the four terms: $\frac{\partial}{\partial t}\phi(\vec{x},t)$, $\frac{\partial}{\partial x}\phi(\vec{x},t)$, $\frac{\partial}{\partial y}\phi(\vec{x},t)$ and $\frac{\partial}{\partial z}\phi(\vec{x},t)$.

Below, we determine the functional derivative of the above action (5.7). In this case we select a variation $f(\vec{x},t)$ which vanishes as well at the end points of the time integration, t_1 and t_2, as at infinity of the spatial integration interval. Consequently:

$$\int_{t_1}^{t_2} dt \int d^3x\; \frac{\partial f}{\partial t}\frac{\delta\mathcal{L}_0}{\delta\partial_t\phi} = -\int_{t_1}^{t_2} dt \int d^3x\; f(\vec{x},t)\frac{\partial}{\partial t}\left(\frac{\delta\mathcal{L}_0}{\delta\partial_t\phi}\right) \quad , \tag{5.8}$$

$$\int_{t_1}^{t_2} dt \int d^3x\; \frac{\partial f}{\partial x}\frac{\delta\mathcal{L}_0}{\delta\partial_x\phi} = -\int_{t_1}^{t_2} dt \int d^3x\; f(\vec{x},t)\frac{\partial}{\partial x}\left(\frac{\delta\mathcal{L}_0}{\delta\partial_x\phi}\right) \quad , \tag{5.9}$$

etcetera.

One obtains then for the functional derivative of the action (5.7) the following expression:

$$\frac{\delta S_0[\mathcal{L}_0]}{\delta\phi(\vec{x},t)} = \frac{\delta\mathcal{L}_0}{\delta\phi(\vec{x},t)} - \frac{\partial}{\partial x^\mu}\left(\frac{\delta\mathcal{L}_0}{\delta\partial_\mu\phi}\right) \quad . \tag{5.10}$$

For an extremum this expression vanishes, which leads to the well-known Euler-Lagrange equations of motion:

$$\frac{\delta\mathcal{L}_0}{\delta\phi(\vec{x},t)} - \frac{\partial}{\partial x^\mu}\left(\frac{\delta\mathcal{L}_0}{\delta\partial_\mu\phi}\right) = 0 \quad . \tag{5.11}$$

Explicitly, one obtains

$$-\left(\mu^2 + \partial^\mu\partial_\mu\right)\phi(x) = 0 \quad , \tag{5.12}$$

which is the Klein-Gordon equation for a massive spinless particle.

When we add a source term, $J(x)\phi(x)$, to the free Lagrangian density (5.6), *i.e.*

$$\mathcal{L}(\phi) = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}\mu^2\phi^2 + J(x)\phi(x) \quad , \tag{5.13}$$

then we obtain for the classical equations of motion

$$-\left(\mu^2 + \partial^\mu\partial_\mu\right)\phi(x) = -J(x) \quad . \tag{5.14}$$

Solutions to this equation are denoted by $\phi_{cl}(x)$.

A specific solution can be found by defining the Feynman propagator, $\Delta_F(x)$, through the equation

$$-\left(\mu^2 + \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\right)\Delta_F(x-y) = i\delta^{(4)}(x-y) \quad . \tag{5.15}$$

We may then cast the classical solution in the form

$$\phi_{cl}(x) = i\int d^4y\, \Delta_F(x-y)\, J(y) \quad . \tag{5.16}$$

In order to obtain an explicit expression for the Feynman propagator, we write a Fourier expansion for $\Delta_F(x)$,

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, \Delta_F(k) \quad . \tag{5.17}$$

Inserting this in equation (5.15), also writing the Fourier expansion of the Dirac delta function on the righthand side of equation (5.15), results in

$$\int \frac{d^4k}{(2\pi)^4}\left(-\mu^2 + k^2\right) e^{-ik(x-y)}\, \Delta_F(k) = i\int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \quad . \tag{5.18}$$

Hence,

$$\Delta_F(k) = \frac{i}{k^2 - \mu^2} \quad . \tag{5.19}$$

5.1.2 The free-field path integral

For the free-field path integral, related to the Lagrangian density of formula (5.6), we obtain

$$\begin{aligned} W_0 &= \int [d\phi]\, e^{i\int d^4x\, \mathcal{L}_0(\phi(x))} \\ &= \int [d\phi]\, e^{i\int d^4x\, \left\{\frac{1}{2}\left(\partial_\mu\phi(x)\right)^2 - \frac{1}{2}\mu^2\phi^2(x)\right\}} \quad . \end{aligned} \tag{5.20}$$

At this stage, it is opportune to mention the following identity.

$$\partial^\mu\left(\phi\partial_\mu\phi\right) = \left(\partial_\mu\phi\right)^2 + \phi\partial^2\phi \quad . \tag{5.21}$$

The lefthand side of relation (5.21), which is a total derivative, vanishes under the integral over space and time, because of the boundary conditions imposed (without saying so) on ϕ. Hence, instead of expression (5.20), we may write

$$W_0 = \int [d\phi]\, e^{i\int d^4x\, \phi(x)\left\{-\frac{1}{2}\partial^2 - \frac{1}{2}\mu^2\right\}\phi(x)} \quad . \tag{5.22}$$

Gaussian integrals

Recall the following elementary integration.

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\alpha x^2 \;+\; \beta x} \;=\; \sqrt{\frac{2\pi}{\alpha}}\, e^{\beta^2/2\alpha} \quad . \tag{5.23}$$

Next, let us consider the N-dimensional Gaussian integral, given by

$$\int_{-\infty}^{\infty} d\phi_1 \;\cdots\; \int_{-\infty}^{\infty} d\phi_N \;\; e^{-\frac{1}{2}\phi_i A_{ij}\phi_j \;+\; B_i\phi_i} \quad . \tag{5.24}$$

Summation is understood wherever indices are repeated. We assume that the matrix A is such that it can be diagonalised by an orthogonal coordinate transformation O, given by

$$\phi_i \;=\; O_{ik}\theta_k \quad \text{where} \quad O^T = O^{-1} \quad . \tag{5.25}$$

The expression $\phi_i A_{ij}\phi_j$ transforms according to

$$\phi_i A_{ij}\phi_j \;=\; O_{ik}\,\theta_k\, A_{ij}O_{j\ell}\,\theta_\ell \;=\; \theta_k\, O^T_{ki}A_{ij}O_{j\ell}\,\theta_\ell \;=\; \theta_k\,(O^TAO)_{k\ell}\,\theta_\ell \quad . \tag{5.26}$$

Moreover, since $O^TAO = O^{-1}AO$ is supposed to be diagonal, all off-diagonal terms in the sum (5.26) vanish. Hence, the expression $\phi_i A_{ij}\phi_j$ transforms according to

$$\phi_i A_{ij}\phi_j \;=\; (O^{-1}AO)_{11}\,\theta_1^2 \;+\; \cdots \;+\; (O^{-1}AO)_{NN}\,\theta_N^2 \quad . \tag{5.27}$$

For the expression $B_i\phi_i$ we have similarly

$$B_i\phi_i \;=\; B_iO_{ik}\,\theta_k \;=\; (BO)_k\,\theta_k \;=\; (BO)_1\,\theta_1 \;+\; \cdots \;+\; (BO)_N\,\theta_N \quad . \tag{5.28}$$

Furthermore, since also $\det(O) = 1$, we have for the volume element of formula (5.24), the result

$$d\phi_1 \;\cdots\; d\phi_N \;=\; \det(O)\, d\theta_1 \;\cdots\; d\theta_N \;=\; d\theta_1 \;\cdots\; d\theta_N \quad . \tag{5.29}$$

Putting everything together, we obtain for (5.24) the expression

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\phi_1 \;\cdots\; \int_{-\infty}^{\infty} d\phi_N \;\; e^{-\frac{1}{2}\phi_i A_{ij}\phi_j \;+\; B_i\phi_i} \;= \\
&= \int_{-\infty}^{\infty} d\theta_1\, e^{-\frac{1}{2}(O^{-1}AO)_{11}\,\theta_1^2 \;+\; (BO)_1\,\theta_1} \;\ldots \\
&\qquad \ldots \int_{-\infty}^{\infty} d\theta_N\, e^{-\frac{1}{2}(O^{-1}AO)_{NN}\,\theta_N^2 \;+\; (BO)_N\,\theta_N} \\
&= \sqrt{\frac{2\pi}{(O^{-1}AO)_{11}}}\; e^{(BO)_1(BO)_1/2(O^{-1}AO)_{11}} \;\ldots \\
&\qquad \ldots \sqrt{\frac{2\pi}{(O^{-1}AO)_{NN}}}\; e^{(BO)_N(BO)_N/2(O^{-1}AO)_{NN}} \quad .
\end{aligned} \tag{5.30}$$

The product of square roots in formula (5.30) gives

$$\sqrt{\frac{(2\pi)^N}{\det(O^{-1}AO)}} \;=\; \sqrt{\frac{(2\pi)^N}{\det(A)}} \quad . \tag{5.31}$$

Whereas for the sum of terms in the exponent, remembering that $O^{-1}AO$ is diagonal and $\det(O) = 1$, we deduce

$$(BO)_1(BO)_1\left[(O^{-1}AO)_{11}\right]^{-1} + \cdots + (BO)_N(BO)_N\left[(O^{-1}AO)_{NN}\right]^{-1} = \tag{5.32}$$

$$= (BO)_k\left[(O^{-1}AO)^{-1}\right]_{kn}(BO)_n = (BO)_k\left[O^{-1}A^{-1}O\right]_{kn}(BO)_n$$

$$= B_i\,O_{ik}\,O^{-1}_{k\ell}\,A^{-1}_{\ell m}\,O_{mn}\,B_j\,O_{jn} = B_i\left[OO^{-1}\right]_{i\ell}A^{-1}_{\ell m}\,O_{mn}\,O^T_{nj}\,B_j$$

$$= B_i\,\delta_{i\ell}\,A^{-1}_{\ell m}\,O_{mn}\,O^{-1}_{nj}\,B_j = B_i\,A^{-1}_{im}\left[OO^{-1}\right]_{mj}B_j$$

$$= B_i\,A^{-1}_{im}\,\delta_{mj}\,B_j = B_i\,A^{-1}_{ij}\,B_j \quad .$$

By substitution of the results (5.31) and (5.32) into relation (5.30), we find for the N-dimensional Gaussian integral (5.24) the expression

$$\int_{-\infty}^{\infty}d\phi_1\cdots\int_{-\infty}^{\infty}d\phi_N\;e^{-\frac{1}{2}\phi_iA_{ij}\phi_j + B_i\phi_i} = \sqrt{\frac{(2\pi)^N}{\det(A)}}\;e^{\frac{1}{2}B_iA^{-1}_{ij}B_j} \quad . \tag{5.33}$$

Continuous indices

We consider here the following replacements

$$A_{ij} \longrightarrow A(x,y) \quad \text{and} \quad \phi_i \longrightarrow \phi(x) \quad , \tag{5.34}$$

where x and y are continuous variables. Under replacements (5.34), summations turn into integrations, *i.e.*

$$B_i\phi_i \longrightarrow \int dx\;B(x)\phi(x) \quad \text{and} \quad \phi_iA_{ij}\phi_j \longrightarrow \int dx\int dy\;\phi(x)A(x,y)\,\phi(y) \quad . \tag{5.35}$$

One may repeat then the procedure of the previous paragraph (5.1.2) and obtain the continuous-indices equivalent of formula (5.33)

$$\int[d\phi]\;e^{-\frac{1}{2}\int dx\int dy\;\phi(x)A(x,y)\,\phi(y) + \int dx\;B(x)\phi(x)} =$$

$$= \sqrt{\frac{(2\pi)^N}{\det(A)}}\;e^{\frac{1}{2}\int dx\int dy\;B(x)A^{-1}(x,y)B(y)} \quad , \tag{5.36}$$

Where we have assumed that the continuous limit,

$$\int_{-\infty}^{\infty}d\phi_1\cdots\int_{-\infty}^{\infty}d\phi_N \longrightarrow \int[d\phi] \quad ,$$

exists for the discrete field formulation of formula (5.33).

Also some attention should be paid to the definition of $\det(A)$, but that is outside the scope of this notes. We will deal with such “normalisation” factors later on. The inverse of the operator A is defined by

$$A_{ik}A^{-1}_{kj} = \delta_{ij} \longrightarrow \int dz\;A(x,z)A^{-1}(z,y) = \delta(x-y) \quad . \tag{5.37}$$

For continuous variables in n dimensions, the generalisation of formula (5.36) is straightforward

$$\int [d\phi]\ e^{-\frac{1}{2}\int d^nx \int d^ny\ \phi(x)A(x,y)\ \phi(y)\ +\ \int d^nx\ B(x)\phi(x)} =$$

$$= \sqrt{\frac{(2\pi)^N}{\det(A)}}\ e^{\frac{1}{2}\int d^nx \int d^ny\ B(x)A^{-1}(x,y)B(y)} \quad . \tag{5.38}$$

For an example, let us study the case

$$K(x,y)\ =\ \delta^{(4)}(x-y)\ \left\{-\partial_x^2\ -\ \mu^2\right\} \quad . \tag{5.39}$$

Its inverse is given by

$$\int d^4z\ K(x,z)K^{-1}(z,y)\ =\ \delta^{(4)}(x-y) \quad . \tag{5.40}$$

By the use of formulas (5.15), (5.17) and (5.19), we obtain for $K^{-1}(x,y)$ the expression

$$K^{-1}(x,y)\ =\ -i\Delta_F(x-y)\ =\ \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x\ -\ y)}}{k^2\ -\ \mu^2} \quad , \tag{5.41}$$

The $i\epsilon$ factor in the propagator

The integrand of expression (5.20) oscillates rapidly under small variations of the fields when not near an extremum. Consequently, the path integral (5.20) will not converge. In order to define a convergent expression one may introduce a small damping factor, parametrized by $\epsilon > 0$, and take the limit $\epsilon \downarrow 0$ at the end of calculations. For the scalar field Lagrangian defined in formula (5.6) this can be done by adding a term, quadratic in the field

$$W_0\ =\ \int [d\phi]\ e^{i\int d^4x\ \left\{\frac{1}{2}\left(\partial_\mu\phi(x)\right)^2\ -\ \frac{1}{2}\mu^2\phi^2(x)\right\}\ -\ \frac{1}{2}\epsilon\int d^4x\ \phi^2(x)} \quad . \tag{5.42}$$

Effectively, this is equivalent to adding a quadratic term to the free field Lagrangian (5.6), *i.e.*

$$\mathcal{L}_0(\phi)\ =\ \frac{1}{2}\left(\partial_\mu\phi\right)^2\ -\ \frac{1}{2}\mu^2\phi^2\ +\ \frac{i}{2}\epsilon\phi^2 \quad . \tag{5.43}$$

For the corresponding propagator (5.19) it amounts in adding a term to μ^2. Hence, we obtain then for the propagator

$$\Delta_F(k)\ =\ \frac{i}{k^2\ -\ \mu^2\ +\ i\epsilon} \quad . \tag{5.44}$$

In the formal manipulations which follow, we will often not mention the $i\epsilon$ factor, in order not to unnecessarily complicate the expressions. However, this factor is not to be forgotten and must be recovered from time to time for explicit calculations.

5.1.3 The free-field generating functional

When we introduce an external source term in expression (5.42), *i.e.*

$$W_0[J] \;=\; \int [d\phi]\; e^{i\int d^4x\;\left\{\frac{1}{2}\left(\partial_\mu\phi(x)\right)^2 \;-\; \frac{1}{2}\mu^2\phi^2(x) \;+\; J(x)\phi(x)\right\}} \quad , \tag{5.45}$$

then we obtain the free-field generating functional. Using formula (5.22), we find

$$\begin{aligned} W_0[J] &= \int [d\phi] e^{i\int d^4x\;\phi(x)\left\{-\frac{1}{2}\partial_x^2 - \frac{1}{2}\mu^2\right\}\phi(x) + i\int d^4x\;J(x)\phi(x)} \\ &= e^{i\int d^4x \int d^4y\;\phi(x)\delta^{(4)}(x-y)\left\{-\frac{1}{2}\partial_y^2 - \frac{1}{2}\mu^2\right\}\phi(y) + i\int d^4x\;J(x)\phi(x)} \quad , \end{aligned} \tag{5.46}$$

which is precisely of the form of formula (5.38) for $A(x,y) = \frac{1}{2}i\delta^{(4)}(x-y)\left\{-\partial_y^2 - \mu^2\right\}$ and $B(x) = iJ(x)$. Hence, by also using expressions (5.41) for $A^{-1}(x,y)$, we may, up to non-essential normalisation factors, cast expression (5.45) into

$$\begin{aligned} W_0[J] &= e^{\int d^4x \int d^4y\; iJ(x)\;\frac{1}{2}i\left\{-i\Delta_F(x-y)\right\}\;iJ(y)} \\ &= e^{-\frac{1}{2}\int d^4x\int d^4y\;J(x)\;\Delta_F(x-y)\;J(y)} \quad . \end{aligned} \tag{5.47}$$

5.2 $\lambda\phi^4$ theory

We will study here $\lambda\phi^4$ theory, for which the interaction Lagrangian reads

$$\mathcal{L}_I(\phi) \;=\; -\frac{\lambda}{4!}\,\phi^4 \quad . \tag{5.48}$$

The constant λ describes the intensity of the interaction. It is assumed to be small, such that perturbative expansions can be made.

