Some notes on group theory.

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Chapter 1

Some intuitive notions of groups.

In this chapter we will discuss the definition and the properties of a group. As an example we will study the symmetric groups S_n of permutations of n objects.

1.1 What is a group?

For our purpose it is sufficient to consider a group G as a set of operations, *i.e.*:

$$G = \{g_1, g_2, \cdots, g_p\}. \tag{1.1}$$

Each element g_i $(i = 1, \cdots, p)$ of the group G (1.1) represents an operation. Groups can be finite as well as infinite. The number of elements of a finite group is called the *order* p of the group.

As an example let us study the group S_3 of permutations of three objects. Let us define as *objects* the letters A, B and C. With those objects one can form "three-letter" words, like ABC and CAB (six possibilities). We define the operation of an element of S_3 on a "three-letter" word, by a reshuffling of the order of the letters in a word, *i.e.*:

$$g_i(\text{word}) = \text{another (or the same) word}. \tag{1.2}$$

Let us denote the elements of S_3 in such a way that from the notation it is immediatly clear how it operates on a "three-letter" word, *i.e.* by:

$$\begin{array}{lll} g_1 = [123] = I, & g_2 = [231], & g_3 = [312], \\ g_4 = [132], & g_5 = [213], & \text{and } g_6 = [321]. \end{array}$$

Table 1.1: The elements of the permutation group S_3.

Then, we might define the action of the elements of the group S_3 by:

$$[123](ABC) = \left\{ \begin{array}{l} \text{the } 1^{\text{st}} \text{ letter remains in the } 1^{\text{st}} \text{ position} \\ \text{the } 2^{\text{nd}} \text{ letter remains in the } 2^{\text{nd}} \text{ position} \\ \text{the } 3^{\text{rd}} \text{ letter remains in the } 3^{\text{rd}} \text{ position} \end{array} \right\} = ABC$$

$$
[231](ABC) = \left\{\begin{array}{l} \text{the } 2^{\text{nd}} \text{ letter moves to the } 1^{\text{st}} \text{ position} \\ \text{the } 3^{\text{rd}} \text{ letter moves to the } 2^{\text{nd}} \text{ position} \\ \text{the } 1^{\text{st}} \text{ letter moves to the } 3^{\text{rd}} \text{ position} \end{array}\right\} = BCA
$$

$$
[312](ABC) = \left\{\begin{array}{l} \text{the } 3^{\text{rd}} \text{ letter moves to the } 1^{\text{st}} \text{ position} \\ \text{the } 1^{\text{st}} \text{ letter moves to the } 2^{\text{nd}} \text{ position} \\ \text{the } 2^{\text{nd}} \text{ letter moves to the } 3^{\text{rd}} \text{ position} \end{array}\right\} = CAB
$$

$$
[132](ABC) = \left\{\begin{array}{l} \text{the } 1^{\text{st}} \text{ letter remains in the } 1^{\text{st}} \text{ position} \\ \text{the } 3^{\text{rd}} \text{ letter moves to the } 2^{\text{nd}} \text{ position} \\ \text{the } 2^{\text{nd}} \text{ letter moves to the } 3^{\text{rd}} \text{ position} \end{array}\right\} = ACB
$$

$$
[213](ABC) = \left\{\begin{array}{l} \text{the } 2^{\text{nd}} \text{ letter moves to the } 1^{\text{st}} \text{ position} \\ \text{the } 1^{\text{st}} \text{ letter moves to the } 2^{\text{nd}} \text{ position} \\ \text{the } 3^{\text{rd}} \text{ letter remains in the } 3^{\text{rd}} \text{ position} \end{array}\right\} = BAC
$$

$$
[321](ABC) = \left\{\begin{array}{l} \text{the } 3^{\text{rd}} \text{ letter moves to the } 1^{\text{st}} \text{ position} \\ \text{the } 2^{\text{nd}} \text{ letter remains in the } 2^{\text{nd}} \text{ position} \\ \text{the } 1^{\text{st}} \text{ letter moves to the } 3^{\text{rd}} \text{ position} \end{array}\right\} = CBA
\tag{1.3}
$$

In equations (1.3) a definition is given for the operation of the group elements of table (1.1) at page 1 of the permutation group S_3. But a group is not only a set of operations. A group must also be endowed with a suitable definition of a *product*. In the case of the example of S_3 such a product might be chosen as follows:

First one observes that the repeated operation of two group elements on a "three-letter" word results again in a "three-letter" word, *for example*:

$$
[213]\left([312](ABC)\right) = [213](CAB) = ACB. \tag{1.4}
$$

Then one notices that such repeated operation is equal to one of the other operations of the group; in the case of the above example (1.4) one finds:

$$
[213]\left([312](ABC)\right) = ACB = [132](ABC). \tag{1.5}
$$

In agreement with the definition of the product function in Analysis for repeated actions of operations on the "variable" ABC, we define here for the product of group operations the following:

$$
\left([213]\circ[312]\right)(ABC) = [213]\left([312](ABC)\right). \tag{1.6}
$$

From this definition and the relation (1.5) one might deduce for the *product* of [213] and [312] the result:

$$
[213]\,[312] = [132]. \tag{1.7}
$$

Notice that, in the final notation of formula (1.7) for the product of two group elements, we dropped the symbol $\circ$, which is used in formula (1.6). The product of group elements must, in general, be defined such, that the product of two elements of the group is itself also an element of the group.

The products between all elements of S_3 are collected in table (1.2) below. Such table is called the *multiplication table* of the group. It contains all information on the *structure* of the group; in the case of table (1.2), of the group S_3.

S_3	b	I	[231]	[312]	[132]	[213]	[321]
a	ab						
I		I	[231]	[312]	[132]	[213]	[321]
[231]		[231]	[312]	I	[321]	[132]	[213]
[312]		[312]	I	[231]	[213]	[321]	[132]
[132]		[132]	[213]	[321]	I	[231]	[312]
[213]		[213]	[321]	[132]	[312]	I	[231]
[321]		[321]	[132]	[213]	[231]	[312]	I

Table 1.2: The multiplication table of the group S_3.