5.2.1 The interaction term

The generating functional for $\lambda\phi^4$ theory reads

$$W[J] = \int [d\phi]\; e^{i\int d^4x\left\{\frac{1}{2}\left(\partial_\mu\phi(x)\right)^2 - \frac{1}{2}\mu^2\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x) + J(x)\phi(x)\right\}} \quad . \tag{5.49}$$

We may elaborate on the above expression (5.49) according to

$$\begin{aligned} W[J] &= \int [d\phi]\; e^{-i\frac{\lambda}{4!}\int d^4x_1\,\phi^4(x_1)}\; e^{i\int d^4x\left\{\frac{1}{2}\left(\partial_\mu\phi(x)\right)^2 - \frac{1}{2}\mu^2\phi^2(x) + J(x)\phi(x)\right\}} = \\ &= \int [d\phi]\;\left\{1 + \left(-i\frac{\lambda}{4!}\right)\int d^4x_1\,\phi^4(x_1) + \frac{1}{2!}\left(-i\frac{\lambda}{4!}\right)^2\int d^4x_1\,\phi^4(x_1)\,\times\right. \\ &\quad \left.\times\int d^4x_2\,\phi^4(x_2) + \dots\right\}\; e^{i\int d^4x\left\{\frac{1}{2}\left(\partial_\mu\phi(x)\right)^2 - \frac{1}{2}\mu^2\phi^2(x) + J(x)\phi(x)\right\}} \end{aligned}$$

$$= \int [d\phi]\, e^{i \int d^4x \left\{ \frac{1}{2} (\partial_\mu \phi(x))^2 - \frac{1}{2}\mu^2 \phi^2(x) + J(x)\phi(x) \right\}} +$$

$$+ \left(-i \frac{\lambda}{4!} \right) \int d^4x_1 \int [d\phi]\, \phi^4(x_1)\, e^{i \int d^4x \left\{ \frac{1}{2} (\partial_\mu \phi(x))^2 - \frac{1}{2}\mu^2 \phi^2(x) + J(x)\phi(x) \right\}} +$$

$$+ \frac{1}{2!} \left(-i \frac{\lambda}{4!} \right)^2 \int d^4x_1 \int d^4x_2$$

$$\int [d\phi]\, \phi^4(x_1)\phi^4(x_2)\, e^{i \int d^4x \left\{ \frac{1}{2} (\partial_\mu \phi(x))^2 - \frac{1}{2}\mu^2 \phi^2(x) + J(x)\phi(x) \right\}} +$$

$$+ \dots$$

which by the use of formula (5.5) can be written in the form

$$W[J] = W_0[J] + \left(-i \frac{\lambda}{4!} \right) \int d^4x_1 \left(-i \frac{\delta}{\delta J(x_1)} \right)^4 W_0[J] +$$

$$+ \frac{1}{2!} \left(-i \frac{\lambda}{4!} \right)^2 \int d^4x_1 \int d^4x_2 \left(-i \frac{\delta}{\delta J(x_1)} \right)^4 \left(-i \frac{\delta}{\delta J(x_2)} \right)^4 W_0[J] + \dots$$

$$= \exp \left\{ \left(-i \frac{\lambda}{4!} \right) \int d^4x_1 \left(-i \frac{\delta}{\delta J(x_1)} \right)^4 \right\} W_0[J]$$

$$= \exp \left\{ \int d^4x\, i\mathcal{L}_I \left(\frac{-i\delta}{\delta J(x)} \right) \right\} W_0[J] \quad . \tag{5.50}$$

Here $\mathcal{L}_I(-i\delta/\delta J(x))$ stands for

$$\mathcal{L}_I \left(\frac{-i\delta}{\delta J(x)} \right) = -\frac{\lambda}{4!} \left[\frac{-i\delta}{\delta J(x)} \right]^4 \quad . \tag{5.51}$$

Moreover, using equation (5.47), we obtain

$$W[J] = \exp \left\{ i \int d^4x\, \mathcal{L}_I \left(\frac{-i\delta}{\delta J(x)} \right) \right\} e^{-\frac{1}{2} \int d^4y \int d^4z\, J(y)\, \Delta_F(y-z)\, J(z)} \quad , \tag{5.52}$$

which is the generating functional for Feynman diagrams.

The derivatives

Let us now study the functional derivatives of the free-field generating functional (5.47). Here, we make the substitution

$$\Delta_F(x-y) \longleftrightarrow -\Delta_F(x-y) \quad , \tag{5.53}$$

which simplifies the formulas in the following for extra factors -1. The free-field generating functional (5.47) takes then the form

$$W_0[J] = e^{+\frac{1}{2} \int d^4x \int d^4y\, J(x)\, \Delta_F(x-y)\, J(y)} \quad .$$

We proceed:

$$W_0[J+\varepsilon j] \ =$$

$$= \ e^{\frac{1}{2}\int d^4x \int d^4y \ \{J(x)+\varepsilon j(x)\} \ \Delta_F(x-y) \ \{J(y)+\varepsilon j(y)\}}$$

$$= \ e^{\frac{1}{2}\int d^4x \int d^4y \ J(x) \ \Delta_F(x-y) \ J(y)} \ \times$$

$$\times \ e^{\frac{1}{2}\varepsilon \int d^4x \int d^4y \ \{j(x) \ \Delta_F(x-y) \ J(y) \ + \ J(x) \ \Delta_F(x-y) \ j(y)\} \ + \ \mathcal{O}(\varepsilon^2)}$$

$$= \ W_0[J] \ e^{\frac{1}{2}\varepsilon \int d^4x \int d^4y \ j(x) \ J(y) \ \{\Delta_F(x-y) \ + \ \Delta_F(y-x)\} \ + \ \mathcal{O}(\varepsilon^2)} \quad .$$

On exploring the symmetry properties of the Feynman propagator (5.41), $\Delta_F(x-y) = \Delta_F(y-x)$, and moreover expanding the exponent, we obtain

$$\frac{W_0[J+\varepsilon j] \ - \ W_0[J]}{\varepsilon} \ = \tag{5.54}$$

$$= \quad W_0[J] \ \left(\int d^4x \int d^4y \ j(x) \ J(y) \ \Delta_F(x-y) \ + \ \mathcal{O}(\varepsilon)\right)$$

$$(\varepsilon \downarrow 0) \quad \int d^4x \ j(x) \ W_0[J] \ \int d^4y J(y) \ \Delta_F(x-y) \quad .$$

Consequently,

$$\frac{\delta W_0[J]}{\delta J(x)} \ = \ W_0[J] \ \int d^4y J(y) \ \Delta_F(x-y) \quad . \tag{5.55}$$

Next, we determine the second derivative

$$\frac{\delta^2 W_0[J]}{\delta J(x)^2} \ = \ \frac{\delta W_0[J]}{\delta J(x)} \ \int d^4y J(y) \ \Delta_F(x-y) \ + \ W_0[J] \ \frac{\delta \int d^4y \ J(y) \ \Delta_F(x-y)}{\delta J(x)} \tag{5.56}$$

$$= \ W_0[J] \ \int d^4y_1 \int d^4y_2 \ J(y_1) \ J(y_2) \ \Delta_F(x-y_1) \ \Delta_F(x-y_2) \ +$$

$$+ \ W_0[J] \ \int d^4y \ \delta^{(4)}(x-y) \ \Delta_F(x-y)$$

$$= \ W_0[J] \ \left\{\int d^4y_1 \int d^4y_2 \ J(y_1) \ J(y_2) \ \Delta_F(x-y_1) \ \Delta_F(x-y_2) \ + \ \Delta_F(0)\right\} \quad .$$

Then, the third derivative

$$\frac{\delta^3 W_0[J]}{\delta J(x)^3} \ = \ W_0[J] \ \left\{\int d^4y_1 \int d^4y_2 \int d^4y_3 \ J(y_1) \ J(y_2) \ J(y_3) \ \times\right. \tag{5.57}$$

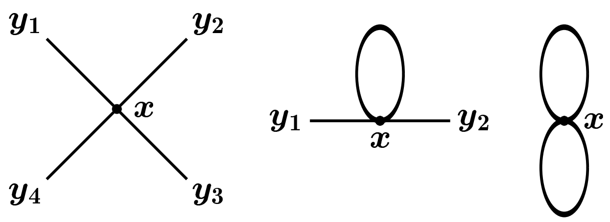
$$\left. \times \ \Delta_F(x-y_1) \ \Delta_F(x-y_2) \ \Delta_F(x-y_3) \ + \ 3\Delta_F(0) \ \int d^4y J(y) \ \Delta_F(x-y)\right\} \quad .$$

And, finally, the fourth derivative

$$\frac{\delta^4 W_0[J]}{\delta J(x)^4} \ = \ W_0[J] \ \left\{\int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4 \ J(y_1) \ J(y_2) \ J(y_3) \ J(y_4) \ \times\right. \tag{5.58}$$

$$\times\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)\ \Delta_F(x-y_4)\ +$$

$$+\ 6\Delta_F(0)\ \int d^4y_1 \int d^4y_2\ J(y_1)\ J(y_2)\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ +\ 4\ (\Delta_F(0))^2 \Big\} \quad .$$



(a) (b) (c)

Figure 5.1: Graphical representation of formula (5.58).

In figure (5.1) we have presented each of the three terms in expression (5.58). The symbol $\Delta_F(0)$ is represented by a loop. Hence, the diagram (a), without any loop, represents the first term of formula (5.58). The diagram (b), with one loop, represents the second term, whereas diagram (c) represents the last term, with two loops.

5.2.2 The full generating functional

The full generating functional $W[J]$ is defined in formula (5.49), but can also be expressed by relation (5.50), or more explicitly by equation (5.52). Using formula (5.51), one may expand the exponential of relation (5.50). This gives

$$W[J]\ =\ W_0[J]\ -\ \frac{\lambda}{4!}\ \int d^4x\ \frac{\delta^4 W_0[J]}{\delta J(x)^4}\ +\ \mathcal{O}(\lambda^2) \quad . \tag{5.59}$$

For the first order term we may substitute formula (5.58). This gives

$$W[J]\ = \tag{5.60}$$

$$=\ W_0[J]\ \Bigg(1\ -\ \frac{\lambda}{4!}\ \int d^4x\ \Big\{ \int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4\ J(y_1)\ J(y_2)\ J(y_3)\ J(y_4)\ \times$$

$$\times\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)\ \Delta_F(x-y_4)\ +$$

$$+\ 6\Delta_F(0)\ \int d^4y_1 \int d^4y_2\ J(y_1)\ J(y_2)\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ +$$

$$+\ 4\ (\Delta_F(0))^2 \Big\}\ +\ \mathcal{O}(\lambda^2) \Bigg) \quad .$$

5.2.3 Feynman diagrams

The n-point Green's function is defined by

$$G^{(n)}(x_1,\dots,x_n)\ =\ \left[\frac{1}{W[J]}\ \frac{(-i)^n \delta^n W[J]}{\delta J(x_1)\ \cdots\ \delta J(x_n)}\right]_{J\,=\,0} \quad . \tag{5.61}$$

Notice that by dividing by $W[J]$ in expression (5.61), we deal with most of the unaccounted-for normalisation factors.

The four-point Green's function to first order in $\boldsymbol{\lambda}$

We first determine, starting from the result (5.60), the fourth derivative of $W[J]$. In order to reduce the amount of work, we first study carefully which terms may survive.

- Any term which contains J at the end of the calculations, vanishes because of the limit $J \to 0$.
- From formula (5.55) we read that the functional derivative of $W_0[J]$ raises the number of J's by one.
- The functional derivative in J for a term of the form $\int d^4y\ J(y)\ \Delta_F(x-y)$ reduces the number of J's by one.

Hence, the term with a product of four J's in formula (5.60) cannot cope with an extra derivative from $W_0[J]$. Consequently, we have for that term

$$
\begin{aligned}
&\frac{\delta}{\delta J(x_1)}\ W_0[J] \int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4\ J(y_1)\ J(y_2)\ J(y_3)\ J(y_4)\ \times \\
&\qquad \times\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)\ \Delta_F(x-y_4) \\
&\to\ 4\ W_0[J] \int d^4y_1 \int d^4y_2 \int d^4y_3\ J(y_1)\ J(y_2)\ J(y_3)\ \times \\
&\qquad \times\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)\ \Delta_F(x-x_1)\quad ,
\end{aligned} \tag{5.62}
$$

where we have only kept the term which will not vanish at the end of our calculation. Repeating the same procedure for the other three derivatives, we obtain

$$
\begin{aligned}
&\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\ W_0[J]\ \times \\
&\qquad \times \int d^4y_1 \int d^4y_2 \int d^4y_3 \int d^4y_4\ J(y_1)\ J(y_2)\ J(y_3)\ J(y_4)\ \times \\
&\qquad \times\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)\ \Delta_F(x-y_4) \\
&\to\ 4!\ W_0[J]\ \Delta_F(x-x_1)\ \Delta_F(x-x_2)\ \Delta_F(x-x_3)\ \Delta_F(x-x_4)\quad .
\end{aligned} \tag{5.63}
$$

For the term with two J's in formula (5.60) we have the following (notice that we start by a change of integration variables $y_1, y_2 \to y_2, y_3$)

$$
\begin{aligned}
&\frac{\delta\ W_0[J] \int d^4y_2 \int d^4y_3\ J(y_2)\ J(y_3)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)}{\delta J(x_1)} \\
&=\ W_0[J] \int d^4y_1 \int d^4y_2 \int d^4y_3\ J(y_1)\ J(y_2)\ J(y_3)\ \times \\
&\qquad \times\ \Delta_F(x_1-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)\ + \\
&\qquad +\ 2\ W_0[J] \int d^4y_2\ J(y_2)\ \Delta_F(x-y_2)\ \Delta_F(x-x_1)\quad .
\end{aligned} \tag{5.64}
$$

The first of the two terms, which stems from the derivative of $W_0[J]$, cannot cope with one more such derivative. The second term still can. Hence,

$$\frac{\delta^2\, W_0[J] \int d^4y_2 \int d^4y_3\ J(y_2)\ J(y_3)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)}{\delta J(x_1)\delta J(x_2)} \tag{5.65}$$

$$\begin{aligned}
\rightarrow\ \ & W_0[J] \int d^4y_2 \int d^4y_3\ J(y_2)\ J(y_3)\ \Delta_F(x_1-x_2)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)\ + \\
& + 2\, W_0[J] \int d^4y_1 \int d^4y_2\ J(y_1)\ J(y_2)\ \Delta_F(x_1-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-x_2)\ + \\
& + 2\, W_0[J] \int d^4y_1 \int d^4y_2\ J(y_1)\ J(y_2)\ \Delta_F(x_2-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-x_1)\ + \\
& + 2\, W_0[J]\ \Delta_F(x-x_2)\ \Delta_F(x-x_1) \quad .
\end{aligned}$$

Next,

$$\frac{\delta^3\, W_0[J] \int d^4y_2 \int d^4y_3\ J(y_2)\ J(y_3)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} \tag{5.66}$$

$$\begin{aligned}
\rightarrow\ \ & 2\, W_0[J] \int d^4y_2\ J(y_2)\ \Delta_F(x_1-x_2)\ \Delta_F(x-y_2)\ \Delta_F(x-x_3)\ + \\
& + 2\, W_0[J] \int d^4y_2\ J(y_2)\ \Delta_F(x_1-x_3)\ \Delta_F(x-y_2)\ \Delta_F(x-x_2)\ + \\
& + 2\, W_0[J] \int d^4y_1\ J(y_1)\ \Delta_F(x_1-y_1)\ \Delta_F(x-x_3)\ \Delta_F(x-x_2)\ + \\
& + 2\, W_0[J] \int d^4y_2\ J(y_2)\ \Delta_F(x_2-x_3)\ \Delta_F(x-y_2)\ \Delta_F(x-x_1)\ + \\
& + 2\, W_0[J] \int d^4y_1\ J(y_1)\ \Delta_F(x_2-y_1)\ \Delta_F(x-x_3)\ \Delta_F(x-x_1)\ + \\
& + 2\, W_0[J] \int d^4y_1\ J(y_1)\ \Delta_F(x_3-y_1)\ \Delta_F(x-x_2)\ \Delta_F(x-x_1) \quad .
\end{aligned}$$

Finally,

$$\frac{\delta^4\, W_0[J] \int d^4y_2 \int d^4y_3\ J(y_2)\ J(y_3)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \tag{5.67}$$

$$\begin{aligned}
\rightarrow\ \ & 2\, W_0[J]\ \Delta_F(x_1-x_2)\ \Delta_F(x-x_4)\ \Delta_F(x-x_3)\ + \\
& + 2\, W_0[J]\ \Delta_F(x_1-x_3)\ \Delta_F(x-x_4)\ \Delta_F(x-x_2)\ + \\
& + 2\, W_0[J]\ \Delta_F(x_1-x_4)\ \Delta_F(x-x_3)\ \Delta_F(x-x_2)\ + \\
& + 2\, W_0[J]\ \Delta_F(x_2-x_3)\ \Delta_F(x-x_4)\ \Delta_F(x-x_1)\ + \\
& + 2\, W_0[J]\ \Delta_F(x_2-x_4)\ \Delta_F(x-x_3)\ \Delta_F(x-x_1)\ + \\
& + 2\, W_0[J]\ \Delta_F(x_3-x_4)\ \Delta_F(x-x_2)\ \Delta_F(x-x_1) \quad .
\end{aligned}$$

For the two terms without any J's in formula (5.60), we may use equation (5.55) for the first derivative

$$\frac{\delta W_0[J]}{\delta J(x_1)} = W_0[J] \int d^4y_1 \; J(y_1) \, \Delta_F(x_1 - y_1) \quad . \tag{5.68}$$

The second derivative gives

$$\begin{aligned} \frac{\delta^2 W_0[J]}{\delta J(x_1) \delta J(x_2)} &= W_0[J] \int d^4y_1 \int d^4y_2 \; J(y_1) \, J(y_2) \, \Delta_F(x_1 - y_1) \, \Delta_F(x_2 - y_2) \; + \\ & \quad + W_0[J] \, \Delta_F(x_1 - x_2) \quad . \end{aligned} \tag{5.69}$$

From here on we do not write the terms which vanish at the end.
The third derivative gives

$$\begin{aligned} \frac{\delta^3 W_0[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3)} &\rightarrow W_0[J] \int d^4y_2 \; J(y_2) \, \Delta_F(x_1 - x_3) \, \Delta_F(x_2 - y_2) + \\ & + W_0[J] \int d^4y_1 \; J(y_1) \, \Delta_F(x_1 - y_1) \, \Delta_F(x_2 - x_3) \; + \\ & + W_0[J] \int d^4y_1 \; J(y_1) \, \Delta_F(x_3 - y_1) \, \Delta_F(x_1 - x_2) \quad . \end{aligned} \tag{5.70}$$

The fourth derivative gives

$$\frac{\delta^4 W_0[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3) \delta J(x_4)} \tag{5.71}$$

$$\begin{aligned} &\rightarrow \; W_0[J] \, \Delta_F(x_1 - x_3) \, \Delta_F(x_2 - x_4) \; + \\ &\quad + W_0[J] \, \Delta_F(x_1 - x_4) \, \Delta_F(x_2 - x_3) \; + \; W_0[J] \, \Delta_F(x_3 - x_4) \, \Delta_F(x_1 - x_2) \quad . \end{aligned}$$

Subsequently, we put everything together.
First the numerator of formula (5.61) for $n = 4$.

$$\left[\frac{\delta^4 W[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3) \delta J(x_4)} \right]_{J\,=\,0} = \tag{5.72}$$

$$\begin{aligned} = \; & W_0[J] \Bigg(\left\{ 1 \; - \; \frac{\lambda}{3!} \int d^4x \; (\Delta_F(0))^2 \right\} \Big\{ \Delta_F(x_1 - x_3) \, \Delta_F(x_2 - x_4) \; + \\ & + \Delta_F(x_1 - x_4) \, \Delta_F(x_2 - x_3) \; + \; \Delta_F(x_3 - x_4) \, \Delta_F(x_1 - x_2) \Big\} \; + \\ & - \lambda \int d^4x \; \Delta_F(x - x_1) \, \Delta_F(x - x_2) \, \Delta_F(x - x_3) \, \Delta_F(x - x_4) \\ & - \frac{\lambda}{2} \int d^4x \; \Delta_F(0) \Big\{ \Delta_F(x_1 - x_2) \, \Delta_F(x - x_4) \, \Delta_F(x - x_3) \; + \\ & + \Delta_F(x_1 - x_3) \, \Delta_F(x - x_4) \, \Delta_F(x - x_2) \; + \; \Delta_F(x_1 - x_4) \, \Delta_F(x - x_3) \, \Delta_F(x - x_2) \; + \end{aligned}$$

$$+ \Delta_F(x_2 - x_3)\,\Delta_F(x - x_4)\,\Delta_F(x - x_1) \;+\; \Delta_F(x_2 - x_4)\,\Delta_F(x - x_3)\,\Delta_F(x - x_1) \;+$$

$$+ \Delta_F(x_3 - x_4)\,\Delta_F(x - x_2)\,\Delta_F(x - x_1)\Big\} \;+\; \mathcal{O}(\lambda^2)\Big) \quad .$$