The product which is defined for a group must have the following properties:

1. The product must be *associative*, *i.e.*:

$$\left(g_i g_j\right) g_k = g_i \left(g_j g_k\right) . \tag{1.8}$$

2. There must exist an identity operation, I. For example the identity operation for S_3 is defined by:

$$I = [123] \quad i.e.: \quad I(ABC) = ABC. \tag{1.9}$$

3. Each element g of the group G must have its inverse operation g^{-1}, such that:

$$gg^{-1} = g^{-1}g = I. \tag{1.10}$$

From the multiplication table we find for S_3 the following results for the inverses of its elements:

$$\begin{array}{rcl} I^{-1} = I \ , & [231]^{-1} = [312] \ , & [312]^{-1} = [231] \ , \\ [132]^{-1} = [132] \ , & [213]^{-1} = [213] \ , & [321]^{-1} = [321] \ . \end{array}$$

Notice that some group elements are their own inverse. Notice also that the group product is in general not commutative. For example:

$$[312][132] = [213] \neq [321] = [132][312].$$

A group for which the group product is commutative is called an *Abelian group.*

Notice moreover from table (1.2) that in each row and in each column appear once all elements of the group. This is a quite reasonable result, since:

$$\text{if } ab = ac \ , \text{ then consequently } b = a^{-1}ac = c \ .$$

1.2 The cyclic notation for permutation groups.

A different way of denoting the elements of a permutation group S_n is by means of cycles (*e.g.* $(ijk\cdots l)(m)$). In a cycle each object is followed by its image, and the image of the last object of a cycle is given by the first object, *i.e.*:

$$(ijk\cdots l)(m) = \left\{ \begin{array}{ccc} i & \rightarrow & j \\ j & \rightarrow & k \\ k & \rightarrow & . \\ & \vdots & \\ . & \rightarrow & l \\ l & \rightarrow & i \\ m & \rightarrow & m \end{array} \right\} \quad \begin{array}{l} \text{for the numbers } i, j, k, \ldots l \\ \text{all different.} \end{array} \tag{1.11}$$

For example, the cyclic notation of the elements of S_3 is given by:

$$\begin{array}{l} [123] = \left\{ \begin{array}{ccc} 1 & \rightarrow & 1 \\ 2 & \rightarrow & 2 \\ 3 & \rightarrow & 3 \end{array} \right\} = (1)(2)(3) \\ [231] = \left\{ \begin{array}{ccc} 2 & \rightarrow & 1 \\ 3 & \rightarrow & 2 \\ 1 & \rightarrow & 3 \end{array} \right\} = (132) = (321) = (213) \\ [312] = \left\{ \begin{array}{ccc} 3 & \rightarrow & 1 \\ 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \end{array} \right\} = (123) = (231) = (312) \\ [132] = \left\{ \begin{array}{ccc} 1 & \rightarrow & 1 \\ 3 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \end{array} \right\} = (1)(23) = (23) = (32) \\ [213] = \left\{ \begin{array}{ccc} 2 & \rightarrow & 1 \\ 1 & \rightarrow & 2 \\ 3 & \rightarrow & 3 \end{array} \right\} = (12)(3) = (12) = (21) \\ [321] = \left\{ \begin{array}{ccc} 3 & \rightarrow & 1 \\ 2 & \rightarrow & 2 \\ 1 & \rightarrow & 3 \end{array} \right\} = (13)(2) = (13) = (31) \end{array} \tag{1.12}$$

As the length of a cycle one might define the amount of numbers which appear in between its brackets. Cycles of length 2, like (12), (13) and (23), are called *transpositions*. A cycle of length l is called an *l-cycle*.

Notice that cycles of length 1 may be omitted, since it is understood that positions in the "n-letter" words, which are not mentioned in the cyclic notation of an element of S_n, are not touched by the operation of that element on those words. This leads, however, to some ambiguity in responding to the questions as to which group a certain group element belongs. For example, in S_6 the meaning of the group element (23) is explicitly given by:

$$(23) = (1)(23)(4)(5)(6).$$

In terms of the cyclic notation, the multiplication table of S_3 is given in table (1.3) below.

S_3	b	I	(132)	(123)	(23)	(12)	(13)
a	ab						
I		I	(132)	(123)	(23)	(12)	(13)
(132)		(132)	(123)	I	(13)	(23)	(12)
(123)		(123)	I	(132)	(12)	(13)	(23)
(23)		(23)	(12)	(13)	I	(132)	(123)
(12)		(12)	(13)	(23)	(123)	I	(132)
(13)		(13)	(23)	(12)	(132)	(123)	I

Table 1.3: The multiplication table of S_3 in the cyclic notation.

As the *order* q of an element of a finite group one defines the number of times it has to be multiplied with itself in order to obtain the identity operator. In table (1.4) is collected the order of each element of S_3. One might notice from that table that the order of an element of the permutation group S_3 (and of any other permutation group) equals the length of its cyclic representation.

element (g)	order (q)	g^q
I	1	I
(132)	3	$(132)(132)(132) = (132)(123) = I$
(123)	3	$(123)(123)(123) = (123)(132) = I$
(23)	2	$(23)(23) = I$
(12)	2	$(12)(12) = I$
(13)	2	$(13)(13) = I$

Table 1.4: The order of the elements of S_3.

Let us study the product of elements of permutation groups in the cyclic notation, in a bit more detail. First we notice for elements of S_3 the following:

$$\begin{aligned}
(12)(23) &= \left\{\begin{array}{ccccc} 1 & \rightarrow & 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 & \rightarrow & 3 \\ 3 & \rightarrow & 2 & \rightarrow & 1 \end{array}\right\} = (123) \\
(123)(31) &= \left\{\begin{array}{ccccc} 1 & \rightarrow & 3 & \rightarrow & 1 \\ 2 & \rightarrow & 2 & \rightarrow & 3 \\ 3 & \rightarrow & 1 & \rightarrow & 2 \end{array}\right\} = (23)
\end{aligned} \tag{1.13}$$

i.e. In multiplying a cycle by a transposition, one can add or take out a number from the cycle, depending on whether it is multiplied from the left or from the right. In the first line of (1.13) the number 1 is added to the cycle (23) by multiplying it by (12) from the left. In the second line the number 1 is taken out of the cycle (123) by multiplying it from the right by (13).