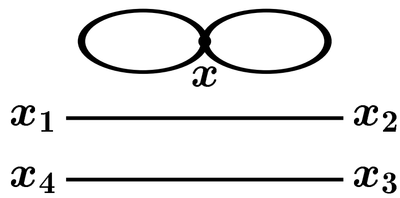
For the denominator of formula (5.61) one has

$$\begin{aligned} \left[\frac{1}{W[J]}\right]_{J\,=\,0} &= \frac{1}{W_0[J]}\left(1 \;-\; \frac{\lambda}{3!}\int d^4x\;\left(\Delta_F(0)\right)^2 \;+\; \mathcal{O}(\lambda^2)\right) \\ &= \frac{1}{W_0[J]}\left(1 \;+\; \frac{\lambda}{3!}\int d^4x\;\left(\Delta_F(0)\right)^2 \;+\; \mathcal{O}(\lambda^2)\right) \quad . \end{aligned} \tag{5.73}$$

In the product of expressions (5.72) and (5.73), we find that $W_0[J]$ drops out. But, more interestingly, also disconnected terms cancel. For example, the term

$$-\frac{\lambda}{3!}\int d^4x\;\left(\Delta_F(0)\right)^2\;\Delta_F(x_1 - x_2)\,\Delta_F(x_4 - x_3) \quad , \tag{5.74}$$

represented by the diagram



is exactly cancelled by a similar term stemming from the denominator. We obtain finally for the four-point Green's function.

$$G^{(4)}(x_1, x_2, x_3, x_4) \;= \tag{5.75}$$

$$\begin{aligned} =\;\; & \Delta_F(x_1 - x_3)\,\Delta_F(x_2 - x_4) \;+ \\ & + \Delta_F(x_1 - x_4)\,\Delta_F(x_2 - x_3) \;+\; \Delta_F(x_3 - x_4)\,\Delta_F(x_1 - x_2) \;+ \\ & - \lambda\int d^4x\;\Delta_F(x - x_1)\,\Delta_F(x - x_2)\,\Delta_F(x - x_3)\,\Delta_F(x - x_4) \\ & - \frac{\lambda}{2}\int d^4x\;\Delta_F(0)\,\Big\{\Delta_F(x_1 - x_2)\,\Delta_F(x - x_4)\,\Delta_F(x - x_3) \;+ \\ & + \Delta_F(x_1 - x_3)\,\Delta_F(x - x_4)\,\Delta_F(x - x_2) \;+\; \Delta_F(x_1 - x_4)\,\Delta_F(x - x_3)\,\Delta_F(x - x_2) \;+ \\ & + \Delta_F(x_2 - x_3)\,\Delta_F(x - x_4)\,\Delta_F(x - x_1) \;+\; \Delta_F(x_2 - x_4)\,\Delta_F(x - x_3)\,\Delta_F(x - x_1) \;+ \\ & + \Delta_F(x_3 - x_4)\,\Delta_F(x - x_2)\,\Delta_F(x - x_1)\Big\} \;+\; \mathcal{O}(\lambda^2) \quad . \end{aligned}$$

The first three terms in (5.75) are the three different ways in which one can connect the four points x_1, x_2, x_3 and x_4. The next term is the four-point vertex, depicted in figure (5.1(a)). The last six terms are the different ways in which a bubble can be connected to the first three terms, as depicted in figure (5.1(b)).

The two-point Green's function to first order in λ

From the discussion in paragraph (5.2.3) we understand now that terms with more than two fields in expression (5.60), cannot contribute. Consequently, we are left with the expression with two J's, which contributes

$$\frac{\delta^2\ W_0[J] \int d^4y_1 \int d^4y_2\ J(y_1)\ J(y_2)\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)}{\delta J(x_1)\delta J(x_2)} \tag{5.76}$$

$$\rightarrow\ \ 2\ W_0[J]\ \Delta_F(x-x_1)\ \Delta_F(x-x_2) \quad .$$

and the terms with no J's, whose contributions can be read from formula (5.69)

$$\frac{\delta^2 W_0[J]}{\delta J(x_1)\delta J(x_2)}\ \rightarrow\ W_0[J]\ \Delta_F(x_1-x_2) \quad . \tag{5.77}$$

For the numerator we have here

$$\left[\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\right]_{J\,=\,0} = W_0[J]\Bigg(\left\{1\ -\ \frac{\lambda}{3!}\int d^4x\ \left(\Delta_F(0)\right)^2\right\}\ \Delta_F(x_1-x_2)\ +$$

$$-\frac{\lambda}{2}\int d^4x\ \Delta_F(0)\ \Delta_F(x-x_1)\ \Delta_F(x-x_2)\ +\ \mathcal{O}(\lambda^2)\Bigg) \tag{5.78}$$

We notice that again the order-λ disconnected 2-loop contribution in (5.78) is exactly cancelled by the corresponding term from the denominator (5.73). The two-point Green's function which results, equals

$$G^{(2)}(x_1,x_2)\ =\ \Delta_F(x_1-x_2)\ -\ \frac{\lambda}{2}\int d^4x\ \Delta_F(0)\ \Delta_F(x-x_1)\ \Delta_F(x-x_2)\ +\ \mathcal{O}(\lambda^2)\ . \tag{5.79}$$

The first term in (5.79) represents the free propagator which connects the points x_1 and x_2. The second term represents a propagator with a bubble attached to it, as depicted in figure (5.1(b)).

5.3 $\lambda\phi^3$ theory

5.3.1 The interaction term

We will study here $\lambda\phi^3$ theory, for which the interaction Lagrangian reads

$$\mathcal{L}_I(\phi)\ =\ -\frac{\lambda}{3!}\ \phi^3 \quad . \tag{5.80}$$

The constant λ describes the intensity of the interaction. It is assumed to be small, such that perturbative expansions can be made.

The generating functional for $\lambda\phi^3$ theory reads

$$W[J]\ =\ \exp\left\{\int d^4x\ i\mathcal{L}_I\left(\frac{-i\delta}{\delta J(x)}\right)\right\}\ W_0[J] \quad . \tag{5.81}$$

Here $\mathcal{L}_I(-i\delta/\delta J(x))$ stands for

$$\mathcal{L}_I\left(\frac{-i\delta}{\delta J(x)}\right) = -\frac{\lambda}{3!}\left[\frac{-i\delta}{\delta J(x)}\right]^3 \quad . \tag{5.82}$$

Moreover, using equation (5.47), we obtain

$$W[J] = \exp\left\{\int d^4x\; i\mathcal{L}_I\left(\frac{-i\delta}{\delta J(x)}\right)\right\} e^{\frac{1}{2}\int d^4y \int d^4z\; J(y)\,\Delta_F(y-z)\,J(z)} \quad , \tag{5.83}$$

which is the generating functional for Feynman diagrams.

The derivatives

The third derivative of the free-field generating functional (5.81) is given in formula (5.57), and reads

$$\frac{\delta^3 W_0[J]}{\delta J(x)^3} = W_0[J]\left\{\int d^4y_1 \int d^4y_2 \int d^4y_3\; J(y_1)\,J(y_2)\,J(y_3)\,\times\right. \tag{5.84}$$

$$\left.\times\,\Delta_F(x-y_1)\,\Delta_F(x-y_2)\,\Delta_F(x-y_3) + 3\Delta_F(0)\int d^4y_1\; J(y_1)\,\Delta_F(x-y_1)\right\} \quad .$$

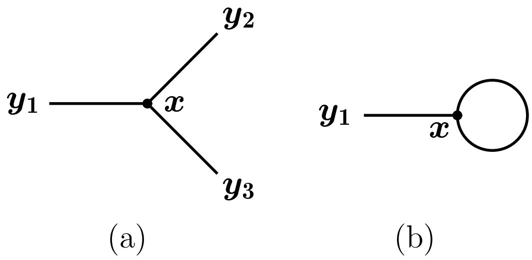


Figure 5.2: Graphical representation of formula (5.58).

In figure (5.1) we have presented each of the terms in expression (5.84). The symbol $\Delta_F(0)$ is represented by a loop. Hence, the diagram (a), without any loop, represents the first term of formula (5.84). The diagram (b), with one loop, represents the second term.

5.3.2 The full generating functional

The full generating functional $W[J]$ is defined in formula (5.49), but can also be expressed by relation (5.50), or more explicitly by equation (5.52). Using formula (5.51), one may expand the exponential of relation (5.50). This gives

$$W[J] = W_0[J] - \frac{\lambda}{3!}\int d^4x\, \frac{\delta^3 W_0[J]}{\delta J(x)^3} + \mathcal{O}(\lambda^2) \quad . \tag{5.85}$$

For the first order term we may substitute formula (5.84). This gives

$$W[J] = W_0[J]\left(1 - \frac{\lambda}{3!}\int d^4x\right. \tag{5.86}$$

$$\left\{\int d^4y_1 \int d^4y_2 \int d^4y_3\ J(y_1)\ J(y_2)\ J(y_3)\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3) + \right.$$

$$\left. +\ 3\Delta_F(0) \int d^4y_1\ J(y_1)\ \Delta_F(x-y_1)\right\} + \mathcal{O}(\lambda^2)\Big) .$$

5.3.3 Feynman diagrams

The n-point Green's function is defined in formula (5.61).

The three-point Green's function to first order in $\boldsymbol{\lambda}$

We first determine, starting from the result (5.86), the third derivative of $W[J]$. The term linear in $W_0[J]$ does not survive three derivatives. The term with a product of three J's in formula (5.60) cannot cope with an extra derivative from $W_0[J]$. Consequently, we have for that term

$$\frac{\delta}{\delta J(x_1)}\ W_0[J] \int d^4y_1 \int d^4y_2 \int d^4y_3\ J(y_1)\ J(y_2)\ J(y_3) \times \tag{5.87}$$

$$\times\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)$$

$$\rightarrow\ 3\ W_0[J] \int d^4y_1 \int d^4y_2\ J(y_1)\ J(y_2)\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-x_1) \quad ,$$

where we have only kept the term which will not vanish at the end of our calculation. Repeating the same procedure for the other two derivatives, we obtain

$$\frac{\delta^3 W_0[J]\ \int d^4y_1 \int d^4y_2 \int d^4y_3\ J(y_1)\ J(y_2)\ J(y_3)\ \Delta_F(x-y_1)\ \Delta_F(x-y_2)\ \Delta_F(x-y_3)}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}$$

$$\rightarrow\ 3!\ W_0[J]\ \Delta_F(x-x_1)\ \Delta_F(x-x_2)\ \Delta_F(x-x_3) \quad . \tag{5.88}$$

For the term with one J in formula (5.86) we have the following (notice that we start by a change of integration variables $y_1 \rightarrow y_2$)

$$\frac{\delta\ W_0[J] \int d^4y_2\ J(y_2)\ \Delta_F(x-y_2)}{\delta J(x_1)} \tag{5.89}$$

$$=\ W_0[J] \int d^4y_1 \int d^4y_2\ J(y_1)\ J(y_2)\ \Delta_F(x_1-y_1)\ \Delta_F(x-y_2)\ + W_0[J]\Delta_F(x-x_1) \quad .$$

The first of the two terms, which stems from the derivative of $W_0[J]$, cannot cope with one more such derivative. The second term still can. Hence,

$$\frac{\delta^2\ W_0[J] \int d^4y_2\ J(y_2)\ \Delta_F(x-y_2)}{\delta J(x_1)\delta J(x_2)} \tag{5.90}$$

$$\rightarrow\ W_0[J] \int d^4y_2\ J(y_2)\ \Delta_F(x_1-x_2)\ \Delta_F(x-y_2)\ +$$

$$+\ W_0[J] \int d^4y_1\ J(y_1)\ \Delta_F(x_1-y_1)\ \Delta_F(x-x_2)\ +$$

$$+\ W_0[J] \int d^4y_1\ J(y_1)\ \Delta_F(y_1-x_2)\ \Delta_F(x-x_1) \quad .$$

Finally,

$$\frac{\delta^3 \, W_0[J] \int d^4y_2 \; J(y_2) \; \Delta_F(x-y_2)}{\delta J(x_1)\delta J(x_2)\delta J(x_3)} \tag{5.91}$$

$$\rightarrow \;\; W_0[J] \left\{ \Delta_F(x_1-x_2) \, \Delta_F(x-x_3) \; + \; \Delta_F(x_1-x_3) \, \Delta_F(x-x_2) \; + \right.$$

$$\left. + \; \Delta_F(x_3-x_2) \, \Delta_F(x-x_1) \right\} \quad .$$

Subsequently, we put everything together.
First the numerator of formula (5.61) for $n=3$.

$$\left[\frac{\delta^3 W[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}\right]_{J\,=\,0} \quad = \tag{5.92}$$

$$= \;\; -\frac{\lambda}{2!} \, W_0[J] \int d^4x \, \Big(\Delta_F(x-x_1) \, \Delta_F(x-x_2) \, \Delta_F(x-x_3) \; +$$

$$+ \; \Delta_F(0) \;\; \{ \Delta_F(x_1-x_2) \, \Delta_F(x-x_3) \; + \; \Delta_F(x_1-x_3) \, \Delta_F(x-x_2) \; +$$

$$+ \;\; \Delta_F(x_3-x_2) \, \Delta_F(x-x_1) \} \; + \; \mathcal{O}(\lambda^2) \Big) \quad .$$

For the denominator of formula (5.61) one has

$$\left[\frac{1}{W[J]}\right]_{J\,=\,0} \;\; = \;\; \frac{1}{W_0[J]} \, \Big(1 \; + \; \mathcal{O}(\lambda^2) \Big) \quad . \tag{5.93}$$

The product of expressions (5.92) and (5.93) reads

$$G^{(4)}(x_1, x_2, x_3) \;\; = \tag{5.94}$$

$$= \;\; -\frac{\lambda}{2!} \; \int d^4x \, \Big(\Delta_F(x-x_1) \, \Delta_F(x-x_2) \, \Delta_F(x-x_3) \; +$$

$$+ \; \Delta_F(0) \;\; \{ \Delta_F(x_1-x_2) \, \Delta_F(x-x_3) \; + \; \Delta_F(x_1-x_3) \, \Delta_F(x-x_2) \; +$$

$$+ \;\; \Delta_F(x_3-x_2) \, \Delta_F(x-x_1) \} \; + \; \mathcal{O}(\lambda^2) \Big) \quad .$$

The first terms in (5.94) is the vertex, depicted in figure (5.2(a)). The last three terms are the different ways in which a bubble can be connected to one of the external points, as depicted in figure (5.2(b)).

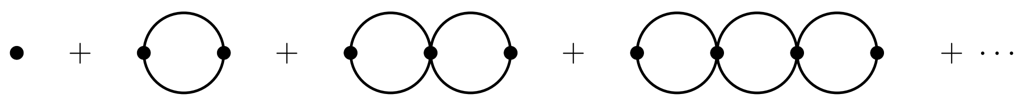
Chapter 6

The Bethe-Salpeter equation

In this chapter we discuss an integral equation for two-particle elastic scattering, which, instead of the perturbative series studied in the previous chapters, orders the interaction graphs in certain subsets.

6.1 The bubble sum

A famous subset, mainly applied to quark physics in the Nambu-Jona/Lasino model, is the bubble sum which contains the following subset of contributions to the amputed vertex function



When we represent the loop-integral by $B\left(p_1+p_2, m^2, \lambda\right)$, where p_1 and p_2 are the external momenta of the incoming particles, and the whole sum of the above contributions by $S\left(p_1+p_2, m^2, \lambda\right)$, then for S one has in formula

$$S \;=\; \lambda \;+\; \lambda^2 B \;+\; \lambda^3 B^2 \;+\; \lambda^4 B^3 \;+\; \cdots \quad , \tag{6.1}$$

which can be casted in the form

$$S \;=\; \lambda \;+\; \lambda B\left\{\lambda \;+\; \lambda^2 B \;+\; \lambda^3 B^2 \;+\; \lambda^4 B^3 \;+\; \cdots\right\} \;=\; \lambda \;+\; \lambda B\, S \quad , \tag{6.2}$$

and thus formally be summed up, to give

$$S\left(p_1+p_2, m^2, \lambda\right) \;=\; \frac{\lambda}{1 \;-\; \lambda B\left(p_1+p_2, m^2, \lambda\right)} \quad . \tag{6.3}$$

Now, this is more or less the only elegant subset of graphs in a theory with a four-particle vertex, since other subsets get rather messy. The series (6.1), moreover, did not lead to an integral equation.

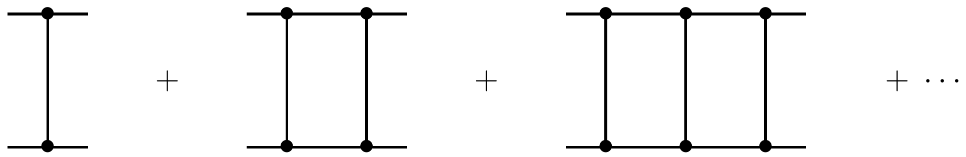
6.2 The ladder sum

However, for a theory with a three-particle vertex, *i.e.* for a theory with an interaction Lagrangian which is of the form

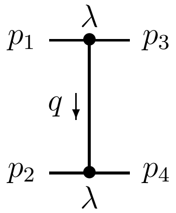
$$\mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\,\varphi^3 \quad , \tag{6.4}$$

one has as the standard example of such approach, the so-called *ladder series*, given by

$$L\left(p_1, p_2, p_3, p_4; q; m^2, \lambda^2\right) = \tag{6.5}$$



Where the external momenta are represented by p_1, p_2, p_3, and p_4. When we denote the exchange momentum by q, then the first term of this series, as depicted by the following graph,

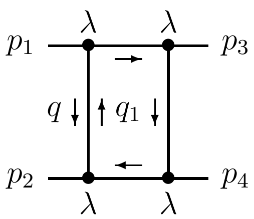


,

represents the truncated one-boson-exchange diagram and is given by the expression

$$G_0(q) = \frac{\lambda^2}{q^2 - m^2} \quad . \tag{6.6}$$

The second term in the ladder series, which is represented by

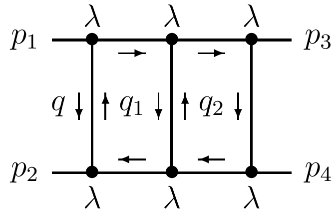


,

contains a loop integral, for which its loop variable is indicated by q_1, and which is given by the expression

$$\int d^4q_1\; G_0\left(q_1 - q\right)\; G_0\left(p_1 + q_1 - q\right)\; G_0\left(p_2 - q_1 + q\right)\; G_0\left(q_1\right) \quad . \tag{6.7}$$

The third term in the ladder series, which is depicted by the following graph



,

contains two loop integrations, for which the loop variables are respectively indicated by q_1 and q_2, and which is given by the expression

$$\int d^4q_1\, G_0(q_1-q)\; G_0(p_1+q_1-q)\; G_0(p_2-q_1+q)\;\times$$

$$\times\int d^4q_2\, G_0(q_2-q_1)\, G_0(p_1+q_2-q)\; G_0(p_2-q_2+q)\; G_0(q_2)\quad . \tag{6.8}$$

The ladder sum is just the sum of all those terms (6.6), (6.7), (6.8), ..., added up to an infinite number of loops.

When we now define the partial sum of the first two terms, (6.6) and (6.7), by

$$L^{(1)}\left(p_1,p_2,p_3,p_4;q;m^2,\lambda^2\right)\;=\;G_0(q)\;+$$

$$+\int d^4q_1\, G_0(q_1-q)\; G_0(p_1+q_1-q)\; G_0(p_2-q_1+q)\; G_0(q_1)\quad ,$$

then we observe that the partial sum of the first three terms, (6.6), (6.7), and (6.8), might have been written in the form

$$L^{(2)}\left(p_1,p_2,p_3,p_4;q;m^2,\lambda^2\right)\;=\;G_0(q)\;+$$

$$+\int d^4q_1\, G_0(q_1-q)\; G_0(p_1+q_1-q)\; G_0(p_2-q_1+q)\; L^{(1)}\left(p_1,p_2,p_3,p_4;q_1;m^2,\lambda^2\right).$$

Repeating this procedure for the first four terms, next for the first five terms, etcetera, we might come to the conclusion that the whole ladder sum can be casted in such form, *i.e.*

$$L\left(p_1,p_2,p_3,p_4;q;m^2,\lambda^2\right)\;=\;G_0(q)\;+ \tag{6.9}$$

$$+\int d^4q_1\, G_0(q_1-q)\; G_0(p_1+q_1-q)\; G_0(p_2-q_1+q)\; L\left(p_1,p_2,p_3,p_4;q_1;m^2,\lambda^2\right)\quad ,$$

which gives indeed an integral equation for the amplitude L of the ladder series contribution to the full amputed four particle Green's function.

The product of the two horizontal propagators in (6.9) is often referred two as the *two-particles propagator* and denoted by

$$G_0^{(2)}(p_1+q_1-q;p_2-q_1+q)\;=\;G_0(p_1+q_1-q)\; G_0(p_2-q_1+q)\quad . \tag{6.10}$$

With this definition we arrive at the well-known form of the Bethe-Salpeter equation for the ladder series contribution to the full amputed four particle Green's function, mnemoniced by

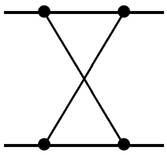
$$L \;=\; G_0 \;+\; G_0 \, G_0^{(2)} \, L \quad . \tag{6.11}$$

6.3 The driving term

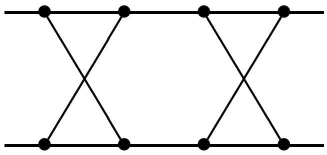
The ladder series (6.5) consists of a certain subset of the set of all one-particle irreducible diagrams for the four-points Green's function. Each diagram is a multiple copy of the one-particle exchange diagram (6.6) connected by two-particles propagators (6.10). For that reason, diagram (6.6) is called the *driving term* of the ladder series.

Another such series might for example be *driven* by the *crossed box* and is given by

$$CB\left(p_1, p_2, p_3, p_4; q; m^2, \lambda^2\right) \;= \qquad (6.12)$$



$+$

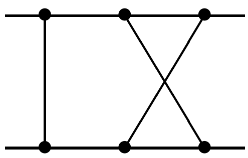


$+ \cdots$

which consists of multiple copies of the first, one-particle irreducible, crossed-box diagram connected by two-particles propagators (6.10). When we denote the crossed-box diagram by B, then we may obtain for the above series (6.12), the following integral equation:

$$CB \;=\; B \;+\; B\, G_0^{(2)}\, CB \quad . \qquad (6.13)$$

A series driven by the sum of the one-particle exchange diagram and the crossed box diagram, contains diagrams of the form



$$. \qquad (6.14)$$

At this stage it is opportune to define *two-particle irreducible* diagrams by those terms of the full amputed four-particle Green's function which cannot be separated into a pair of diagrams by cutting two lines whose momenta sum to the full incoming four-momentum $p_1 + p_2$.

When we denote the sum of all two-particle irreducible diagrams by V and, moreover, substitute the two propagators of the two-particles Green's function by full propagators, S'_F, which are defined in (2.37), then the full amputed four-particle Green's function, or vertex function, Λ, which is defined in (3.14), satisfies the following integral equation:

$$\Lambda \;=\; V \;+\; V\, S'^{(2)}_F\, \Lambda \quad , \qquad (6.15)$$

where $S'^{(2)}_F$ represents the obvious generalization of the two-particles Green's function for free propagators, $G_0^{(2)}$, to the two-particles Green's function with two full propagators. The complete driving term for the full amputed vertex function, Λ, is thus the sum, V, of all two-particle irreducible diagrams.

Chapter 7

Fermions

7.1 Fermions

The Lagrangian density for free massive fermions is given by

$$\mathcal{L}(x) \;=\; i\bar{\psi}(x)\,\gamma^{\mu}\partial_{\mu}\,\psi(x) \;-\; m\,\bar{\psi}(x)\,\psi(x) \quad . \tag{7.1}$$

In quantum theory, the field $\psi(x)$ and its canonical conjugate momentum

$$\pi(x) \;=\; \frac{\delta\mathcal{L}}{\delta\left(\partial^{0}\psi(x)\right)} \;=\; i\bar{\psi}(x)\,\gamma^{0} \;=\; i\psi^{\dagger}(x) \quad , \tag{7.2}$$

are postulated to be operators. In order to arrive at the Pauli exclusion principle, the field operators must satisfy equal-time *anti*commutation relations

$$\left\{\,\psi_{\alpha}(\vec{x},t)\,,\,\psi^{\dagger}_{\beta}(\vec{y},t)\,\right\} \;=\; \delta_{\alpha\beta}\,\delta^{(3)}\left(\vec{x}-\vec{y}\,\right) \tag{7.3}$$

$$\left\{\,\psi_{\alpha}(\vec{x},t)\,,\,\psi_{\beta}(\vec{y},t)\,\right\} \;=\; \left\{\,\psi^{\dagger}_{\alpha}(\vec{x},t)\,,\,\psi^{\dagger}_{\beta}(\vec{y},t)\,\right\} \;=\; 0 \quad ,$$

where α and β are the Dirac indices of the four-component fermion spinors.

7.2 Dirac spinors

We first define the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{7.4}$$

for which one easily deduces

$$\{\sigma_i\,,\,\sigma_j\} = 2\delta_{ij} \quad , \quad \mathrm{Tr}\left(\sigma_i\sigma_j\right) = 2\delta_{ij} \quad , \quad [\sigma_i\,,\,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k \quad , \tag{7.5}$$

$$\text{with} \quad \varepsilon_{ijk} \text{ totally antisymmetric, and } \varepsilon_{123} = 1 \quad .$$

A further property of the Pauli matrices, is given by

$$\left(\vec{\sigma}\cdot\vec{k}\,\right)^{2} = \sigma_i\sigma_j k_i k_j = \frac{1}{2}\,\{\sigma_i\,,\,\sigma_j\}\,k_i k_j = \mathbf{1}\,\delta_{ij}k_i k_j = \mathbf{1}\,\vec{k}^{\,2} \quad . \tag{7.6}$$

Then, by the use of the matrices (7.4), we define the gamma matrices

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad , \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \ , \tag{7.7}$$

in the Dirac representation, for which one easily deduces

$$\{\gamma^\mu \, , \, \gamma^\nu\} = 2g^{\mu\nu} \quad \text{with} \quad g^{\mu\nu} = \text{diagonal}(\,1\,,\,-1\,,\,-1\,,\,-1\,) \ , \tag{7.8}$$

$$\text{and} \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

Furthermore,

$$\left\{\gamma^\mu \, , \, \gamma^5\right\} = 0 \ . \tag{7.9}$$

The Dirac equation is given by

$$(\, i\gamma^\mu \partial_\mu \, - \, m\,)\, \psi(x) \; = \; 0 \ , \tag{7.10}$$

or in momentum space

$$(\, \not{p} \, - \, m\,)\, u(p,s) \; = \; 0 \quad \text{and} \quad (\, \not{p} + \, m\,)\, v(p,s) \; = \; 0 \ ,$$

where $\not{p} = \gamma^\mu p_\mu$, and where $u(p,s)$ and $v(p,s)$ are Dirac spinors, given by ($E = \sqrt{\vec{p}^{\,2} + m^2}$)

$$\frac{u(p)}{\sqrt{E+m}} = \begin{pmatrix} \mathbf{1} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \quad \text{and} \quad \frac{v(p)}{\sqrt{E+m}} = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \mathbf{1} \end{pmatrix} \ , \tag{7.11}$$

or, explicitly, by the use of formula (7.4)

$$\frac{u(p)}{\sqrt{E+m}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{p_3}{E+m} & \dfrac{p_1 - ip_2}{E+m} \\ \dfrac{p_1 + ip_2}{E+m} & \dfrac{-p_3}{E+m} \end{pmatrix} \quad \text{and} \quad \frac{v(p)}{\sqrt{E+m}} = \begin{pmatrix} \dfrac{p_3}{E+m} & \dfrac{p_1 - ip_2}{E+m} \\ \dfrac{p_1 + ip_2}{E+m} & \dfrac{-p_3}{E+m} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

The first column of $u(p)$ represents spin *up*, the second spin *down*, whereas the first column of $v(p)$ represents spin *down*, the second spin *up*.

We introduce, furthermore, the adjoint spinors

$$\bar{\psi} \; = \; \psi^{\dagger}\gamma^0 \ , \tag{7.12}$$

which read explicitly,

$$\frac{\bar{u}(p)}{\sqrt{E+m}} = \begin{pmatrix} 1 & 0 & \dfrac{-p_3}{E+m} & \dfrac{-p_1 + ip_2}{E+m} \\ 0 & 1 & \dfrac{-p_1 - ip_2}{E+m} & \dfrac{p_3}{E+m} \end{pmatrix} = \left(\mathbf{1} \, , \; -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right) \ ,$$

and

$$\frac{\bar{v}(p)}{\sqrt{E+m}} = \begin{pmatrix} \dfrac{p_3}{E+m} & \dfrac{p_1 - ip_2}{E+m} & -1 & 0 \\ \dfrac{p_1 + ip_2}{E+m} & \dfrac{-p_3}{E+m} & 0 & -1 \end{pmatrix} = \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \ , \, -\mathbf{1} \right) \ .$$

7.2.1 Properties of the Dirac spinors

Below we study some of the most important properties of the Dirac spinors. Many more can be found in textbooks.

The Lorentz invariant $\bar{\psi}\psi$

We determine for the first column of $u(p)$

$$\begin{aligned}
\bar{u}(p,1)\cdot u(p,1) &= (E+m)\left(\,1\,,\;\; 0\,,\;\; \frac{-p_3}{E+m}\,,\;\; \frac{-p_1+ip_2}{E+m}\,\right)\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_3}{E+m} \\ \dfrac{p_1+ip_2}{E+m}\end{pmatrix} \\
&= (E+m)\left\{1\,+\,0\,+\,\frac{-p_3^2}{(E+m)^2}\,+\,\frac{-p_1^2-p_2^2}{(E+m)^2}\right\} \\
&= \frac{(E+m)^2\,-\,\vec{p}^{\,2}}{E+m}\;=\;2m\quad . \qquad (7.13)
\end{aligned}$$

For the second row of $\bar{u}(p)$ and the first column of $u(p)$, we obtain

$$\begin{aligned}
\bar{u}(p,2)\cdot u(p,1) &= (E+m)\left(\,0\,,\;\; 1\,,\;\; \frac{-p_1-ip_2}{E+m}\,,\;\; \frac{p_3}{E+m}\,\right)\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_3}{E+m} \\ \dfrac{p_1+ip_2}{E+m}\end{pmatrix} \\
&= (E+m)\left\{0\,+\,0\,+\,\frac{-p_1p_3-ip_2p_3}{(E+m)^2}+\frac{p_3p_1+ip_3p_2}{(E+m)^2}\right\} \\
&= 0\quad . \qquad (7.14)
\end{aligned}$$

We may combine formulae (7.13) and (7.14), together with the other two possibilities, in

$$\begin{aligned}
\bar{u}(p)\cdot u(p) &= (E+m)\left(\,\mathbf{1}\,,\;\; -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\begin{pmatrix} \mathbf{1} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix} \\
&= (E+m)\left\{\mathbf{1}\times\mathbf{1}\,-\,\frac{(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2}\right\}\;=\;(E+m)\left\{\mathbf{1}\,-\,\frac{\mathbf{1}\,\vec{p}^{\,2}}{(E+m)^2}\right\} \\
&= \frac{(E+m)^2-\vec{p}^{\,2}}{E+m}\,\mathbf{1}\;=\;2m\,\mathbf{1}\quad , \qquad (7.15)
\end{aligned}$$

where we used relations (7.6), (7.11) and (7.12).

Similarly,

$$\begin{aligned}
\bar{v}(p)\cdot u(p) &= (E+m)\left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \quad , \; -\mathbf{1} \right) \left(\begin{array}{c} \mathbf{1} \\ \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{array} \right) \\
&= \frac{\vec{\sigma}\cdot\vec{p}}{E+m} - \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \;=\; 0 \quad .
\end{aligned} \tag{7.16}$$

In general, we find

$$\begin{aligned}
&\bar{u}(p,s)\cdot u(p,s'\,) \;=\; -\bar{v}(p,s)\cdot v(p,s'\,) \;=\; 2m\,\delta_{ss'} \\
\text{and}\quad &\bar{u}(p,s)\cdot v(p,s'\,) \;=\; \bar{v}(p,s)\cdot u(p,s'\,) \;=\; 0 \quad .
\end{aligned} \tag{7.17}$$

The Lorentz vector $\psi^\dagger\psi$

We also determine the probability density for $u(p)$, given by

$$u^\dagger(p)\cdot u(p) \;=\; (E+m)\left(\mathbf{1}\,, \; \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right) \left(\begin{array}{c} \mathbf{1} \\ \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{array} \right) \;=\; 2E\,\mathbf{1} \quad . \tag{7.18}$$

Consequently, for the various spin combinations one has

$$u^\dagger(p,s)\cdot u(p,s'\,) \;=\; 2E\,\delta_{ss'} \quad \text{and, similarly,} \quad v^\dagger(p,s)\cdot v(p,s'\,) \;=\; 2E\,\delta_{ss'} \quad .$$

Furthermore

$$u^\dagger(p)\cdot v(p) \;=\; (E+m)\left(\mathbf{1}\,, \; \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right) \left(\begin{array}{c} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \mathbf{1} \end{array} \right) \;=\; 2\vec{\sigma}\cdot\vec{p} \quad . \tag{7.19}$$

Consequently, for the various spin combinations one has

$$\begin{aligned}
&u^\dagger(p,1)\cdot v(p,1) \;=\; -u^\dagger(p,2)\cdot v(p,2) \;=\; 2p^3 \quad . \\
\text{and}\quad &u^\dagger(p,1)\cdot v(p,2) \;=\; \left(u^\dagger(p,2)\cdot v(p,1)\right)^* \;=\; 2\left(p^1 - ip^2\right) \quad .
\end{aligned}$$

The spin sum

Next, we consider the following matrix which we construct from the first column of $u(p)$ and the first row of $\bar{u}(p)$.

$$u(p,1)\,\bar{u}(p,1) \;=\; (E+m)\left(\begin{array}{c} 1 \\ \\ 0 \\ \\ \dfrac{p_3}{E+m} \\ \dfrac{p_1+ip_2}{E+m} \end{array} \right) \left(1\,, \quad 0\,, \quad \frac{-p_3}{E+m}\,, \quad \frac{-p_1+ip_2}{E+m} \right)$$

$$= (E+m)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{p_3}{E+m} & 0 & 0 & 0 \\ \frac{p_1+ip_2}{E+m} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & \frac{-p_3}{E+m} & \frac{-p_1+ip_2}{E+m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= (E+m)\begin{pmatrix} 1 & 0 & \frac{-p_3}{E+m} & \frac{-p_1+ip_2}{E+m} \\ 0 & 0 & 0 & 0 \\ \frac{p_3}{E+m} & 0 & \frac{-p_3^2}{(E+m)^2} & \frac{-p_3p_1+ip_3p_2}{(E+m)^2} \\ \frac{p_1+ip_2}{E+m} & 0 & \frac{-p_1p_3-ip_2p_3}{(E+m)^2} & \frac{-p_1^2-p_2^2}{(E+m)^2} \end{pmatrix}, \tag{7.20}$$

and, moreover, a similar matrix which we construct from the second column of $u(p)$ and the second row of $\bar{u}(p)$.

$$u(p,2)\,\bar{u}(p,2) \;=\; (E+m)\begin{pmatrix} 0 \\ 1 \\ \frac{p_1-ip_2}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix}\left(\,0\,,\;\;1\,,\;\;\frac{-p_1-ip_2}{E+m}\,,\;\;\frac{p_3}{E+m}\,\right)$$

$$= (E+m)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{-p_1-ip_2}{E+m} & \frac{p_3}{E+m} \\ 0 & \frac{p_1-ip_2}{E+m} & \frac{-p_1^2-p_2^2}{(E+m)^2} & \frac{p_1p_3-ip_2p_3}{(E+m)^2} \\ 0 & \frac{-p_3}{E+m} & \frac{p_3p_1+ip_3p_2}{(E+m)^2} & \frac{-p_3^2}{(E+m)^2} \end{pmatrix}. \tag{7.21}$$

The sum of the two matrices (7.20) and (7.21) yields

$$\sum_s u(p,s)\,\bar{u}(p,s) \;=\; u(p,1)\,\bar{u}(p,1) \;+\; u(p,2)\,\bar{u}(p,2) \;=$$

$$= (E+m)\begin{pmatrix} 1 & 0 & \frac{-p_3}{E+m} & \frac{-p_1+ip_2}{E+m} \\ 0 & 1 & \frac{-p_1-ip_2}{E+m} & \frac{p_3}{E+m} \\ \frac{p_3}{E+m} & \frac{p_1-ip_2}{E+m} & \frac{-\vec{p}^{\,2}}{(E+m)^2} & 0 \\ \frac{p_1+ip_2}{E+m} & \frac{-p_3}{E+m} & 0 & \frac{-\vec{p}^{\,2}}{(E+m)^2} \end{pmatrix}$$

$$= (E+m)\begin{pmatrix} \mathbf{1} & \dfrac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} & \dfrac{-E^2+m^2}{(E+m)^2}\,\mathbf{1} \end{pmatrix} = \begin{pmatrix} (E+m)\,\mathbf{1} & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & (-E+m)\,\mathbf{1} \end{pmatrix}$$

$$= m + E\gamma^0 - \vec{\gamma}\cdot\vec{p} = m + \gamma^\mu p_\mu \quad . \tag{7.22}$$

In order to understand fully the result (7.22), we consider the following. The spinor $u(p,s)$ can be represented by a column of four components. In the expression (7.11) we join the two possible spin states in a matrix with two columns of each four components

$$u(p) = \begin{pmatrix} u_1(p,1) & u_1(p,2) \\ u_2(p,1) & u_2(p,2) \\ u_3(p,1) & u_3(p,2) \\ u_4(p,1) & u_4(p,2) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \\ u_{41} & u_{42} \end{pmatrix} \quad . \tag{7.23}$$