These results for S_3 can be generalized to any permutation group S_n, as follows:

For the numbers i_1, i_2, ..., i_k all different:

$$(i_1\ i_2)(i_2\ \cdots\ i_k) = \left\{\begin{array}{ccccc} i_1 & \rightarrow & i_1 & \rightarrow & i_2 \\ i_2 & \rightarrow & i_3 & \rightarrow & i_3 \\ i_3 & \rightarrow & i_4 & \rightarrow & i_4 \\ & & \vdots & & \\ i_k & \rightarrow & i_2 & \rightarrow & i_1 \end{array}\right\} = (i_1\ i_2\ \cdots\ i_k)$$

$$(i_1\ i_2\ i_3)(i_3\ \cdots\ i_k) = \left\{\begin{array}{ccccc} i_1 & \rightarrow & i_1 & \rightarrow & i_2 \\ i_2 & \rightarrow & i_2 & \rightarrow & i_3 \\ i_3 & \rightarrow & i_4 & \rightarrow & i_4 \\ & & \vdots & & \\ i_k & \rightarrow & i_3 & \rightarrow & i_1 \end{array}\right\} = (i_1\ i_2\ \cdots\ i_k)$$

$$\text{etc} \tag{1.14}$$

$$\text{and :}\quad (i_1\ \cdots\ i_k)(i_k\ i_1) = \left\{\begin{array}{ccccc} i_1 & \rightarrow & i_k & \rightarrow & i_1 \\ i_2 & \rightarrow & i_2 & \rightarrow & i_3 \\ i_3 & \rightarrow & i_3 & \rightarrow & i_4 \\ & & \vdots & & \\ i_k & \rightarrow & i_1 & \rightarrow & i_2 \end{array}\right\} = (i_2\ \cdots\ i_k)$$

$$(i_1\ \cdots\ i_k)(i_k\ i_2\ i_1) = \left\{\begin{array}{ccccc} i_1 & \rightarrow & i_k & \rightarrow & i_1 \\ i_2 & \rightarrow & i_1 & \rightarrow & i_2 \\ i_3 & \rightarrow & i_3 & \rightarrow & i_4 \\ & & \vdots & & \\ i_k & \rightarrow & i_2 & \rightarrow & i_3 \end{array}\right\} = (i_3\ \cdots\ i_k)$$

$$\text{etc} \tag{1.15}$$

From the above procedures (1.14) and (1.15) one might conclude that starting with only its transpositions, *i.e.* (12), (13), ..., $(1n)$, (23), ..., $(2n)$, ..., $(n-1, n)$, it is possible, using group multiplication, to construct all other group elements of the group S_n, by adding numbers to the cycles, removing and interchanging them. However, the same can be achieved with a more restricted set of transpositions, *i.e.*:

$$\{(12), (23), (34), \ldots, (n-1, n)\}. \tag{1.16}$$

For example, one can construct the other elements of S_3 starting with the transpositions (12) and (23), as follows:

$$I = (12)(12),\quad (132) = (23)(12),\quad (123) = (12)(23) \quad \text{and} \quad (13) = (12)(23)(12)\quad .$$

This way one obtains all elements of S_3 starting out with two transpositions and using the product operation which is defined on the group.

For S_4 we may restrict ourselves to demonstrating the construction of the remaining transpositions, starting from the set $\{(12), (23), (34)\}$, because all other group elements can readily be obtained using repeatingly the procedures (1.14) and (1.15). The result is shown below:

$$\begin{aligned}
(13) &= (12)(23)(12) \ , \\
(14) &= (12)(23)(12)(34)(12)(23)(12) \ \text{and} \\
(24) &= (23)(12)(34)(12)(23) \ .
\end{aligned}$$

For other permutation groups the procedure is similar as here outlined for S_3 and S_4.

There exist, however, more interesting properties of cycles: The product of all elements of the set of transpositions shown in (1.16), yields the following group element of S_n:

$$(12)(23)(34)\cdots(n-1,n) = (123\cdots n) \ .$$

It can be shown that the complete permutation group S_n can be *generated* by the following two elements:

$$(12) \ \text{and} \ (123\cdots n). \tag{1.17}$$

We will show this below for S_4. Now, since we know already that it is possible to construct this permutation group starting with the set $\{(12), (23), (34)\}$, we only must demonstrate the construction of the group elements (23) and (34) using the generator set $\{(12), (1234)\}$, *i.e.*:

$$\begin{aligned}
(23) &= (12)(1234)(12)(1234)(1234)(12)(1234)(1234)(12) \ \text{and} \\
(34) &= (12)(1234)(1234)(12)(1234)(1234)(12) \ .
\end{aligned}$$

Similarly, one can achieve comparable results for any symmetric group S_n.

1.3 Partitions and Young diagrams

The cyclic structure of a group element of S_n can be represented by a partition of n. A *partition* of n is a set of positive integer numbers:

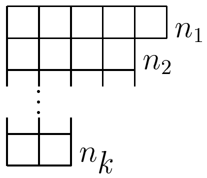
$$[n_1,\ n_2,\ \cdots,\ n_k] \tag{1.18}$$

with the following properties:

$$n_1 \geq n_2 \geq \cdots \geq n_k \ \ , \ \ \text{and} \ \ \ n_1 + n_2 + \cdots + n_k = n.$$

Such partitions are very often visualized by means of Young diagrams as shown below.

A way of representing a partition $[n_1,\ n_2,\ \cdots,\ n_k]$, is by means of a *Young diagram*, which is a figure of n boxes arranged in horizontal rows: n_1 boxes in the upper row, n_2 boxes in the second row, $\cdots$, n_k boxes in the k-th and last row, as shown in the figure.



Let us assume that a group element of S_n consists of a n_1-cycle, followed by a n_2-cycle, followed by ..., followed by a n_k-cycle. And let us moreover assume that those cycles are ordered according to their lengths, such that the largest cycle (n_1) comes first and the smallest cycle (n_k) last. In that case is the cyclic structure of this particular group element represented by the partition (1.18) or, alternatively, by the above Young diagram. In table (1.5) are represented the partitions and Young diagrams for the several cyclic structures of S_3.

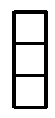
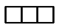
Cyclic notation	partition	Young diagram
(1)(2)(3)	$[1^3]$=[111]	
(132) (123)	[3]	
(1)(23) (13)(2) (12)(3)	[21]	

Table 1.5: The representation by partitions and Young diagrams of the cyclic structures of the permutation group S_3.

1.4 Subgroups and cosets

It might be noticed from the multiplication table (1.3) at page 5 of S_3, that if one restricts oneself to the first three rows and columns of the table, then one finds the group A_3 of symmetric permutations of three objects. In table (1.6) those lines and columns are collected.

A_3	b	I	(132)	(123)
a	ab			
I		I	(132)	(123)
(132)		(132)	(123)	I
(123)		(123)	I	(132)

Table 1.6: The multiplication table of the group A_3 of symmetric permutations of three objects.

Notice that $A_3 = \{I,\ (231)\ (321)\}$ satisfies all conditions for a group: existence of a product, existence of an identity operator, associativity and the existence of the inverse of each group element. A_3 is called a *subgroup* of S_3.

Other subgroups of S_3 are the following sets of operations:

$$\{I,\ (12)\}\,,\quad \{I,\ (23)\}\,,\quad \text{and}\quad \{I,\ (13)\}\,. \tag{1.19}$$

A *coset* of a group G with respect to a subgroup S is a set of elements of G which is obtained by multiplying one element g of G with all elements of the subgroup S. There are left- and right cosets, given by:

$$gS \ \text{ represents a left coset and } \ Sg \ \text{ a right coset.} \tag{1.20}$$

In order to find the right cosets of S_3 with respect to the subgroup A_3, we only have to study the first three lines of the multiplication table (1.3) at page 5 of S_3. Those lines are for our convenience collected in table (1.7) below.

A_3	b	I	(132)	(123)	(23)	(12)	(13)
a	ab						
I		I	(132)	(123)	(23)	(12)	(13)
(132)		(132)	(123)	I	(13)	(23)	(12)
(123)		(123)	I	(132)	(12)	(13)	(23)

Table 1.7: The right cosets of S_3 with respect to its subgroup A_3.

Each column of the table forms a right coset A_3g of S_3. We find from table (1.7) two different types of cosets, *i.e.*: $\{I, (132), (123)\}$ and $\{(23), (12), (13)\}$. Notice that no group element is contained in both types of cosets and also that each group element of S_3 appears at least in one of the cosets of table (1.7).

This is valid for any group:

1. Either $Sg_i = Sg_j$, or no group element in Sg_i coincides with a group element of Sg_j.

This may be understood as follows: Let the subgroup S of the group G be given by the set of elements $\{g_1, g_2, \ldots, g_k\}$. Then we know that for a group element g_α of the subgroup S the operator products g_1g_α, g_2g_α, ..., g_kg_α are elements of S too. Moreover we know that in the set $\{g_1g_\alpha, g_2g_\alpha, \ldots, g_kg_\alpha\}$ each element of S appears once and only once. So, the subgroup S might alternatively be given by this latter set of elements. Now, if for two group elements g_α and g_β of the subgroup S and two elements g_i and g_j of the group G one has $g_\alpha g_i = g_\beta g_j$, then also $g_1g_\alpha g_i = g_1g_\beta g_j$, $g_2g_\alpha g_i = g_2g_\beta g_j$, $\cdots$ $g_kg_\alpha g_i = g_kg_\beta g_j$. Consequently, the two cosets are equal, *i.e.*

$$\{g_1g_\alpha g_i,\, g_2g_\alpha g_i,\, \ldots,\, g_kg_\alpha g_i\} = \{g_1g_\beta g_i,\, g_2g_\beta g_i,\, \ldots,\, g_kg_\beta g_i\} \quad ,$$

or equivalently $Sg_i = Sg_j$.

2. Each group element g appears at least in one coset. This is not surprising, since I belongs always to the subgroup S and thus comes $g = Ig$ at least in the coset Sg.

3. The number of elements in each coset is equal to the order of the subgroup S.

As a consequence of the above properties for cosets, one might deduce that the number of elements of a subgroup S is always an integer fraction of the number of elements of the group G, *i.e.*:

$$\frac{\text{order of } G}{\text{order of } S} = \text{positive integer.} \tag{1.21}$$

A subgroup S is called *normal* or *invariant*, if for all elements g in the group G the following holds:

$$g^{-1}Sg = S. \tag{1.22}$$

One might check that A_3 is an invariant subgroup of S_3.

1.5 Equivalence classes

Two elements a and b of a group G are said to be *equivalent*, when there exists a third element g of G such that:

$$b = g^{-1}\, a\, g\ . \tag{1.23}$$

This definition of equivalence has the following properties:

1. Each group element g of G is equivalent to itself, according to:

$$g = Ig = g^{-1}\, g\, g\ . \tag{1.24}$$

2. If a is equivalent to b and a is also equivalent to c, then b and c are equivalent, because according to the definition of equivalence (1.23) there must exist group elements g and h of G such that:

$$c = h^{-1}ah = h^{-1}g^{-1}bgh = (gh)^{-1}b(gh)\ , \tag{1.25}$$

and because the product gh must represent an element of the group G.

Because of the above properties (1.24) and (1.25), a group G can be subdivided into sets of equivalent group elements, the so-called *equivalence classes*.

S_3	b	I	(132)	(123)	(23)	(12)	(13)
a	$b^{-1}ab$						
I		I	I	I	I	I	I
(132)		(132)	(132)	(132)	(123)	(123)	(123)
(123)		(123)	(123)	(123)	(132)	(132)	(132)
(23)		(23)	(13)	(12)	(23)	(13)	(12)
(12)		(12)	(23)	(13)	(13)	(12)	(23)
(13)		(13)	(12)	(23)	(12)	(23)	(13)

Table 1.8: The equivalence operation for all elements of the group S_3.

In table (1.8) we have constructed all possible group elements $g^{-1}ag$ which are equivalent to group element a of the group S_3. Every horizontal line in that table contains the group elements of one equivalence class of S_3.

Notice from table (1.8) that group elements which have the same cyclic structure are in the same equivalence class of S_3. Whereas, group elements with different cyclic structures are in different equivalence classes of S_3.

In table (1.5) at page 8 are separated the group elements of S_3 which belong to different equivalence classes, as well as the representation of their cyclic structure by partitions and Young diagrams. From that table it might be clear that for S_3 and similarly for the other permutation groups, the equivalence classes can be represented by partitions and Young diagrams.

As an example of an other permutation group, we collect in table (1.9) the equivalence classes and their representations by partitions and Young diagrams of the group S_4.

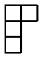
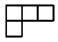
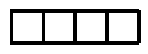
Equivalence class	partition	Young diagram
(1)(2)(3)(4)	[1111]	
(1)(2)(34) (1)(23)(4) (1)(24)(3) (12)(3)(4) (13)(2)(4) (14)(2)(3)	[211]	
(12)(34) (13)(24) (14)(23)	[22]	
(1)(234) (1)(243) (123)(4) (132)(4) (124)(3) (142)(3) (134)(2) (143)(2)	[31]	
(1234) (1243) (1324) (1342) (1423) (1432)	[4]	

Table 1.9: The equivalence classes and their representation by partitions and Young diagrams of the permutation group S_4.

Chapter 2

Representations of finite groups.

In this chapter we will discuss the definition and the properties of matrix representations of finite groups.

2.1 What is a matrix representation of a group?

An n-dimensional matrix representation of a group element g of the group G, is given by a transformation $D(g)$ of an n-dimensional (complex) vector space V_n into itself, *i.e.*:

$$D(g) \ : \ V_n \ \rightarrow \ V_n. \tag{2.1}$$

The matrix for $D(g)$ is known, once the transformation of the basis vectors $\hat{e}_i$ $(i = 1, \cdots, n)$ of V_n is specified.