Now, when we take the product of $u(p)$ and $\bar{u}(p)$, then we obtain

$$u(p)\bar{u}(p) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \\ u_{41} & u_{42} \end{pmatrix} \begin{pmatrix} \bar{u}_{11} & \bar{u}_{21} & \bar{u}_{31} & \bar{u}_{41} \\ \bar{u}_{12} & \bar{u}_{22} & \bar{u}_{32} & \bar{u}_{42} \end{pmatrix} = \tag{7.24}$$

$$= \begin{pmatrix} u_{11}\bar{u}_{11}+u_{12}\bar{u}_{12} & u_{11}\bar{u}_{21}+u_{12}\bar{u}_{22} & u_{11}\bar{u}_{31}+u_{12}\bar{u}_{32} & u_{11}\bar{u}_{41}+u_{12}\bar{u}_{42} \\ u_{21}\bar{u}_{11}+u_{22}\bar{u}_{12} & u_{21}\bar{u}_{21}+u_{22}\bar{u}_{22} & u_{21}\bar{u}_{31}+u_{22}\bar{u}_{32} & u_{21}\bar{u}_{41}+u_{22}\bar{u}_{42} \\ u_{31}\bar{u}_{11}+u_{32}\bar{u}_{12} & u_{31}\bar{u}_{21}+u_{32}\bar{u}_{22} & u_{31}\bar{u}_{31}+u_{32}\bar{u}_{32} & u_{31}\bar{u}_{41}+u_{32}\bar{u}_{42} \\ u_{41}\bar{u}_{11}+u_{42}\bar{u}_{12} & u_{41}\bar{u}_{21}+u_{42}\bar{u}_{22} & u_{41}\bar{u}_{31}+u_{42}\bar{u}_{32} & u_{41}\bar{u}_{41}+u_{42}\bar{u}_{42} \end{pmatrix} \quad .$$

The matrix elements of the matrices (7.20) and (7.21) are respectively given by

$$[u(p,1)\,\bar{u}(p,1)]_{\alpha\beta} = u_\alpha(p,1)\,\bar{u}_\beta(p,1) = u_{\alpha 1}\bar{u}_{\beta 1}$$

$$\text{and} \quad [u(p,2)\,\bar{u}(p,2)]_{\alpha\beta} = u_\alpha(p,2)\,\bar{u}_\beta(p,2) = u_{\alpha 2}\bar{u}_{\beta 2} \quad .$$

Their sum (7.22) follows thus by considering

$$[u(p,1)\,\bar{u}(p,1) + u(p,2)\,\bar{u}(p,2)]_{\alpha\beta} = u_{\alpha 1}\bar{u}_{\beta 1} + u_{\alpha 2}\bar{u}_{\beta 2} \quad ,$$

which represents exactly the element $(\alpha,\,\beta)$ of the matrix shown in formula (7.24). Consequently, using the lefthand side of equation (7.24), we may also determine the spin sum (7.22) from

$$\left[\sum_s u(p,s)\,\bar{u}(p,s)\right]_{\alpha\beta} = [u(p,1)\,\bar{u}(p,1) + u(p,2)\,\bar{u}(p,2)]_{\alpha\beta} =$$

$$= \left[(E+m) \begin{pmatrix} \mathbf{1} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \begin{pmatrix} \mathbf{1}\, , & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \right]_{\alpha\beta}$$

$$= \left[(E+m) \begin{pmatrix} \mathbf{1} & \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \frac{-(\vec{\sigma}\cdot\vec{p})^2}{(E+m)^2} \end{pmatrix} \right]_{\alpha\beta} = [\not{p} + m]_{\alpha\beta} \quad . \tag{7.25}$$

When we repeat the calculus (7.25) for $v(p)$ (7.11), we obtain

$$\sum_s v(p,s)\,\bar{v}(p,s) = \not{p} - m \quad . \tag{7.26}$$

Matrix elements

In calculus, one often encounters the following expression for the matrix elements of a 4×4 matrix A sandwiched between spinors.

$$\sum_{s_i,s_f} \left|\bar{u}(p_f,s_f)\; A\, u(p_i,s_i)\right|^2 \quad . \tag{7.27}$$

The product of a 4×4 matrix A with a 4-component column, results in a 4-component column, with matrix elements

$$[A\, u(p_i,s_i)]_\alpha = A_{\alpha\beta}\, u_\beta(p_i,s_i) \quad , \tag{7.28}$$

where a summation from 1 to 4 over the repeated index β is understood. The product of the column (7.28) with the 4-component row $\bar{u}(p_f,s_f)$, results in the sum

$$\bar{u}_\alpha(p_f,s_f)\; [A\, u(p_i,s_i)]_\alpha = \bar{u}_\alpha(p_f,s_f)\; A_{\alpha\beta}\, u_\beta(p_i,s_i) \quad , \tag{7.29}$$

where a summation from 1 to 4 over both repeated indices α and β is understood.

For the expression (7.27), we find then

$$\sum_{s_i,s_f} \left|\bar{u}(p_f,s_f)\; A\, u(p_i,s_i)\right|^2 = \sum_{s_i,s_f} [\bar{u}(p_f,s_f)\; A\, u(p_i,s_i)]\; [\bar{u}(p_f,s_f)\; A\, u(p_i,s_i)]^\dagger =$$

$$= \sum_{s_i,s_f} \bar{u}_\alpha(p_f,s_f)\; A_{\alpha\beta}\, u_\beta(p_i,s_i)\, u^\dagger_\gamma(p_i,s_i)\; A^\dagger_{\gamma\delta}\, \bar{u}^\dagger_\delta(p_f,s_f)$$

$$= \sum_{s_i,s_f} \bar{u}_\alpha(p_f,s_f)\; A_{\alpha\beta}\, u_\beta(p_i,s_i)\, \left(\bar{u}(p_i,s_i)\,\gamma^0\right)_\gamma\; A^\dagger_{\gamma\delta}\, \left(\gamma^0 u(p_f,s_f)\right)_\delta$$

$$= \sum_{s_i,s_f} \bar{u}_\alpha(p_f,s_f)\; A_{\alpha\beta}\, u_\beta(p_i,s_i)\; \bar{u}_\sigma(p_i,s_i)\,\gamma^0_{\sigma\gamma}\, A^\dagger_{\gamma\delta}\,\gamma^0_{\delta\tau} u_\tau(p_f,s_f)$$

$$= \sum_{s_i,s_f} A_{\alpha\beta}\, u_\beta(p_i,s_i)\; \bar{u}_\sigma(p_i,s_i)\,\gamma^0_{\sigma\gamma}\, A^\dagger_{\gamma\delta}\,\gamma^0_{\delta\tau} u_\tau(p_f,s_f)\; \bar{u}_\alpha(p_f,s_f)$$

$$= A_{\alpha\beta} \left\{ \sum_{s_i} u_\beta(p_i,s_i)\; \bar{u}_\sigma(p_i,s_i) \right\} \gamma^0_{\sigma\gamma}\, A^\dagger_{\gamma\delta}\,\gamma^0_{\delta\tau} \left\{ \sum_{s_f} u_\tau(p_f,s_f)\; \bar{u}_\alpha(p_f,s_f) \right\}$$

$$= A_{\alpha\beta}\, \{\, \not{p}_i \,+\, m \,\}_{\beta\sigma}\, \gamma^0_{\sigma\gamma}\, A^\dagger_{\gamma\delta}\, \gamma^0_{\delta\tau}\, \{\, \not{p}_i \,+\, m \,\}_{\tau\alpha}$$

$$= \mathrm{Tr}\left(A\, \{\, \not{p}_i \,+\, m \,\}\, \gamma^0\, A^\dagger\, \gamma^0\, \left\{\, \not{p}_f \,+\, m \,\right\} \right) \quad . \tag{7.30}$$

In deducing the result (7.30), we used formulae (7.7), (7.12) and (7.25), and furthermore

$$u^\dagger(p,s) = u^\dagger(p,s)\gamma^0\gamma^0 = \bar{u}(p,s)\gamma^0 \quad \text{and} \quad \bar{u}^\dagger(p,s) = \left(u^\dagger(p,s)\gamma^0 \right)^\dagger = \gamma^0 u(p,s).$$

When we repeat the calculus (7.30) for $v(p)$ (7.11), also using formula (7.26), we obtain

$$\sum_{s_i,s_f} \left| \bar{v}\,(p_f, s_f)\; A\, v\,(p_i, s_i) \right|^2 \;=\; \mathrm{Tr}\left(A\, \{\, \not{p}_i \,-\, m \,\}\, \gamma^0\, A^\dagger\, \gamma^0\, \left\{\, \not{p}_f \,-\, m \,\right\} \right) \quad . \tag{7.31}$$

7.2.2 Dirac traces

Repeatedly, one encounters in calculus traces of products of gamma matrices (7.7). Here we will study some properties. We start by studying the product of n gamma matrices, using relation (7.9).

$$\begin{aligned} \gamma^{\mu_1} \cdots \gamma^{\mu_n} \;&=\; \gamma^{\mu_1} \cdots \gamma^{\mu_n}\, \gamma^5\, \gamma^5 \;=\; -\,\gamma^{\mu_1} \cdots \gamma^{\mu_{n-1}}\, \gamma^5\, \gamma^{\mu_n}\, \gamma^5 \;= \\ &=\; (-1)^2\, \gamma^{\mu_1} \cdots \gamma^{\mu_{n-2}}\, \gamma^5\, \gamma^{\mu_{n-1}}\, \gamma^{\mu_n}\, \gamma^5 \\ &\;\;\vdots \\ &=\; (-1)^n\, \gamma^5\, \gamma^{\mu_1} \cdots \gamma^{\mu_n}\, \gamma^5 \quad . \end{aligned} \tag{7.32}$$

Then, by also using $\mathrm{Tr}(BA) = \mathrm{Tr}(AB)$, we find for the trace of the product of n gamma matrices

$$\begin{aligned} \mathrm{Tr}\left(\gamma^{\mu_1} \cdots \gamma^{\mu_n} \right) \;&=\; \mathrm{Tr}\left(\gamma^{\mu_1} \cdots \gamma^{\mu_n}\, \gamma^5\, \gamma^5 \right) \;=\; (-1)^n\, \mathrm{Tr}\left(\gamma^5\, \gamma^{\mu_1} \cdots \gamma^{\mu_n}\, \gamma^5 \right) \;= \\ &=\; (-1)^n\, \mathrm{Tr}\left(\gamma^{\mu_1} \cdots \gamma^{\mu_n}\, \gamma^5\, \gamma^5 \right) \;=\; (-1)^n\, \mathrm{Tr}\left(\gamma^{\mu_1} \cdots \gamma^{\mu_n} \right) \quad . \end{aligned} \tag{7.33}$$

Hence, for n odd, we must conclude that the trace (7.33) vanishes.

Next, by also using the anticommutation relation (7.8) for gamma matrices, we deduce

$$\mathrm{Tr}\left(\not{a}\not{b} \right) \;=\; a_\mu b_\nu \mathrm{Tr}\left(\gamma^\mu\, \gamma^\nu \right) \;=\; \frac{1}{2}\, a_\mu b_\nu \mathrm{Tr}\left(\{ \gamma^\mu \,,\, \gamma^\nu \} \right) \;=\; a_\mu b_\nu g^{\mu\nu}\, \mathrm{Tr}\left(\mathbf{1}_{4\times 4} \right) \;=\; 4\, a \cdot b. \tag{7.34}$$

7.3 Coulomb scattering

Let us consider the case of electrons scattered off a static source which is placed at the center of our coordinate system. The static source causes the electron to deflect, hence change the direction of its linear momentum, but does not affect its total energy, hence does not change the modulus of its linear momentum.

In general, an incoming electron in the spin state s_i is described by a wave packet of the form

$$\psi_i(x) \;=\; \mathcal{N} \int d^3k\, \phi\left(\vec{k}\,, s_i \right)\, e^{-ikx} \;=\; \mathcal{N} \int d^3k\, \phi\left(\vec{k}\,, s_i \right)\, e^{-iE\left(\vec{k} \right) \,+\, i\vec{k}\cdot\vec{x}} \quad , \tag{7.35}$$

where $\mathcal{N}$ represents a normalisation factor which we will determine later on. Here, we assume that the incoming (and outgoing) electron has a sharp momentum distribution, which we approximate by a Dirac delta function, *i.e.*

$$\psi_i(x) \; = \; \mathcal{N} \int d^3k \, u\left(\vec{k} \, , s_i\right) \, \delta^{(3)}\left(\vec{k} \, - \, \vec{p}_i\right) \, e^{-ikx} \; = \; \mathcal{N} \, u\left(\vec{p}_i \, , s_i\right) \, e^{-ip_ix} \quad . \tag{7.36}$$

The wave function $\psi_i(x)$ is a solution of the Dirac equation, (7.11) given by

$$\left(i\gamma^\mu \partial_\mu \, - \, m\right) \psi_i(x) \; = \; 0 \quad , \tag{7.37}$$

which leads for the expression (7.36) to

$$\left(\gamma^\mu p_{i,\mu} \, - \, m\right) \, u\left(\vec{p}_i \, , s_i\right) \; = \; 0 \quad . \tag{7.38}$$

The latter equation is solved by the relation (7.11) for u.

With respect to the normalisation factor $\mathcal{N}$, we must determine

$$\int d^3x \, \psi_i^\dagger(x) \, \psi_i(x) \; = \; |\mathcal{N}|^2 \int d^3x \, u^\dagger\left(\vec{p}_i \, , s_i\right) \, u\left(\vec{p}_i \, , s_i\right) \quad .$$

Also using relations (7.18), we end up with

$$\int d^3x \, \psi_i^\dagger(x) \, \psi_i(x) \; = \; 2E_i \; |\mathcal{N}|^2 \int d^3x \, 1 \quad , \tag{7.39}$$

which is a divergent expression.

In order to deal with the result (7.39), we place the scattering experiment in a big box of volume V. We will find later on that the size of V has no influence on the final result for the scattering cross section. We obtain then for the properly normalised wave function which describes the incoming electron

$$\psi_i(x) \; = \; = \; \sqrt{\frac{1}{2E_iV}} \, u\left(\vec{p}_i \, , s_i\right) \, e^{-ip_ix} \quad . \tag{7.40}$$

For the outgoing electron, we have similarly

$$\psi_f(x) \; = \; = \; \sqrt{\frac{1}{2E_fV}} \, u\left(\vec{p}_f \, , s_f\right) \, e^{-ip_fx} \quad . \tag{7.41}$$

The interaction of the electron with the electromagnetic field of any source, is given by

$$ie \, \bar{\psi}_f(x) \, \gamma^\mu A_\mu(x) \, \psi_i(x) \quad . \tag{7.42}$$

The matrix element which describes the transition from the initial state i to the final state f, is determined from

$$T_{fi} \; = \; ie \int d^4x \, \bar{\psi}_f(x) \, \gamma^\mu A_\mu(x) \, \psi_i(x) \quad \text{for} \quad f \neq i \quad . \tag{7.43}$$

For the wave functions (7.40) and (7.41) this gives

$$T_{f \neq i} \; = \; ie \, \frac{1}{2V} \sqrt{\frac{1}{E_fE_i}} \int d^4x \, \bar{u}\left(\vec{p}_f \, , s_f\right) \, \gamma^\mu A_\mu(x) \, u\left(\vec{p}_i \, , s_i\right) \, e^{i\left(p_f \, - \, p_i\right) x} \quad . \tag{7.44}$$

The electromagnetic field of a static source with electric charge Ze is given by the Coulomb potential, *i.e.*

$$A_0(x) \;=\; \frac{Ze}{4\pi\,|\vec{x}\,|} \quad \text{and} \quad \vec{A}(x) \;=\; 0 \quad . \tag{7.45}$$

This leads for the transition matrix element (7.44) to

$$T_{f\neq i} \;=\; ie\,\frac{1}{2V}\sqrt{\frac{1}{E_f E_i}}\;\bar{u}\,(\vec{p}_f\,,s_f)\;\gamma^0\,u\,(\vec{p}_i\,,s_i)\int d^4x\,\frac{Ze}{4\pi\,|\vec{x}\,|}\,e^{i\,(p_f\,-\,p_i)\,x} \quad . \tag{7.46}$$

The space integral in (7.46) gives

$$\int d^3x\,\frac{Ze}{4\pi\,|\vec{x}\,|}\,e^{-i\,(\vec{p}_f\,-\,\vec{p}_i\,)\,\cdot\,\vec{x}} \;=\; \frac{Ze}{|\vec{p}_f - \vec{p}_i\,|^2} \quad ,$$

whereas the time integral yields

$$\int dt\,e^{i\,(E_f\,-\,E_i)\;t} \;=\; 2\pi\,\delta\,(E_f - E_i) \quad .$$

For formula (7.46) we obtain

$$T_{f\neq i} \;=\; i\,\frac{Ze^2}{2V}\sqrt{\frac{1}{E_f E_i}}\,\frac{\bar{u}\,(\vec{p}_f\,,s_f)\;\gamma^0\,u\,(\vec{p}_i\,,s_i)}{|\vec{p}_f - \vec{p}_i\,|^2}\,2\pi\,\delta\,(E_f - E_i) \quad . \tag{7.47}$$

The probability for a transtion $\vec{p}_i \to \vec{p}_f$ to occur is given by the modulus squared of the matrix element (7.47) times the number of states available.

7.3.1 Number of states

For a free Schrödinger particle which, in one dimension, is confined to the interval running from $x = 0$ to $x = X$, we have one-particle wave functions

$$\sin\,(kx) \quad ,$$

which satisfy the boundary condition at $x = 0$ and for which the boundary condition at $x = X$ yields the spectrum

$$kX \;=\; n\pi \quad \text{for} \quad n = 0,\,\pm 1,\,\pm 2,\,\ldots \quad .$$

However, the solution for $n = 0$ vanishes, hence does not make part of the spectrum of free Schrödinger particles. Furthermore,

$$\sin\,(-kx) \;=\; -\sin\,(kx) \quad \text{and} \quad \sin\,(kx) \quad ,$$

represent the same particle distribution. Moreover,

$$\sin\,(kx) \;=\; \frac{1}{2i}\left(e^{ikx}\; - \;e^{-ikx}\right) \quad ,$$

implying that 50% represents a wave in the forward direction, an other 50% a wave in the backward direction. Consequently, for the full spectrum of free Schrödinger particles confined to the interval $(0, X)$, we may restrict the values of n to positive integers, such that

$$n = \frac{kX}{2\pi} \quad \text{for} \quad n = 1,\, 2,\, \ldots \quad . \tag{7.48}$$

It is an easy task to determine the number of states in an interval $(k\ ,\ k + dk)$. The result reads

$$N(k, k + dk) = n(k + dk) - n(k) = \frac{X}{2\pi}\, dk \quad . \tag{7.49}$$

In a two-dimensional box, given by the area

$$0 < x < X \quad \text{for} \quad 0 < y < Y \quad ,$$

we have one-particle wave functions

$$\sin\left(k_x x\right) \sin\left(k_y y\right) \quad ,$$

which satisfy the boundary condition at $(x = 0\,,\, y = 0)$ and for which the boundary conditions at $x = X$ and $y = Y$ yield the spectrum

$$k_x X = n_x \pi \quad \text{and} \quad k_y Y = n_y \pi \quad \text{for} \quad n_x, n_y = 0, \pm 1 \pm 2 \ldots \quad ,$$

hence, folowing the same reasoning as before in the one-dimensional case, we obtain

$$n_x = \frac{k_x X}{2\pi} \quad \text{and} \quad n_y = \frac{k_y Y}{2\pi} \quad . \tag{7.50}$$

The number of states for which the x-component of the linear momentum is in the interval $(k_x\ ,\ k_x + dk_x)$, whereas its y-component of the linear momentum is in the interval $(k_y\ ,\ k_y + dk_y)$, is given by

$$dn_x dn_y = \frac{XY}{(2\pi)^2}\, dk_x dk_y \quad . \tag{7.51}$$

The generalisation to three dimensions is straightforward

$$dN = \frac{V}{(2\pi)^3}\, d^3k \quad , \tag{7.52}$$

where V is the volume of the box where the particle is confined.