2.2 The word-representation $D^{(w)}$ of S_3.

Instead of the "three-letter" words introduced in section (1.1), we represent those words here by vectors of a three-dimensional vector space, *i.e.*:

$$ABC \ \Rightarrow \left(\begin{array}{c} A \\ B \\ C \end{array}\right) = A\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right) + B\left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right) + C\left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right). \tag{2.2}$$

Moreover, the operations (1.12) are "translated" into vector transformations. For example, the operation of (12) on a "three-letter" word, given in (1.3), is here translated into the transformation:

$$D^{(w)}((12))\left(\begin{array}{c} A \\ B \\ C \end{array}\right) = \left(\begin{array}{c} B \\ A \\ C \end{array}\right). \tag{2.3}$$

Then, assuming arbitrary (complex) values for A, B and C, the transformations $D^{(w)}(g)$ can be represented by matrices. In the above example one finds:

$$D^{(w)}((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.4}$$

Similarly, one can represent the other operations of S_3 by matrices $D^{(w)}(g)$. In table (2.1) below, the result is summarized.

g	$D^{(w)}(g)$	g	$D^{(w)}(g)$	g	$D^{(w)}(g)$
I	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(132)	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	(123)	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
(23)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	(12)	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(13)	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Table 2.1: The word-representation $D^{(w)}$ of S_3.

The structure of a matrix representation D of a group G reflects the structure of the group, *i.e.* the group product is preserved by the matrix representation. Consequently, we have for any two group elements a and b of G the following property of their matrix representations $D(a)$ and $D(b)$:

$$D(a)D(b) = D(ab). \tag{2.5}$$

For example, for $D^{(w)}$ of S_3, using table (2.1) for the word-representation and multiplication table (1.3) at page 5, we find for the group elements (12) and (23):

$$\begin{aligned} D^{(w)}((12))D^{(w)}((23)) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= D^{(w)}((123)) = D^{(w)}((12)(23)). \end{aligned} \tag{2.6}$$

Notice from table (2.1) that all group elements of S_3 are represented by a different matrix. A representation with that property is called a *faithful* representation.

Notice also from table (2.1) that I is represented by the unit matrix. This is always the case, for any representation.

2.3 The regular representation.

In the *regular representation* $D^{(r)}$ of a finite group G, the matrix which represents a group element g is defined in close analogy with the product operation of g on all elements of the group. The dimension of the vector space is in the case of the regular representation equal to the order p of the group G. The elements $\hat{e}_i$ of the orthonormal basis $\{\hat{e}_1, \cdots, \hat{e}_p\}$ of the vector space are somehow related to the group elements g_i of G.

The operation of the matrix which represents the transformation $D^{(r)}(g_k)$ on a basis vector $\hat{e}_j$ is defined as follows:

$$D^{(r)}(g_k)\ \hat{e}_j = \sum_{i=1}^{p} \delta(g_k g_j, g_i)\hat{e}_i, \tag{2.7}$$

where the "Kronecker delta function" is here defined by:

$$\delta(g_k g_j, g_i) = \begin{cases} 1 & \text{if} \quad g_k g_j = g_i \\ \\ 0 & \text{if} \quad g_k g_j \neq g_i \end{cases} .$$

Let us study the regular representation for the group S_3. Its dimension is equal to the order of S_3, *i.e.* equal to 6. So, each element g of S_3 is represented by a 6×6 matrix $D^{(r)}(g)$.

For the numbering of the group elements we take the one given in table (1.1) at page 1.

As an explicit example, let us construct the matrix $D^{(r)}(g_3 = (123))$. From the multiplication table (1.2) at page 3 we have the following results:

$$\begin{array}{lll} g_3g_1 = g_3 \ , & g_3g_2 = g_1 \ , & g_3g_3 = g_2 \ , \\ g_3g_4 = g_5 \ , & g_3g_5 = g_6 \ , & g_3g_6 = g_4 \ . \end{array}$$

Using these results and formula (2.7), we find for the orthonormal basis vectors $\hat{e}_i$ of the six-dimensional vector space, the set of transformations:

$$\begin{aligned} D^{(r)}(g_3)\hat{e}_1 &= \hat{e}_3 \\ D^{(r)}(g_3)\hat{e}_2 &= \hat{e}_1 \\ D^{(r)}(g_3)\hat{e}_3 &= \hat{e}_2 \\ D^{(r)}(g_3)\hat{e}_4 &= \hat{e}_5 \\ D^{(r)}(g_3)\hat{e}_5 &= \hat{e}_6 \\ D^{(r)}(g_3)\hat{e}_6 &= \hat{e}_4. \end{aligned} \tag{2.8}$$

Consequently, the matrix which belongs to this set of transformations, is given by:

$$D^{(r)}(g_3) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.9}$$

The other matrices can be determined in a similar way. The resulting regular representation $D^{(r)}$ of S_3 is shown in table (2.2) below.

g	$D^{(r)}(g)$	g	$D^{(r)}(g)$
I	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(23)	$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
(132)	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	(12)	$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
(123)	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	(13)	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Table 2.2: The regular representation $D^{(r)}$ of S_3.

Notice from table (2.2) that, as in the previous case, each group element of S_3 is represented by a different matrix. The regular representation is thus a faithful representation.

From formula (2.7) one might deduce the following general equation for the matrix elements of the regular representation of a group:

$$\left[D^{(r)}(g_k)\right]_{ij} = \delta(g_k g_j, g_i). \tag{2.10}$$

2.4 The symmetry group of the triangle.

In this section we study those transformations of the 2-dimensional plane into itself, which leave invariant the equilateral triangle shown in figure (2.1).

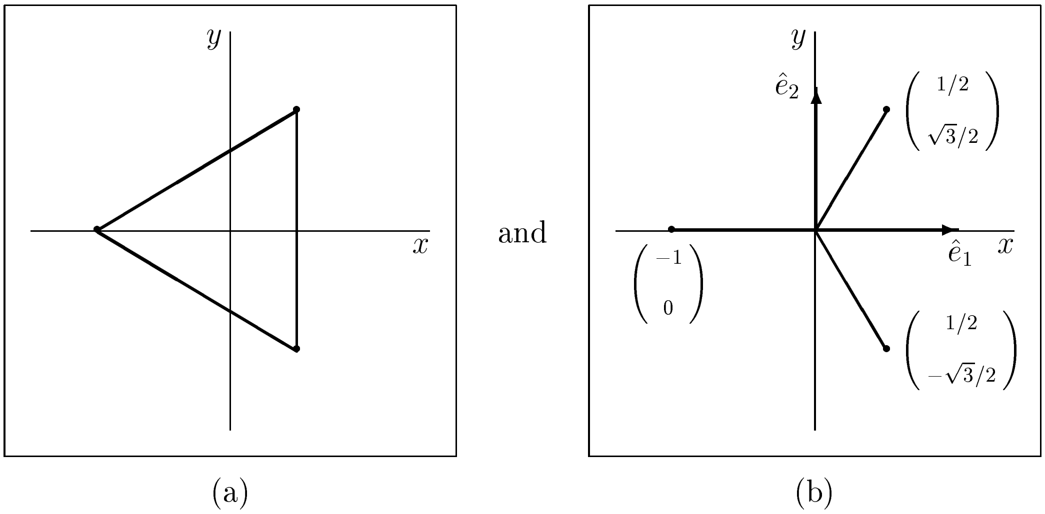


Figure 2.1: The equilateral triangle in the plane. The centre of the triangle (a) coincides with the origin of the coordinate system. In (b) are indicated the coordinates of the corners of the triangle and the unit vectors in the plane.