7.3.2 Transition probability

Using formulae (7.47) and (7.52), we obtain for the transition probability into a state which has its final linear momentum in the volume

$$(k_{f,1}\ ,\ k_{f,1} + dk_{f,1}) \quad , \quad (k_{f,2}\ ,\ k_{f,2} + dk_{f,2}) \quad , \quad (k_{f,3}\ ,\ k_{f,3} + dk_{f,3}) \quad , \tag{7.53}$$

the transition probability

$$\left|T_{f \neq i}\right|^2 \frac{V}{(2\pi)^3}\, d^3p_f =$$

$$= \left(\frac{Ze^2}{2V}\right)^2 \frac{1}{E_f E_i} \frac{\left|\bar{u}\left(\vec{p}_f\,,\, s_f\right) \gamma^0\, u\left(\vec{p}_i\,,\, s_i\right)\right|^2}{\left|\vec{p}_f - \vec{p}_i\right|^4} \left|2\pi\delta\left(E_f - E_i\right)\right|^2 \frac{V}{(2\pi)^3}\, d^3p_f \quad . \tag{7.54}$$

The square of the Dirac delta function can be handled as follows. Instead of considering an infinite interval of time for a transition to take place, we may, more realistically, but mathematically more difficult, have considered a finite interval of time of a period T. The Dirac delta function becomes then

$$2\pi\delta\left(E_f - E_i\right) \;=\; \lim_{T\to\infty} \int_{-T/2}^{T/2} dt\, e^{i\left(E_f - E_i\right)\, t} \quad . \tag{7.55}$$

Furthermore, since both delta functions express the same, We may put in one of the two $E_f = E_i$. In that case, we find for (7.55) the result

$$2\pi\delta(0) \;=\; \lim_{T\to\infty} \int_{-T/2}^{T/2} dt\, 1 \;=\; T \quad , \tag{7.56}$$

whereas, formula (7.54) turns into

$$\begin{aligned} &\left|T_{f\neq i}\right|^2 \frac{V}{(2\pi)^3}\, d^3p_f \;= \\ &= \left(\frac{Ze^2}{2V}\right)^2 \frac{1}{E_fE_i} \frac{\left|\bar{u}\left(\vec{p}_f\,, s_f\right)\, \gamma^0\, u\left(\vec{p}_i\,, s_i\right)\right|^2}{\left|\vec{p}_f - \vec{p}_i\,\right|^4}\, 2\pi\delta\left(E_f - E_i\right) \frac{VT}{(2\pi)^3}\, d^3p_f \quad . \end{aligned} \tag{7.57}$$

For the transition rate, which is the transition probability per unit of time, we have now

$$\begin{aligned} &\frac{\left|T_{f\neq i}\right|^2}{T} \frac{V}{(2\pi)^3}\, d^3p_f \;= \\ &= \left(\frac{Ze^2}{2V}\right)^2 \frac{1}{E_fE_i} \frac{\left|\bar{u}\left(\vec{p}_f\,, s_f\right)\, \gamma^0\, u\left(\vec{p}_i\,, s_i\right)\right|^2}{\left|\vec{p}_f - \vec{p}_i\,\right|^4}\, 2\pi\delta\left(E_f - E_i\right) \frac{V}{(2\pi)^3}\, d^3p_f \quad . \end{aligned} \tag{7.58}$$

7.3.3 Flux of incoming particles

In order to find the differential cross section, we must still determine the flux of incoming particles. Let us choose $\vec{p}_i$ in the direction of the positive z axis. The flux of incoming particles is then, also using (7.7), (7.11) and (7.40), given by

$$\begin{aligned} J^3(x) \;&=\; \bar{\psi}_i(x)\, \gamma^3\, \psi_i(x) \;= \\ &= \frac{1}{2E_iV}\, \bar{u}\left(\vec{p}_i\,, s_i\right)\, \gamma^3\, u\left(\vec{p}_i\,, s_i\right) \;=\; \frac{1}{2E_iV}\, 2p_{i,z} \quad . \end{aligned} \tag{7.59}$$

For a general direction, we find for the flux

$$\vec{J}_i \;=\; \frac{\vec{p}_i}{E_iV} \;=\; \frac{\vec{v}_i}{V} \quad , \tag{7.60}$$

where $\vec{v}_i$ represents the velocity of the incoming particles.

7.3.4 Differential cross section

The cross section for a final state which describes an outgoing electron with linear momentum $\vec{p}_f$, is defined as the transition rate (7.58) divided by the modulus of the flux of the incoming electrons.

$$
\begin{aligned}
d\sigma \; &= \; \frac{\left|T_{f\neq i}\right|^2}{T\left|\vec{J}_i\right|}\,\frac{V}{(2\pi)^3}\,d^3p_f \; = \\
&= \left(\frac{Ze^2}{2}\right)^2 \frac{1}{E_f\left|\vec{p}_i\right|}\,\frac{\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2}{\left|\vec{p}_f-\vec{p}_i\,\right|^4}\,2\pi\delta\left(E_f-E_i\right)\frac{1}{(2\pi)^3}\,d^3p_f \\
&= \left(\frac{Ze^2}{4\pi}\right)^2 \frac{1}{E_f\left|\vec{p}_i\right|}\,\frac{\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2}{\left|\vec{p}_f-\vec{p}_i\,\right|^4}\,\delta\left(E_f-E_i\right)\,p_f^2dp_fd\Omega \quad .
\end{aligned}
\tag{7.61}
$$

Using, moreover

$$
p_fdp_f \; = \; E_fdE_f \quad ,
$$

we arrive at the differential cross section

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} \; &= \; \int dE_f \left(\frac{Ze^2}{4\pi}\right)^2 \frac{1}{E_f\left|\vec{p}_i\right|}\,\frac{\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2}{\left|\vec{p}_f-\vec{p}_i\,\right|^4}\,\delta\left(E_f-E_i\right)\,p_fE_f \\
&= \; \left(\frac{Ze^2}{4\pi}\right)^2 \frac{\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2}{\left|\vec{p}_f-\vec{p}_i\,\right|^4} \quad .
\end{aligned}
\tag{7.62}
$$

7.3.5 Averaging over spins

When in experiment one has no information on the polarisation of the electrons, then one must avarage over the possible spin states of the incoming electrons and sum over the possible spin states of the outgoing electrons.

$$
\frac{d\sigma}{d\Omega} \; = \; \left(\frac{Ze^2}{4\pi}\right)^2 \frac{1}{2}\sum_{s_i,s_f} \frac{\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2}{\left|\vec{p}_f-\vec{p}_i\,\right|^4} \quad .
\tag{7.63}
$$

$\boldsymbol{\sum_{s_i,s_f}\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2}$

In order to determine the matrix element

$$
\sum_{s_i,s_f}\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2 \quad ,
$$

we remember formula (7.30) for $A=\gamma^0$. This gives (${\gamma^0}^\dagger=\gamma^0$)

$$
\sum_{s_i,s_f}\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^0\,u\left(\vec{p}_i\,,s_i\right)\right|^2 \; = \; \mathrm{Tr}\left(\gamma^0\,\left\{m\;+\;\not{p}_i\right\}\,\gamma^0\,\gamma^0\,\gamma^0\,\left\{m\;+\;\not{p}_f\right\}\right) \quad .
$$

Furthermore, applying formulas (7.33), (7.34) and $(\gamma^0)^2 = \mathbf{1}$

$$\sum_{s_i, s_f} \left| \bar{u}\left(\vec{p}_f\,, s_f\right) \gamma^0 \, u\left(\vec{p}_i\,, s_i\right) \right|^2 \; = \; \mathrm{Tr}\left(\gamma^0 \, \{ m \, + \, \not{p}_i \} \, \gamma^0 \, \left\{ m \, + \, \not{p}_f \, \right\} \right) \; =$$

$$= \; \mathrm{Tr}\left(m^2 \mathbf{1} \, + \, m\not{p}_f \, + \, m\gamma^0 \not{p}_i \gamma^0 \, + \, \gamma^0 \not{p}_i \gamma^0 \not{p}_f \right) \; = \; 4m^2 \, + \, \mathrm{Tr}\left(\gamma^0 \gamma^\mu \gamma^0 \gamma^\nu \right) \, p_{i\mu} \, p_{f\nu}$$

$$= \; 4m^2 \, + \, \mathrm{Tr}\left(\left(\left\{ \gamma^0 \,, \, \gamma^\mu \right\} \, - \, \gamma^\mu \gamma^0 \right) \gamma^0 \gamma^\nu \right) \, p_{i\mu} \, p_{f\nu}$$

$$= \; 4m^2 \, + \, \mathrm{Tr}\left(2g^{0\mu} \gamma^0 \gamma^\nu \, - \, \gamma^\mu \gamma^\nu \right) \, p_{i\mu} \, p_{f\nu} \; = \; 4m^2 \, + \, 8E_i E_f \, - \, 4p_i \cdot p_f \quad . \tag{7.64}$$

7.3.6 Differential cross section continued

For the total energy E of an electron which scatters off a fixed target, and its linear momentum $|\vec{p}\,|$, one has

$$E \; = \; E_i \; = \; E_f \quad \text{and} \quad |\vec{p}\,| \; = \; |\vec{p}_i| \; = \; |\vec{p}_f| \quad .$$

Furthermore, we define the scattering angle θ by

$$\vec{p}_i \cdot \vec{p}_f \; = \; |\vec{p}\,|^2 \cos(\theta) \quad .$$

Hence,

$$4m^2 + 8E_i E_f - 4p_i \cdot p_f = 4m^2 + 4E^2 + 4\,|\vec{p}\,|^2 \cos(\theta) =$$

$$= 8E^2 - 4\,|\vec{p}\,|^2 \left(1 - \cos(\theta)\right) = 8E^2 - 8\,|\vec{p}\,|^2 \sin^2\left(\frac{\theta}{2}\right) \quad . \tag{7.65}$$

Also,

$$|\vec{p}_f - \vec{p}_i\,|^2 = 2\,|\vec{p}\,|^2 \left(1 - \cos(\theta)\right) = 4\,|\vec{p}\,|^2 \sin^2\left(\frac{\theta}{2}\right) \quad . \tag{7.66}$$

For the differential cross section (7.63) for Coulomb scattering, we deduce then ($\beta^2 = |\vec{p}\,|^2 / E^2$)

$$\frac{d\sigma}{d\Omega} \; = \; \left(\frac{Ze^2}{4\pi}\right)^2 \frac{E^2 - |\vec{p}\,|^2 \sin^2\left(\frac{\theta}{2}\right)}{4\,|\vec{p}\,|^4 \sin^4\left(\frac{\theta}{2}\right)} \; = \; \left(\frac{Ze^2}{4\pi}\right)^2 \frac{1 - \beta^2 \sin^2\left(\frac{\theta}{2}\right)}{4\beta^2\,|\vec{p}\,|^2 \sin^4\left(\frac{\theta}{2}\right)} \quad . \tag{7.67}$$

In the literature, one refers to expression (7.67) by Mott cross section. In the limit $\beta \to 0$, *i.e.* for nonrelativistic velocities, one obtains the Rutherford cross section, given by

$$\frac{d\sigma}{d\Omega} \; = = \; \left(\frac{Ze^2}{4\pi}\right)^2 \frac{1}{4\beta^2\,|\vec{p}\,|^2 \sin^4\left(\frac{\theta}{2}\right)} \quad . \tag{7.68}$$

7.3.7 Positron scattering

All that changes in the previous calculus, when applied to the description of the scattering of a positron off a fixed (heavy) charge, is that $u(p)$ must be substituted by $v(p)$ and the electric charge reversed. Formula (7.67) changes for positrons to

$$\frac{d\sigma}{d\Omega} = \left(\frac{-Ze^2}{4\pi}\right)^2 \frac{\left|\bar{v}\left(\vec{p}_f\,, s_f\right)\, \gamma^0\, v\left(\vec{p}_i\,, s_i\right)\right|^2}{\left|\vec{p}_f - \vec{p}_i\,\right|^4} \quad . \tag{7.69}$$

The relevant matrix element is studied in formula (7.31). In analogy with formula (7.64), we find

$$\left|\bar{v}\left(\vec{p}_f\,, s_f\right)\, \gamma^0\, v\left(\vec{p}_i\,, s_i\right)\right|^2 = \mathrm{Tr}\left(\gamma^0\, \{\, \not{p}_i \, - \, m\,\}\, \gamma^0 \left\{\, \not{p}_f \, - \, m\,\right\}\right) \quad . \tag{7.70}$$

Now, since only the quadratic terms contribute to the trace because of the result (7.33), we find

$$\left|\bar{v}\left(\vec{p}_f\,, s_f\right)\, \gamma^0\, v\left(\vec{p}_i\,, s_i\right)\right|^2 = 4m^2 \, + \, 8E_iE_f \, - \, 4p_i \cdot p_f \quad , \tag{7.71}$$

which is equal to the result (7.64) for electron scattering in a Coulomb potential. The only difference between an electron and its antiparticle, positron, is the electric charge, expressed by the minus sign in formula (7.69).

7.4 The electron propagator

We determine the electron propagator from the Dirac equation (see formula 7.10) for the Greens function S_F

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} \, - \, m\right) S_F(x, x') = \delta^{(4)}(x - x'\,) \quad , \tag{7.72}$$

which is readily solved by

$$S_F(x, x') = \lim_{\epsilon \downarrow 0} \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}\, e^{-ip\,\cdot\,(x\,-\,x'\,)} \quad . \tag{7.73}$$

The Feynman propagator for free electrons is thus given by

$$S_F(p) = \frac{\not{p} + m}{p^2 - m^2} = \frac{1}{\not{p} - m} \quad , \tag{7.74}$$

where the $i\epsilon$ term is implicitly understood.

7.5 The photon propagator

The photon propagator follows from the Maxwell equations for the electromagnetic field

$$\partial_\alpha \partial^\alpha\, A^\mu(x) = J^\mu(x) \quad , \tag{7.75}$$

that is

$$\frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x_\alpha}\, D_F(x, x'\,) = \delta^{(4)}(x - x'\,) \quad , \tag{7.76}$$

which is solved by

$$D_F(x,x') \;=\; \lim_{\epsilon\downarrow 0}\int \frac{d^4p}{(2\pi)^4}\,\frac{-1}{p^2+i\epsilon}\,e^{-ip\cdot(x-x')}\quad . \tag{7.77}$$

The Feynman propagator for free photons is thus given by

$$D_F(p) \;=\; \frac{-1}{p^2}\quad , \tag{7.78}$$

where the $i\epsilon$ term is implicitly understood.

7.6 Electron-muon scattering

A muon has the same properties as an electron or a positron. The only difference between electrons and muons is their mass.

$$m_{e^-}c^2 = m_{e^+}c^2 = 0.51\ \text{MeV}\quad , \quad m_{\mu^-}c^2 = m_{\mu^+}c^2 = 106\ \text{MeV}\quad . \tag{7.79}$$

Consequently, the only difference in the wave equation and the wave functions of electrons and muons, lies in their mass. Here we consider the elastic scattering of an electron and a muon.

The lowest order Feynman diagram for elastic electron-muon scattering is shown in figure (7.1). The interaction is represented by one-photon exchange.

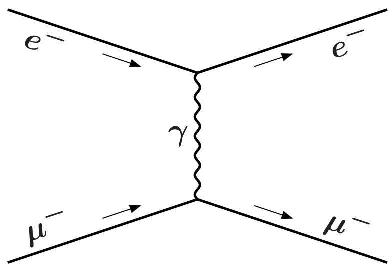


Figure 7.1: The lowest-order one-photon-exchange Feynman diagram for elastic electron-muon scattering.

We denote the momenta and spin states by respectivily p_i and s_i for the incoming electron, by respectivily P_i and S_i for the incoming muon, by respectivily p_f and s_f for the outgoing electron, and by respectivily P_f and S_f for the outgoing muon. For the mass of the electron we write m, whereas the mass of the muon is represented by M.

The lowest-order contribution (7.1) to the amplitude for elastic electron-muon scattering, has the following interpretation. The intensity of the coupling is given by the charge of the particle at each of the two vertices At the electron vertex, the current

$$\bar{u}\,(\vec{p}_f\,,s_f)\;\gamma^\mu\,u\,(\vec{p}_i\,,s_i)$$

couples to the electromagnetic field (photon, for short), whereas at the muon vertex, the current

$$\bar{u}\left(\vec{P}_f\,,S_f\right)\;\gamma^\mu\,u\left(\vec{P}_i\,,S_i\right)$$

couples to the photon. The structure of the currents is completely identical, since the only difference between electrons and muons resides in their masses (7.79). The momentum flow which passes through the photon line is given by

$$p_f - p_i \quad ,$$

which, by momentum conservation is equal to

$$P_i - P_f \quad .$$

The photon propagator is given in formula (7.78).

Following the above considerations, the transition matrix element $\mathcal{M}_{fi}$ for the lowest order diagram (7.1) is given by

$$\mathcal{M}_{fi} \;=\; \bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^\mu\,u\left(\vec{p}_i\,,s_i\right)\,\frac{-e^2}{\left(p_f-p_i\right)^2}\,\bar{u}\left(\vec{P}_f\,,S_f\right)\,\gamma_\mu\,u\left(\vec{P}_i\,,S_i\right) \quad . \tag{7.80}$$

In the expression for the differential cross section, the transition matrix element (7.80) comes squared. When the polarisations of the electron and the muon are not measured, we must average over the initial spin states and sum over the final spin states. We obtain then

$$\left|\overline{\mathcal{M}}_{fi}\right|^2 \;=\; \frac{1}{4}\sum_{\substack{s_i,\,s_f\\ S_i,\,S_f}}\left|\bar{u}\left(\vec{p}_f\,,s_f\right)\,\gamma^\mu\,u\left(\vec{p}_i\,,s_i\right)\,\frac{-e^2}{\left(p_f-p_i\right)^2}\,\bar{u}\left(\vec{P}_f\,,S_f\right)\,\gamma_\mu\,u\left(\vec{P}_i\,,S_i\right)\right|^2 \quad , \tag{7.81}$$

which, by the use formula (7.30) for $A=\gamma^\mu$, gives ($\gamma^0\gamma^{\mu\dagger}=\gamma^\mu\gamma^0$)

$$\left|\overline{\mathcal{M}}_{fi}\right|^2 \;=\; \frac{1}{4}\mathrm{Tr}\left(\gamma^\mu\left\{\not{p}_i+m\right\}\gamma^\nu\left\{\not{p}_f+m\right\}\right)\mathrm{Tr}\left(\gamma_\mu\left\{\not{P}_i+M\right\}\gamma_\nu\left\{\not{P}_f+M\right\}\right)\frac{e^4}{q^4} \quad , \tag{7.82}$$

where $q=p_f-p_i$. Using, moreover, formulae (7.33) and (7.34), we find

$$\left|\overline{\mathcal{M}}_{fi}\right|^2 \;= \tag{7.83}$$

$$= 4\left[p_f^\mu p_i^\nu + p_f^\nu p_i^\mu + g^{\mu\nu}\left(m^2 - p_f\cdot p_i\right)\right]\left[P_{f\,\mu}P_{i\nu} + P_{f\,\nu}P_{i\mu} + g_{\mu\nu}\left(M^2 - P_f\cdot P_i\right)\right]\frac{e^4}{q^4}$$

$$= 8\left[\left(P_f\cdot p_f\right)\left(P_i\cdot p_i\right) + \left(P_f\cdot p_i\right)\left(P_i\cdot p_f\right) - m^2\left(P_f\cdot P_i\right) - M^2\left(p_f\cdot p_i\right) + 2m^2M^2\right]\frac{e^4}{q^4} \quad .$$

7.7 Electron-photon scattering

Electron-photon scattering has been studied extensively in the past century. For the lowest-order contribution, we imagine that the incoming photon is absorbed by the electron, whereas the outgoing photon is radiated off the electron. In figure (7.2) we show the two distinct possibilities. The difference between the two diagrams resides in the momentum which is carried by the intermediate electron. In one diagram, the intermediate

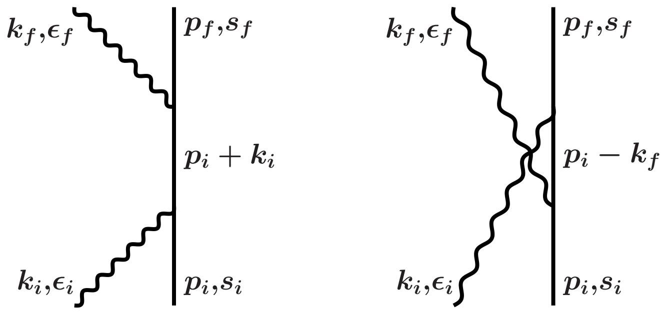


Figure 7.2: The lowest-order Feynman diagrams for elastic electron-photon, or Compton, scattering.

electron carries the sum of the incoming electron momentum p_i and the incoming photon momentum k_i,

$$p_i + k_i = p_f + k_f \quad , \tag{7.84}$$

in the other diagram, it carries the difference of the incoming electron momentum p_i and the outgoing photon momentum k_f,

$$p_i - k_f = p_f - k_i \quad .$$

We denote the momenta and spin states by respectivily p_i and s_i for the incoming electron, by respectivily k_i and ϵ_i for the incoming photon, by respectivily p_f and s_f for the outgoing electron, and by respectivily k_f and ϵ_f for the outgoing photon. For the mass of the electron we write m, whereas the photon is massless.

The photon field is assumed to be in a plane wave

$$A^\mu(x,k) \propto \epsilon^\mu \left(e^{-ikx} + e^{ikx} \right) \quad , \tag{7.85}$$

where ϵ^μ represents its polarisation. We assume transversally polarised photons, *i.e.*

$$\epsilon_\mu \, k^\mu \;=\; 0 \quad . \tag{7.86}$$

The EM interaction between the photon and the electron is given by formula (7.42). Hence, it is an easy task to write down the matrix element for the lowest-order contribution to the transition probability in electron-photon scattering.

$$\mathcal{M}_{fi} \;=\; \bar{u}\left(\vec{p}_f\,,s_f\right) \left\{ (-i\not{\epsilon}_f) \frac{-ie^2}{\not{p}_i + \not{k}_i - m} (-i\not{\epsilon}_i) + (-i\not{\epsilon}_i) \frac{-ie^2}{\not{p}_i - \not{k}_f - m} (-i\not{\epsilon}_f) \right\} u\left(\vec{p}_i\,,s_i\right) \quad . \tag{7.87}$$

In the rest frame of the electron, where $p_i = (m, \vec{0}\,)$, we may (gauge freedom!) select for the polarisations of the incoming and outgoing photons

$$\epsilon_{i,f} = (0, \vec{\epsilon}_{i,f}) \quad \text{with} \quad \epsilon_i \cdot k_i = \epsilon_f \cdot k_f = 0 \quad \text{and} \quad \epsilon_{i,f} \cdot p_i = 0 \quad . \tag{7.88}$$

This choice, with formula (7.8), leads to

$$\begin{aligned} \not{\epsilon}_{i,f}\not{p}_i &= \gamma^\mu \epsilon_{i,f\,\mu} \gamma^\nu p_{i\,\nu} = \left(\{\, \gamma^\mu \,,\, \gamma^\nu \,\} \,-\, \gamma^\nu \gamma^\mu \right) \epsilon_{i,f\,\mu} p_{i\,\nu} = \\ &= 2g^{\mu\nu} \epsilon_{i,f\,\mu} p_{i\,\nu} \,-\, \not{p}_i \not{\epsilon}_{i,f} = 0 \,-\, \not{p}_i \not{\epsilon}_{i,f} = -\, \not{p}_i \not{\epsilon}_{i,f} \quad . \end{aligned} \tag{7.89}$$

Similarly,

$$\not\epsilon_i \not k_i = - \not k_i \not\epsilon_i \quad \text{and} \quad \not\epsilon_f \not k_f = - \not k_f \not\epsilon_f \quad . \tag{7.90}$$

Moreover, by the use of the Dirac equation (7.10), and the results (7.89) and (7.90), we deduce

$$\frac{1}{\not p_i + \not k_i - m} \not\epsilon_i \, u(\vec{p}_i, s_i) = \frac{\not p_i + \not k_i + m}{(p_i + k_i)^2 - m^2} \not\epsilon_i \, u(\vec{p}_i, s_i) = \tag{7.91}$$

$$= \not\epsilon_i \frac{-\not p_i - \not k_i + m}{(p_i + k_i)^2 - m^2} u(\vec{p}_i, s_i) = \not\epsilon_i \frac{-\not k_i}{(p_i + k_i)^2 - m^2} u(\vec{p}_i, s_i) = \not\epsilon_i \frac{-\not k_i}{2p_i \cdot k_i} u(\vec{p}_i, s_i) \ ,$$

where we also used

$$p_i^2 = p_f^2 = m^2 \quad \text{and} \quad k_i^2 = k_f^2 = 0 \quad . \tag{7.92}$$

Similarly, we find furthermore

$$\frac{1}{\not p_i - \not k_f - m} \not\epsilon_f \, u(\vec{p}_i, s_i) = \not\epsilon_f \frac{\not k_f}{-2p_i \cdot k_f} u(\vec{p}_i, s_i) \quad . \tag{7.93}$$

Substitution of the results (7.91) and (7.93) into the expression (7.87) yields

$$\mathcal{M}_{fi} = -ie^2 \bar{u}(\vec{p}_f, s_f) \left\{ \not\epsilon_f \not\epsilon_i \frac{\not k_i}{2p_i \cdot k_i} + \not\epsilon_i \not\epsilon_f \frac{\not k_f}{2p_i \cdot k_f} \right\} u(\vec{p}_i, s_i) \quad . \tag{7.94}$$

For unpolarized electrons we must average over the initial spins s_i and sum over the final spins s_f

$$\left| \overline{\mathcal{M}}_{fi} \right|^2 = \frac{1}{2} \sum_{s_i, s_f} \left| -ie^2 \bar{u}(\vec{p}_f, s_f) \left\{ \not\epsilon_f \not\epsilon_i \frac{\not k_i}{2p_i \cdot k_i} + \not\epsilon_i \not\epsilon_f \frac{\not k_f}{2p_i \cdot k_f} \right\} u(\vec{p}_i, s_i) \right|^2 \quad , \tag{7.95}$$

which, by the use formula (7.30) for

$$A = -ie^2 \left\{ \not\epsilon_f \not\epsilon_i \frac{\not k_i}{2p_i \cdot k_i} + \not\epsilon_i \not\epsilon_f \frac{\not k_f}{2p_i \cdot k_f} \right\}$$

gives ($\gamma^0 \gamma^{\mu\dagger} = \gamma^\mu \gamma^0$)

$$\left| \overline{\mathcal{M}}_{fi} \right|^2 = \frac{1}{2} e^4 \, \mathrm{Tr} \left(\left\{ \frac{\not\epsilon_f \not\epsilon_i \not k_i}{2p_i \cdot k_i} + \frac{\not\epsilon_i \not\epsilon_f \not k_f}{2p_i \cdot k_f} \right\} \{ \not p_i + m \} \left\{ \frac{\not k_i \not\epsilon_i \not\epsilon_f}{2p_i \cdot k_i} + \frac{\not k_f \not\epsilon_f \not\epsilon_i}{2p_i \cdot k_f} \right\} \{ \not p_f + m \} \right) \quad . \tag{7.96}$$

According to expression (7.33) the trace of a product of an odd number of gamma matrices vanishes. Consequently, we obtain from relation (7.96) the following.

$$\begin{aligned} \left| \overline{\mathcal{M}}_{fi} \right|^2 = \frac{1}{8} e^4 \, \mathrm{Tr} \Bigg\{ & \frac{\not\epsilon_f \not\epsilon_i \not k_i \not p_i \not k_i \not\epsilon_i \not\epsilon_f \not p_f + m^2 \not\epsilon_f \not\epsilon_i \not k_i \not k_i \not\epsilon_i \not\epsilon_f}{(p_i \cdot k_i)^2} + \\ & + \frac{\not\epsilon_f \not\epsilon_i \not k_i \not p_i \not k_f \not\epsilon_f \not\epsilon_i \not p_f + m^2 \not\epsilon_f \not\epsilon_i \not k_i \not k_f \not\epsilon_f \not\epsilon_i + \not\epsilon_i \not\epsilon_f \not k_f \not p_i \not k_i \not\epsilon_i \not\epsilon_f \not p_f + m^2 \not\epsilon_i \not\epsilon_f \not k_f \not k_i \not\epsilon_i \not\epsilon_f}{(p_i \cdot k_i)(p_i \cdot k_f)} + \\ & + \frac{\not\epsilon_i \not\epsilon_f \not k_f \not p_i \not k_f \not\epsilon_f \not\epsilon_i \not p_f + m^2 \not\epsilon_i \not\epsilon_f \not k_f \not k_f \not\epsilon_f \not\epsilon_i}{(p_i \cdot k_f)^2} \Bigg\} \quad . \end{aligned} \tag{7.97}$$

In the following we will pass through the calculus of the various traces of expression (7.97) thereby using relations (7.34), (7.88), (7.89), (7.90) and, furthermore, the properties given by

$$\not{a}\not{b} = a_\mu b_\nu \gamma^\mu \gamma^\nu = a_\mu b_\nu \left(\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu \gamma^\mu\right) = a_\mu b_\nu \left(2g^{\mu\nu} - \gamma^\nu \gamma^\mu\right) = 2a \cdot b - \not{b}\not{a} \tag{7.98}$$

and

$$\not{a}\not{a} = a_\mu a_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} a_\mu a_\nu \{\gamma^\mu, \gamma^\nu\} = a_\mu a_\nu g^{\mu\nu} = a^2 \quad . \tag{7.99}$$

We start by eliminating two of the traces by using relations (7.92) and (7.99), namely

$$\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f\right) = k_i^2 \mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_i \not{\epsilon}_f\right) = 0 \tag{7.100}$$

and

$$\mathrm{Tr}\left(\not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right) = k_f^2 \mathrm{Tr}\left(\not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_f \not{\epsilon}_i\right) = 0 \quad . \tag{7.101}$$

Next, also using relations (7.98) and (7.99), we determine the non-vanishing trace of the first term in expression (7.97), which is given by

$$\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right) = \mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \{2p_i \cdot k_i - \not{k}_i \not{p}_i\} \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right) =$$

$$= \mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \left\{(2p_i \cdot k_i) \not{k}_i - k_i^2 \not{p}_i\right\} \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right) = 2p_i \cdot k_i \, \mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right)$$

Here, we use relation (7.90) and the fact that

$$\not{\epsilon}\not{\epsilon} = -\vec{\epsilon}^{\,2} = -1 \tag{7.102}$$

for linearly polarized spin 1 photons, which leads to

$$\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right) = -2p_i \cdot k_i \, \mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_i \not{k}_i \not{\epsilon}_f \not{p}_f\right) = 2p_i \cdot k_i \, \mathrm{Tr}\left(\not{\epsilon}_f \not{k}_i \not{\epsilon}_f \not{p}_f\right)$$

Once more using relations (7.98) and (7.102) gives

$$\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right) = 2p_i \cdot k_i \, \mathrm{Tr}\left(\not{\epsilon}_f \left(2\epsilon_f \cdot k_i - \not{\epsilon}_f \not{k}_i\right) \not{p}_f\right)$$

$$= p_i \cdot k_i \left\{4\epsilon_f \cdot k_i \, \mathrm{Tr}\left(\not{\epsilon}_f \not{p}_f\right) - 2\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_f \not{k}_i \not{p}_f\right)\right\}$$

$$= p_i \cdot k_i \left\{4\epsilon_f \cdot k_i \, \mathrm{Tr}\left(\not{\epsilon}_f \not{p}_f\right) + 2\mathrm{Tr}\left(\not{k}_i \not{p}_f\right)\right\} \tag{7.103}$$

For the non-vanishing trace of the third term in expression (7.97), one just has to substitute $\epsilon_i \Leftrightarrow \epsilon_f$ and $k_i \Leftrightarrow k_f$ in expression (7.103). We obtain then

$$\mathrm{Tr}\left(\not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{p}_f\right) = p_i \cdot k_f \left\{4\epsilon_i \cdot k_f \, \mathrm{Tr}\left(\not{\epsilon}_i \not{p}_f\right) + 2\mathrm{Tr}\left(\not{k}_f \not{p}_f\right)\right\} \tag{7.104}$$

Consequently, for the sum of the first and the third terms in expression (7.97) we obtain

$$\frac{\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right) + m^2 \not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f}{\left(p_i \cdot k_i\right)^2} + \frac{\mathrm{Tr}\left(\not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{p}_f\right) + m^2 \not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{k}_f \not{\epsilon}_f \not{\epsilon}_i}{\left(p_i \cdot k_f\right)^2}$$

$$= \frac{4\epsilon_f \cdot k_i \, \mathrm{Tr}\left(\not{\epsilon}_f \not{p}_f\right) + 2\mathrm{Tr}\left(\not{k}_i \not{p}_f\right)}{p_i \cdot k_i} + \frac{4\epsilon_i \cdot k_f \, \mathrm{Tr}\left(\not{\epsilon}_i \not{p}_f\right) + 2\mathrm{Tr}\left(\not{k}_f \not{p}_f\right)}{p_i \cdot k_f} \quad . \tag{7.105}$$

Here, we insert total energy-momentum conservation (7.84). Also using relations (7.88), (7.89) and (7.90), one has

$$\mathrm{Tr}\left(\not{\epsilon}_f \not{p}_f\right) = 4\epsilon_f \cdot k_i \qquad \text{and} \qquad \mathrm{Tr}\left(\not{\epsilon}_i \not{p}_f\right) = -4\epsilon_i \cdot k_f \tag{7.106}$$

and, furthermore, because of equations (7.92),

$$\begin{aligned}(p_f + k_f)^2 = (p_i + k_i)^2 \qquad &\Longrightarrow \qquad k_f \cdot p_f = k_i \cdot p_i \\ (p_f - k_i)^2 = (p_i - k_f)^2 \qquad &\Longrightarrow \qquad k_i \cdot p_f = k_f \cdot p_i \quad .\end{aligned} \tag{7.107}$$

For the sum (7.105) of the first and the third terms in expression (7.97) one finds then

$$\begin{aligned}&\frac{\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \not{p}_f\right) + m^2 \not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f}{(p_i \cdot k_i)^2} + \frac{\mathrm{Tr}\left(\not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{p}_f\right) + m^2 \not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{k}_f \not{\epsilon}_f \not{\epsilon}_i}{(p_i \cdot k_f)^2} \\ &= \frac{16\left(\epsilon_f \cdot k_i\right)^2 + 8k_f \cdot p_i}{p_i \cdot k_i} + \frac{-16\left(\epsilon_i \cdot k_f\right)^2 + 8k_i \cdot p_i}{p_i \cdot k_f}\end{aligned} \tag{7.108}$$

For the trace of the second term in expression (7.97), we proceed as follows.
We begin by using the fact that traces of products of an odd number of gamma matrices vanish. Hence

$$\begin{aligned}&\mathrm{Tr}\left(\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{p}_f + m^2 \not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i + \not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{p}_i \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \not{p}_f + m^2 \not{\epsilon}_i \not{\epsilon}_f \not{k}_f \not{k}_i \not{\epsilon}_i \not{\epsilon}_f\right) = \\ &= \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_f + m\right)\right\} + \mathrm{Tr}\left\{\not{\epsilon}_i \not{\epsilon}_f \not{k}_f \left(\not{p}_i + m\right) \not{k}_i \not{\epsilon}_i \not{\epsilon}_f \left(\not{p}_f + m\right)\right\}\end{aligned} \tag{7.109}$$

Next, we concentrate on the first term of the righthand side of equation (7.109). Here, we first insert total energy-momentum conservation, which reads $p_f = p_i + k_i - k_f$, to obtain

$$\begin{aligned}&\mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_f + m\right)\right\} = \\ &= \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_i + \not{k}_i - \not{k}_f + m\right)\right\} \\ &= \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_i + m\right)\right\} + \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{k}_i - \not{k}_f\right)\right\} \\ &= \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_i + m\right)\right\} + \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{k}_i - \not{k}_f\right)\right\}\end{aligned} \tag{7.110}$$

Here, we use again the fact that traces of a product of an odd number of gamma matrices vanish (7.33).

$$\begin{aligned}&\mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_f + m\right)\right\} = \\ &= \mathrm{Tr}\left\{\not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_i + m\right) \not{\epsilon}_f \not{\epsilon}_i\right\} + \\ &+ \mathrm{Tr}\left\{\not{k}_i \not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} - \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{k}_f\right\}\end{aligned}$$

Notice that here we also used the property of invariance under cyclic permutations for traces, *i.e.* $\mathrm{Tr}(AB\ldots YZ) = \mathrm{Tr}(ZAB\ldots Y)$.