A rotation, R, of 120° of the plane maps the triangle into itself, but maps the unit vectors $\hat{e}_1$ and $\hat{e}_2$ into $\hat{e}'_1$ and $\hat{e}'_2$ as is shown in figure (2.2).

From the coordinates of the transformed unit vectors, one can deduce the matrix for this rotation, which is given by:

$$R = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} . \tag{2.11}$$

Twice this rotation (*i.e.* R^2) also maps the triangle into itself. The matrix for that operation is given by:

$$R^2 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} . \tag{2.12}$$

Three times (*i.e.* R^3) leads to the identity operation, I.

Reflection, P, around the x-axis also maps the triangle into itself. In that case the unit vector $\hat{e}_1$ maps into $\hat{e}_1$ and $\hat{e}_2$ maps into $-\hat{e}_2$. So, the matrix for reflection around the x-axis is given by:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.13}$$

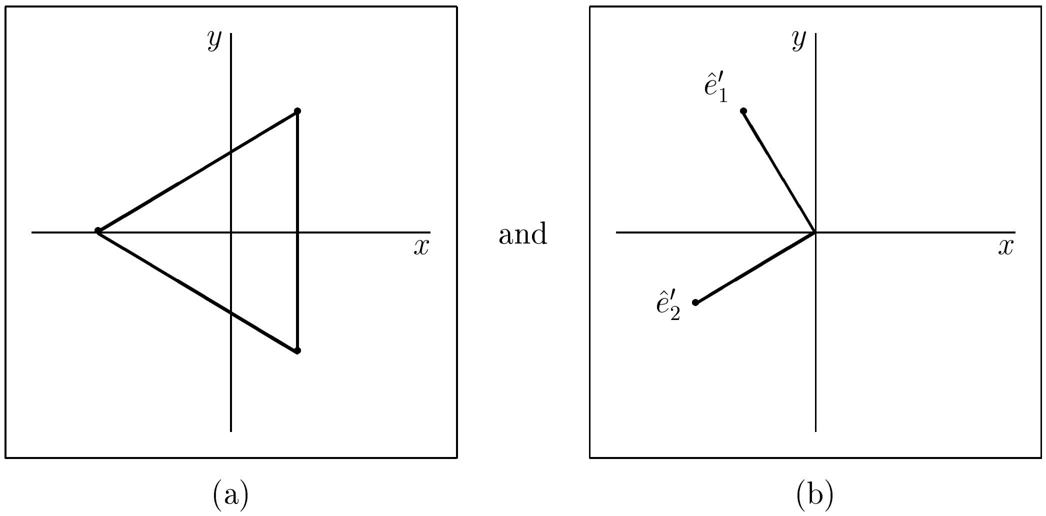


Figure 2.2: The effect of a rotation of 120° of the plane: In (a) the transformed triangle is shown. In (b) are indicated the transformed unit vectors.

The other possible invariance operations for the triangle of figure (2.1) at page 17 are given by:

$$PR = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ & \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \quad \text{and} \quad PR^2 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ & \\ \sqrt{3}/2 & 1/2 \end{pmatrix} . \tag{2.14}$$

This way we obtain the six invariance operations for the triangle, *i.e.*:

$$\left\{ I, R, R^2, P, PR, PR^2 \right\} . \tag{2.15}$$

Now, let us indicate the corners of the triangle of figure (2.1) at page 17 by the letters A, B and C, as indicated in figure (2.3) below.

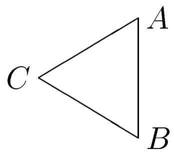


Figure 2.3: The position of the corners A, B and C of the triangle.

Then, the invariance operations (2.15) might be considered as rearrangements of the three letters A, B and C.

For example, with respect to the positions of the corners, the rotation R defined in formula (2.11) is represented by the rearrangement of the three letters A, B and C which is shown in figure (2.4).

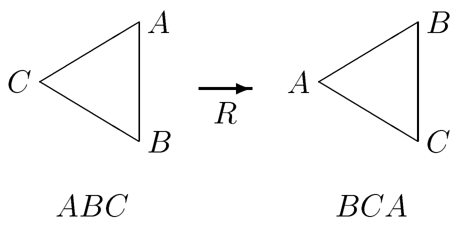


Figure 2.4: The position of the corners A, B and C of the triangle after a rotation of $120°$.

We notice, that with respect to the order of the letters A, B and C, a rotation of $120°$ has the same effect as the operation (132) of the permutation group S_3 (compare formula 1.3). Consequently, one may view the matrix R of equation (2.11) as a two-dimensional representation of the group element (132) of S_3.

Similarly, one can compare the other group elements of S_3 with the above transformations (2.15). The resulting two-dimensional representation $D^{(2)}$ of the permutation group S_3 is shown in table (2.3) below.

g	$D^{(2)}(g)$	g	$D^{(2)}(g)$
I	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	(23)	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$
(132)	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	(12)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(123)	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	(13)	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$

Table 2.3: The two-dimensional representation $D^{(2)}$ of S_3.

Notice that $D^{(2)}$ is a faithful representation of S_3.

2.5 One-dimensional and trivial representations.

Every group G has an one-dimensional *trivial representation* $D^{(1)}$ for which each group element g is represented by 1, *i.e.*:

$$D^{(1)}(g)\hat{e}_1 = \hat{e}_1. \tag{2.16}$$

This representation satisfies all requirements for a (one-dimensional) matrix representation.

For example: if $g_i g_j = g_k$, then

$$D^{(1)}(g_i)D^{(1)}(g_j) = 1 \cdot 1 = 1 = D^{(1)}(g_k) = D^{(1)}(g_i g_j),$$

in agreement with the requirement formulated in equation (2.5).

For the permutation groups S_n exists yet another one-dimensional representation $D^{(1')}$, given by:

$$D^{(1')}(g) = \begin{cases} 1 & \text{for even permutations} \\ -1 & \text{for odd permutations} \end{cases} . \tag{2.17}$$

Other groups might have other (complex) one-dimensional representations. For example the subgroup A_3 of even permutations of S_3 can be represented by the following complex numbers:

$$D(I) = 1 \ , \quad D((132)) = e^{2\pi i/3} \ \text{ and } \ D((123)) = e^{-2\pi i/3} \ , \tag{2.18}$$

or also by the non-equivalent representation, given by:

$$D(I) = 1 \ , \quad D((132)) = e^{-2\pi i/3} \ \text{ and } \ D((123)) = e^{2\pi i/3} \ . \tag{2.19}$$

2.6 Reducible representations.

It might be clear from the preceding sections that the number of representations of a group is unlimited, even for a finite group like S_3. So, the question comes up whether there exists any order in this jungle of representations. Evidently, the answer to this question is positive.

Let us first discuss equivalent matrix representations of a group. Consider two different n-dimensional representations, $D^{(\alpha)}$ and $D^{(\beta)}$, of the group G. If there exists a non-singular $n \times n$ matrix S, such that for all group elements g yields:

$$D^{(\alpha)}(g) = S^{-1}\, D^{(\beta)}(g)\, S\ , \tag{2.20}$$

then the two representations $D^{(\alpha)}$ and $D^{(\beta)}$ are said to be *equivalent*.

For all practical purposes, equivalent representations of a group G are considered to be the same representation of G. In fact, the non-singular $n \times n$ matrix S represents a simple basis transformation in the n-dimensional vector space V_n:

Let $\vec{v}$ and $\vec{w}$ represent two vectors of V_n, such that for the transformation induced by the representation of the group element g of G, follows:

$$D^{(\beta)}(g)\, \vec{v} = \vec{w}. \tag{2.21}$$

A different choice of basis in V_n can be represented by a matrix S, for which:

$$S\ :\ \hat{e}_i\ \ (i = 1, \cdots, n)\ \ \rightarrow\ \ \hat{e}'_j\ \ (j = 1, \cdots, n). \tag{2.22}$$

At the new basis $\hat{e}'_j$ the components of the vectors $\vec{v}$ and $\vec{w}$ are different. The relation between the initial and new components may symbolically be written by:

$$\vec{v}\ \rightarrow\ S^{-1}\vec{v}\ \ \text{and}\ \ \vec{w}\ \rightarrow\ S^{-1}\vec{w}. \tag{2.23}$$

Inserting the above relations (2.23) into the equation (2.21), one finds:

$$D^{(\beta)}(g)\, \vec{v} = \vec{w}\ \rightarrow\ S^{-1}D^{(\beta)}(g)SS^{-1}\, \vec{v} = S^{-1}\, \vec{w}. \tag{2.24}$$

In comparing the result (2.24) to the formula (2.20), we find that $D^{(\alpha)}(g)$ is just the representation of the group element g of G at the new basis in V_n.

As an example, let us study the three-dimensional word-representation $D^{(w)}$ of the permutation group S_3, given in table (2.1) at page 14. For the basis transformation S of (2.20) we take the following anti-orthogonal matrix:

$$S = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -\sqrt{2/3} & 0 \end{pmatrix}. \tag{2.25}$$

The inverse of this basis transformation is given by:

$$S^{-1} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -\sqrt{2/3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} . \tag{2.26}$$

For each group element g of S_3 we determine the representation given by:

$$D^{(wS)}(g) = S^{-1}\, D^{(w)}(g)\, S\ , \tag{2.27}$$

which is equivalent to the word-representation of S_3. The result is collected in table (2.4) below.

g	$S^{-1}D^{(w)}(g)S$	g	$S^{-1}D^{(w)}(g)S$
I	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(23)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$
(132)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}$	(12)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
(123)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}$	(13)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix}$

Table 2.4: The three-dimensional representation $D^{(wS)} = S^{-1}D^{(w)}S$ of S_3.

From the table (2.4) for the three-dimensional representation which is equivalent to the word-representation of S_3 shown in table (2.1) at page 14, we observe the following:

1. For all group elements g of S_3 their representation $D^{(wS)}(g) = S^{-1}D^{(w)}(g)S$ has the form:

$$D^{(wS)}(g) = S^{-1}D^{(w)}(g)S = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & 2\times 2 \\ 0 & \text{matrix} \end{pmatrix} . \tag{2.28}$$

2. The 2×2 submatrices of the matrices $D^{(wS)}(g) = S^{-1}D^{(w)}(g)S$ shown in table (2.4) are equal to the two-dimensional representations $D^{(2)}(g)$ of the group elements g of S_3 as given in table (2.3) at page 19.

As a consequence of the above properties, the 3-dimensional word-representation $D^{(w)}$ of S_3 can be subdivided into two sub-representations of S_3.

In order to show this we take the following orthonormal basis for the three-dimensional vector space V_3 of the word representation:

$$\hat{a} = \begin{pmatrix} 1/\sqrt{3} \\ \\ 1/\sqrt{3} \\ \\ 1/\sqrt{3} \end{pmatrix} \quad , \quad \hat{e}_1 = \begin{pmatrix} 1/\sqrt{6} \\ \\ 1/\sqrt{6} \\ \\ -\sqrt{2/3} \end{pmatrix} \quad , \quad \hat{e}_2 = \begin{pmatrix} 1/\sqrt{2} \\ \\ -1/\sqrt{2} \\ \\ 0 \end{pmatrix} . \tag{2.29}$$

For this basis, using table (2.1) at page 14, we find for all group elements g of S_3, that:

$$D^{(w)}(g)\ \hat{a} = \hat{a} = D^{(1)}(g)\hat{a} . \tag{2.30}$$

Consequently, at the one-dimensional subspace of V_3 which is spanned by the basis vector $\hat{a}$, the word-representation is equal to the trivial representation (2.16).

At the two-dimensional subspace spanned by the basis vectors $\hat{e}_1$ and $\hat{e}_2$ of (2.29), one finds for vectors $\vec{v}$ defined by:

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 = \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} , \tag{2.31}$$

that for all group elements g of S_3 yields:

$$D^{(w)}(g)\vec{v} = D^{(2)}(g) \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} . \tag{2.32}$$

For example:

$$D^{(w)}((123))\vec{v} \quad = \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left\{ v_1 \begin{pmatrix} 1/\sqrt{6} \\ \\ 1/\sqrt{6} \\ \\ -\sqrt{2/3} \end{pmatrix} + v_2 \begin{pmatrix} 1/\sqrt{2} \\ \\ -1/\sqrt{2} \\ \\ 0 \end{pmatrix} \right\}$$

$$
\begin{aligned}
&= v_1 \begin{pmatrix} -\sqrt{2/3} \\ \\ 1/\sqrt{6} \\ \\ 1/\sqrt{6} \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ \\ 1/\sqrt{2} \\ \\ -1/\sqrt{2} \end{pmatrix} \\
&= (-\frac{1}{2}v_1 + \frac{\sqrt{3}}{2}v_2)\hat{e}_1 + (-\frac{\sqrt{3}}{2}v_1 - \frac{1}{2}v_2)\hat{e}_2 \\
&= \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} = D^{(2)}((123))\vec{v}\,. \qquad (2.33)
\end{aligned}
$$

For the other group elements g of S_3 one may verify that $D^{(w)}(g)$ also satisfies (2.32).