We continue by applying relation (7.98).

$$
\begin{aligned}
&\mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_f + m\right)\right\} = \\
&\quad = \mathrm{Tr}\left\{\not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_i + m\right)\right\} + \\
&\qquad + \mathrm{Tr}\left\{\left(2k_i \cdot \epsilon_f - \not{\epsilon}_f \not{k}_i\right) \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} - \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \left(2k_f \cdot \epsilon_i - \not{k}_f \not{\epsilon}_i\right)\right\} \\
&\quad = \mathrm{Tr}\left\{\left(\not{p}_i + m\right) \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i\right\} + \\
&\qquad + 2k_i \cdot \epsilon_f \, \mathrm{Tr}\left\{\not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} - \mathrm{Tr}\left\{\not{\epsilon}_f \not{k}_i \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} + \\
&\qquad - 2k_f \cdot \epsilon_i \, \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f\right\} + \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{k}_f \not{\epsilon}_i\right\}
\end{aligned}
$$

Then we apply relations (7.34), (7.89), (7.90), (7.92), (7.98), (7.99) and (7.102).

$$
\begin{aligned}
&\mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \left(\not{p}_i + m\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_f + m\right)\right\} = \\
&\quad = \mathrm{Tr}\left\{\not{p}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i\right\} + m^2 \mathrm{Tr}\left\{\not{k}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i\right\} + \\
&\qquad + 2k_i \cdot \epsilon_f \, \mathrm{Tr}\left\{\not{\epsilon}_i \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f\right\} + \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} + \\
&\qquad - 2k_f \cdot \epsilon_i \, \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f\right\} - \mathrm{Tr}\left\{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} \\
&\quad = \mathrm{Tr}\left\{\not{p}_i \left(2p_i \cdot k_i - \not{p}_i \not{k}_i\right) \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i\right\} + m^2 \mathrm{Tr}\left\{\not{k}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i\right\} + \\
&\qquad - 2k_i \cdot \epsilon_f \, \mathrm{Tr}\left\{\not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f\right\} + 2k_f \cdot \epsilon_i \, \mathrm{Tr}\left\{\not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f\right\} \\
&\quad = 2p_i \cdot k_i \, \mathrm{Tr}\left\{\not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i\right\} + \mathrm{Tr}\left\{\left(-\not{p}_i \not{p}_i + m^2\right) \not{k}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_f \not{\epsilon}_i\right\} + \\
&\qquad - 2k_i \cdot \epsilon_f \, \mathrm{Tr}\left\{\not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f\right\} + 2k_f \cdot \epsilon_i \, \mathrm{Tr}\left\{\not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f\right\} \\
&\quad = 2p_i \cdot k_i \, \mathrm{Tr}\left\{\not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(2\epsilon_f \cdot \epsilon_i - \not{\epsilon}_i \not{\epsilon}_f\right)\right\} + \\
&\qquad - 2k_i \cdot \epsilon_f \, \mathrm{Tr}\left\{\not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f\right\} + 2k_f \cdot \epsilon_i \, \mathrm{Tr}\left\{\not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f\right\} \\
&\quad = 4\left(p_i \cdot k_i\right)\left(\epsilon_f \cdot \epsilon_i\right) \mathrm{Tr}\left\{\not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} - 2p_i \cdot k_i \, \mathrm{Tr}\left\{\not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \not{\epsilon}_i \not{\epsilon}_f\right\} + \\
&\qquad - 2k_i \cdot \epsilon_f \, \mathrm{Tr}\left\{\not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f\right\} + 2k_f \cdot \epsilon_i \, \mathrm{Tr}\left\{\not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f\right\} \\
&\quad = 4\left(p_i \cdot k_i\right)\left(\epsilon_f \cdot \epsilon_i\right) \mathrm{Tr}\left\{\not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i\right\} - 2p_i \cdot k_i \, \mathrm{Tr}\left\{\not{p}_i \not{k}_f\right\} + \\
&\qquad - 2k_i \cdot \epsilon_f \, \mathrm{Tr}\left\{\not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f\right\} + 2k_f \cdot \epsilon_i \, \mathrm{Tr}\left\{\not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f\right\}
\end{aligned}
$$

$$= 4\,(p_i \cdot k_i)\,(\epsilon_f \cdot \epsilon_i)\ \mathrm{Tr}\left\{ \not{p}_i \not{k}_f \not{\epsilon}_f \not{\epsilon}_i \right\} - 8\,(p_i \cdot k_i)\,(p_i \cdot k_f) +$$

$$-2k_i \cdot \epsilon_f\, \mathrm{Tr}\left\{ \not{k}_i \not{p}_i \not{k}_f \not{\epsilon}_f \right\} + 2k_f \cdot \epsilon_i\, \mathrm{Tr}\left\{ \not{\epsilon}_i \not{k}_i \not{p}_i \not{k}_f \right\}$$

In the above expression we find three traces of the type $\mathrm{Tr}\,(\not{a}\not{b}\not{c}\not{d})$. This can be handled by moving the $\not{a}$ matrix to the right as follows.

$$\mathrm{Tr}\,(\not{a}\not{b}\not{c}\not{d}) =$$

$$= \mathrm{Tr}\,((2a \cdot b - \not{b}\not{a})\,\not{c}\not{d}) = 2\,(a \cdot b)\,\mathrm{Tr}\,(\not{c}\not{d}) - \mathrm{Tr}\,(\not{b}\not{a}\not{c}\not{d})$$

$$= 8\,(a \cdot b)\,(c \cdot d) - \mathrm{Tr}\,(\not{b}\,(2a \cdot c - \not{c}\not{a})\,\not{d}) = 8\,(a \cdot b)\,(c \cdot d) - 2\,(a \cdot c)\,\mathrm{Tr}\,(\not{b}\not{d}) + \mathrm{Tr}\,(\not{b}\not{c}\not{a}\not{d})$$

$$= 8\,(a \cdot b)\,(c \cdot d) - 8\,(a \cdot c)\,(b \cdot d) + \mathrm{Tr}\,(\not{b}\not{c}\,(2a \cdot d - \not{d}\not{a}))$$

$$= 8\,(a \cdot b)\,(c \cdot d) - 8\,(a \cdot c)\,(b \cdot d) + 2\,(a \cdot d)\,\mathrm{Tr}\,(\not{b}\not{c}) - \mathrm{Tr}\,(\not{b}\not{c}\not{d}\not{a})$$

$$= 8\,(a \cdot b)\,(c \cdot d) - 8\,(a \cdot c)\,(b \cdot d) + 8\,(a \cdot d)\,(b \cdot c) - \mathrm{Tr}\,(\not{a}\not{b}\not{c}\not{d})$$

which results in

$$\mathrm{Tr}\,(\not{a}\not{b}\not{c}\not{d}) = 4\,(a \cdot b)\,(c \cdot d) - 4\,(a \cdot c)\,(b \cdot d) + 4\,(a \cdot d)\,(b \cdot c) \quad . \tag{7.111}$$

Applying this result to the previous expression, we obtain

$$\mathrm{Tr}\left\{ \not{\epsilon}_f \not{\epsilon}_i \not{k}_i\,(\not{p}_i + m)\,\not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_f + m\right) \right\} =$$

$$= 16\,(p_i \cdot k_i)\,(\epsilon_f \cdot \epsilon_i)\,\{(p_i \cdot k_f)\,(\epsilon_f \cdot \epsilon_i) - (p_i \cdot \epsilon_f)\,(k_f \cdot \epsilon_i) + (p_i \cdot \epsilon_i)\,(k_f \cdot \epsilon_f)\} +$$

$$-8\,(p_i \cdot k_i)\,(p_i \cdot k_f) +$$

$$-8k_i \cdot \epsilon_f\,\{(p_i \cdot k_i)\,(k_f \cdot \epsilon_f) - (k_i \cdot k_f)\,(p_i \cdot \epsilon_f) + (k_i \cdot \epsilon_f)\,(p_i \cdot k_f)\} +$$

$$+8k_f \cdot \epsilon_i\,\{(k_i \cdot \epsilon_i)\,(p_i \cdot k_f) - (p_i \cdot \epsilon_i)\,(k_i \cdot k_f) + (k_f \cdot \epsilon_i)\,(p_i \cdot k_i)\}$$

Here various terms vanish because of (7.88).

$$\mathrm{Tr}\left\{ \not{\epsilon}_f \not{\epsilon}_i \not{k}_i\,(\not{p}_i + m)\,\not{k}_f \not{\epsilon}_f \not{\epsilon}_i \left(\not{p}_f + m\right) \right\} =$$

$$= 16\,(p_i \cdot k_i)\,(\epsilon_f \cdot \epsilon_i)\,\{(p_i \cdot k_f)\,(\epsilon_f \cdot \epsilon_i)\} +$$

$$-8\,(p_i \cdot k_i)\,(p_i \cdot k_f) - 8k_i \cdot \epsilon_f\,\{(k_i \cdot \epsilon_f)\,(p_i \cdot k_f)\} + 8k_f \cdot \epsilon_i\,\{(k_f \cdot \epsilon_i)\,(p_i \cdot k_i)\}$$

$$= 8\,(p_i \cdot k_i)\,(p_i \cdot k_f)\left\{2\,(\epsilon_f \cdot \epsilon_i)^2 - 1\right\} + 8\,(p_i \cdot k_i)\,(k_f \cdot \epsilon_i)^2 - 8\,(p_i \cdot k_f)\,(k_i \cdot \epsilon_f)^2 \tag{7.112}$$

Next, we study the second term of the righthand side of equation (7.109). We first insert total energy-momentum conservation, which reads $p_f = p_i + k_i - k_f$, to obtain

$$\mathrm{Tr}\left\{ \not{\epsilon}_i \not{\epsilon}_f \not{k}_f\,(\not{p}_i + m)\,\not{k}_i \not{\epsilon}_i \not{\epsilon}_f \left(\not{p}_f + m\right) \right\} =$$

$$= \mathrm{Tr}\left\{ \not\epsilon_i \not\epsilon_f \not k_f \left(\not p_i + m \right) \not k_i \not\epsilon_i \not\epsilon_f \left(\not p_i + \not k_i - \not k_f + m \right) \right\}$$

$$= \mathrm{Tr}\left\{ \not\epsilon_i \not\epsilon_f \not k_f \left(\not p_i + m \right) \not k_i \not\epsilon_i \not\epsilon_f \left(\not p_i + m \right) \right\} + \mathrm{Tr}\left\{ \not\epsilon_i \not\epsilon_f \not k_f \left(\not p_i + m \right) \not k_i \not\epsilon_i \not\epsilon_f \left(\not k_i - \not k_f \right) \right\}$$

$$= \mathrm{Tr}\left\{ \not\epsilon_i \not\epsilon_f \not k_f \left(\not p_i + m \right) \not k_i \not\epsilon_i \not\epsilon_f \left(\not p_i + m \right) \right\} + \mathrm{Tr}\left\{ \not\epsilon_i \not\epsilon_f \not k_f \not p_i \not k_i \not\epsilon_i \not\epsilon_f \left(\not k_i - \not k_f \right) \right\} \quad (7.113)$$

When we make the substitution $\epsilon_i \Leftrightarrow \epsilon_f$ and $k_i \Leftrightarrow k_f$ to the lefthand side of the equation (7.113), *i.e.*

$$\mathrm{Tr}\left\{ \not\epsilon_f \not\epsilon_i \not k_i \left(\not p_i + m \right) \not k_f \not\epsilon_f \not\epsilon_i \left(\not p_f + m \right) \right\}$$

then we find that the result equals the lefthand side of equation (7.110). However, if we do the same substitution in the righthand side of the equation (7.113), *i.e.*

$$\mathrm{Tr}\left\{ \not\epsilon_f \not\epsilon_i \not k_i \left(\not p_i + m \right) \not k_f \not\epsilon_f \not\epsilon_i \left(\not p_i + m \right) \right\} + \mathrm{Tr}\left\{ \not\epsilon_f \not\epsilon_i \not k_i \not p_i \not k_f \not\epsilon_f \not\epsilon_i \left(\not k_f - \not k_i \right) \right\}$$

then we find that the second term acquires a minus sign. Consequently, we may use the result (7.110) for the second term of the righthand side of equation (7.109) by substituting $\epsilon_i \Leftrightarrow \epsilon_f$ and $k_i \Leftrightarrow k_f$. But, we must add a minus sign to the terms which stem from the second term in the righthand side of equation (7.113). We find then

$$\mathrm{Tr}\left\{ \not\epsilon_i \not\epsilon_f \not k_f \left(\not p_i + m \right) \not k_i \not\epsilon_i \not\epsilon_f \left(\not p_f + m \right) \right\} = \quad (7.114)$$

$$= 8 \left(p_i \cdot k_f \right) \left(p_i \cdot k_i \right) \left\{ 2 \left(\epsilon_i \cdot \epsilon_f \right)^2 - 1 \right\} - 8 \left(p_i \cdot k_f \right) \left(k_i \cdot \epsilon_f \right)^2 + 8 \left(p_i \cdot k_i \right) \left(k_f \cdot \epsilon_i \right)^2$$

Joining the results (7.112) and (7.114) we obtain for expression (7.109) the following.

$$\mathrm{Tr}\left(\not\epsilon_f \not\epsilon_i \not k_i \not p_i \not k_f \not\epsilon_f \not\epsilon_i \not p_f + m^2 \not\epsilon_f \not\epsilon_i \not k_i \not k_f \not\epsilon_f \not\epsilon_i + \not\epsilon_i \not\epsilon_f \not k_f \not p_i \not k_i \not\epsilon_i \not\epsilon_f \not p_f + m^2 \not\epsilon_i \not\epsilon_f \not k_f \not k_i \not\epsilon_i \not\epsilon_f \right)$$

$$= 16 \left(p_i \cdot k_f \right) \left(p_i \cdot k_i \right) \left\{ 2 \left(\epsilon_i \cdot \epsilon_f \right)^2 - 1 \right\} - 16 \left(p_i \cdot k_f \right) \left(k_i \cdot \epsilon_f \right)^2 + 16 \left(p_i \cdot k_i \right) \left(k_f \cdot \epsilon_i \right)^2 \quad (7.115)$$

For the transition probability (7.97), we find, by the use of (7.108) and (7.115), finally

$$\left| \overline{\mathcal{M}}_{fi} \right|^2 = \frac{1}{8} e^4 \left\{ \frac{16 \left(\epsilon_f \cdot k_i \right)^2 + 8 k_f \cdot p_i}{p_i \cdot k_i} + \frac{-16 \left(\epsilon_i \cdot k_f \right)^2 + 8 k_i \cdot p_i}{p_i \cdot k_f} + \right.$$

$$\left. + 16 \left\{ 2 \left(\epsilon_i \cdot \epsilon_f \right)^2 - 1 \right\} - \frac{16 \left(k_i \cdot \epsilon_f \right)^2}{p_i \cdot k_i} + \frac{16 \left(k_f \cdot \epsilon_i \right)^2}{p_i \cdot k_f} \right\}$$

$$= e^4 \left\{ \frac{p_i \cdot k_f}{p_i \cdot k_i} + \frac{p_i \cdot k_i}{p_i \cdot k_f} + 2 \left[2 \left(\epsilon_i \cdot \epsilon_f \right)^2 - 1 \right] \right\} \quad (7.116)$$

When we take the electron initially at rest and write k, k' for the photon energy of respectively the incident photon and the scattered photon, then we may simplify the expression to

$$\left| \overline{\mathcal{M}}_{fi} \right|^2 = e^4 \left\{ \frac{k'}{k} + \frac{k}{k'} + 2 \left[2 \left(\epsilon_i \cdot \epsilon_f \right)^2 - 1 \right] \right\} \quad (7.117)$$

which result had been obtained by O. Klein and Y. Nishina in 1929, namely

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{k'}{k}\right)^2 \left\{\frac{k'}{k} + \frac{k}{k'} + 2\left[2\left(\epsilon_i \cdot \epsilon_f\right)^2 - 1\right]\right\} \quad .$$

We may, moreover, relate k' to k and the scattering angle ϑ_{ph} of the outgoing photon by considering, for example, the yz plane as the plane described by the outgoing photon and the outgoing electron with scattering angle ϑ_e, thereby assuming that the incident photon moves along the z axis. For the various four-momenta we define then

$$p_i = (m, 0, 0, 0) \quad , \quad p_f = \left(E_f, 0, |\vec{p}_f| \sin(\vartheta_e), |\vec{p}_f| \cos(\vartheta_e)\right) \quad , \quad E_f = \sqrt{m^2 + \vec{p}_f^{\;2}}$$

$$k_i = (k, 0, 0, k) \quad \text{and} \quad k_f = \left(k', 0, k' \sin(\vartheta_{ph}), k' \cos(\vartheta_{ph})\right)$$

Energy-momentum conservation gives

$$\begin{aligned} m + k &= E_f + k' \\ 0 &= |\vec{p}_f| \sin(\vartheta_e) + k' \sin(\vartheta_{ph}) \\ k &= |\vec{p}_f| \cos(\vartheta_e) + k' \cos(\vartheta_{ph}) \end{aligned}$$

We determine

$$\begin{aligned} & k'^2 + k^2 - 2kk' \cos(\vartheta_{ph}) = \\ & \quad = k'^2 \sin^2(\vartheta_{ph}) + \left(k - k' \cos(\vartheta_{ph})\right)^2 = \vec{p}_f^{\;2} \sin^2(\vartheta_e) + \vec{p}_f^{\;2} \cos^2(\vartheta_e) \\ & \quad = \vec{p}_f^{\;2} = E_f^2 - m^2 = (m + k - k')^2 - m^2 = k^2 + k'^2 + 2mk - 2mk' - 2kk' \quad . \end{aligned}$$

Hence

$$\left(m + k - k \cos(\vartheta_{ph})\right) k' = mk \iff k' = \frac{k}{1 + 2\frac{k}{m} \sin^2(\vartheta_{ph}/2)} \quad .$$

Bibliography

[1] *Quantum Mechanics*, Eugen Merzbacher, 2nd edition (Wiley, NewYork).

[2] *Methods of Theoretical Physics*, Philip M. Morse and Herman Feshbach, McGraw-Hill book company. inc. (NY 1953).

[3] *Spectra and Decay Properties of Pseudo-scalar and Vector Mesons in a Multichannel quark Model*, George Rupp, Doctoral Thesis, Nijmegen (1982).

[4] *A low-lying scalar meson nonet in a unitarized meson model*, by E. van Beveren, T.A. Rijken, K. Metzger, C. Dullemond, G. Rupp and J.E. Ribeiro, Zeitschrift für Physik **C30**, 615-620 (1986)

[5] *Bateman Manuscript Project, Higher Transcendental Functions*, Erdélyi et al., eds., vol.1 (McGraw-Hill, New York).

[6] *General Mechanics*, J.J. de Swart, lecture notes of the Center for Theoretical Physics of the Nijmegen University, January 1981.

[7] W. Bauer, Journal für Mathematik **LVI** (1859), pp. 104-106; see also: G.N. Watson, *A treatise on the Theory of Bessel Functions*, section 4.32.

[8] *Handbook of Feynman path integrals*, C. Grosche and F. Steiner, Springer Verlag (Berlin, 1998).

[9] *Electrodynamics*, Richard P. Feynman.

[10] *Relativistic Quantum Fields*, James D. Bjorken and Sidney D. Drell, McGraw-Hill Book Company (NY 1965).

[11] *Quantum Field Theory*, C. Itzykson and J.-B Zuber, McGraw-Hill Book Company (NY 1980).

[12] *Gauge theory of elementary particle physics*, Ta-Pei Cheng and Ling-Fong Li, Clarendon Press (Oxford, 1984).

[13] M. D. Scadron, *Advanced quantum theory and its applications through Feynman diagrams*, Berlin, Germany: Springer (1991) 410 p. (Texts and monographs in physics).

[14] *Quantum Field Theory*, George Sterman, Cambridge University Press (Cambridge 1993).

[15] *Field theory (a modern primer)*, Pierre Ramond, Redwood City, USA: Addison-Wesley (1989), Frontiers in Physics 74.

[16] *Path integral methods in quantum field theory*, R. J. Rivers, Cambridge University Press (Cambridge 1987).

[17] *Quantum mechanics and path integrals*, Richard P. Feynman and A. R. Hibbs, McGraw-Hill Book Company (NY 1965).

[18] *Diagrammar*, Gerard 't Hooft and Tiny Veltman, CERN publication Cern 73-9, Laboratory I, Theoretical Studies Division, 3/9/1973.

[19] *Regularization and renormalization of gauge fields*, Gerard 't Hooft and Tiny Veltman, Nuclear Physics B44 at pages 189 to 213 (1972).

[20] *Aspects of symmetry*, Sidney Coleman, Cambridge University Press (Cambridge 1985).

[21] E. Fermi, *Trends To A Theory Of Beta Radiation. (In Italian)*, Nuovo Cim. **11**, 1 (1934).

E. Fermi, *An Attempt Of A Theory Of Beta Radiation. 1*, Z. Phys. **88**, 161 (1934).

[22] Sheldon L. Glashow, *Partial Symmetries Of Weak Interactions*, Nucl. Phys. **22**, 579 (1961).

[23] Steven Weinberg, *A Model Of Leptons*, Phys. Rev. Lett. **19**, 1264 (1967).

[24] Abdus Salam and J. C. Ward, *Electromagnetic And Weak Interactions*, Phys. Lett. **13**, 168 (1964).