In general, an n-dimensional representation $D^{(\alpha)}$ of the group G on a complex vector space V_n is said to be *reducible* if there exists a basis transformation S in V_n such that for all group elements g of G yields:

$$
S^{-1}D^{(\alpha)}(g)S = \begin{pmatrix} D^{(a)}(g) & 0 & 0 & \cdots & 0 \\ \\ 0 & D^{(b)}(g) & 0 & \cdots & 0 \\ \\ \vdots & \vdots & \vdots & & \vdots \\ \\ 0 & 0 & 0 & \cdots & D^{(z)}(g) \end{pmatrix} , \qquad (2.34)
$$

where $D^{(a)}$, $D^{(b)}$, $\cdots$, $D^{(z)}$ are representations of dimension smaller than n, such that:

$$
dim(a) + dim(b) + \cdots + dim(z) = n. \qquad (2.35)
$$

The vector space V_n can in that case be subdivided into smaller subspaces V_α (α=a, b, $\ldots$, z) in each of which $D^{(\alpha)}(g)$ is represented by a matrix which has a dimension smaller than n, for all group elements g of G. If one selects a vector $\vec{v}_\alpha$ in one of the subspaces V_α, then the transformed vector $\vec{v}_{\alpha,\,g} = D^{(\alpha)}(g)\vec{v}_\alpha$ is also a vector of the same subspace V_α for any group element g of G.

2.7 Irreducible representations (Irreps).

An n-dimensional representation $D^{(\beta)}$ of a group G which cannot be reduced by the procedure described in the previous section (2.6), is said to be *irreducible*.

Irreducible representations or *irreps* are the building blocks of representations. Other representations can be constructed out of irreps, following the procedure outlined in the previous section and summarized in formula (2.34).

The two-dimensional representation $D^{(2)}$ of S_3 given in table (2.3) at page 19 is an example of an irreducible representation of the permutation group S_3. Other examples of irreps of S_3 are the trivial representation $D^{(1)}$ given in equation (2.16) and the one-dimensional representation $D^{(1')}$ given in equation (2.17).

The one-dimensional representations $D^{(1)}$ and the ones shown in equations (2.18) and (2.19) are irreps of the group A_3 defined in table (1.6) at page 9.

If D is an irrep of the group $G = \{g_1, \ldots, g_p\}$ in the vector space V, then for any arbitrary vector $\vec{v}$ in V the set of vectors $\{\vec{v}_1 = D(g_1)\vec{v}, \ldots, \vec{v}_n = D(g_n)\vec{v}\}$ may not form a subspace of V but must span the whole vector space V.

2.8 Unitary representations.

One may always restrict oneself to unitary representations, since according to the *theorem of Maschke*:

$$\text{Each representation is equivalent to an } \textit{unitary representation.} \tag{2.36}$$

A representation D is an unitary representation of a group G when all group elements g of G are represented by unitary matrices $D(g)$. A matrix A is said to be *unitary* when its inverse equals its complex conjugated transposed, *i.e.*:

$$A^\dagger A = AA^\dagger = \mathbf{1}. \tag{2.37}$$

Instead of proving the theorem of Maschke (2.36) we just construct an unitary equivalent representation for an arbitrary representation D of a group G of order p. First recall the following properties of matrix multiplication:

$$(AB)^\dagger = B^\dagger A^\dagger \text{ and } (AB)^{-1} = B^{-1}A^{-1} \quad . \tag{2.38}$$

Next, let us study the matrix T, given by:

$$T = \sum_{k=1}^{p} D^\dagger(g_k)D(g_k) \quad . \tag{2.39}$$

One can easily verify, using (2.38), that this matrix is Hermitean, *i.e.*:

$$T^\dagger = \sum_{k=1}^{p} \{D^\dagger(g_k)D(g_k)\}^\dagger = \sum_{k=1}^{p} D^\dagger(g_k)D(g_k) = T \quad .$$

It moreover satisfies the following property (using once more 2.38):

$$\begin{aligned} D^\dagger(g_i)TD(g_i) &= \sum_{k=1}^{p} D^\dagger(g_i)D^\dagger(g_k)D(g_k)D(g_i) = \sum_{k=1}^{p} D^\dagger(g_kg_i)D(g_kg_i) \\ &= \sum_{j=1}^{p} D^\dagger(g_j)D(g_j) \;=\; T \;, \end{aligned} \tag{2.40}$$

where we also used the fact that in multiplying one particular group element g_i with all group elements g_k $(k = 1, \ldots p)$, one obtains all group elements g_j only once.

Next, we define a basis transformation S such that:

1. $S^\dagger = S$, and **2.** $S^2 = T^{-1}$.

S represents a transformation from a basis orthonormal with respect to the scalar product given by:

$$(\,\vec{x}\,,\,\vec{y}\,) = \sum_{j=1}^{p} \left\langle D(g_j)\vec{x}\,||\,D(g_j)\vec{y}\right\rangle \;,$$

to a basis orthonormal with respect to the scalar product given by:

$$\langle\vec{x}\,||\,\vec{y}\rangle \;.$$

With the definition of S the property (2.40) can be rewritten by:

$$D^\dagger(g_i)S^{-2}D(g_i) = S^{-2} \;.$$

Multiplying this line from the right by $D^{-1}(g_i)S$ and from the left by S, one obtains:

$$SD^\dagger(g_i)S^{-1} = S^{-1}D^{-1}(g_i)S \;,$$

which, using the fact that S is Hermitean and using (2.38), finally gives:

$$(S^{-1}D(g_i)S)^\dagger = (S^{-1}D(g_i)S)^{-1} \;.$$

Leading to the conclusion that the matrix representation $S^{-1}DS$, equivalent to D, is unitary.

As a consequence of (2.36) and because of the fact that equivalent representations are considered the "same" representations as discussed in section (2.6), we may in the following restrict ourselves to *unitary representations* of groups.

One can easily verify that $D^{(1)}$, $D^{(1')}$, $D^{(2)}$, $D^{(w)}$, $D^{(wS)}$ and $D^{(r)}$ are unitary representations of S_3.

Chapter 3

The standard irreps of S_n.

In this chapter we will discuss the standard construction of irreps for the symmetric groups S_n.

3.1 Standard Young tableaux and their associated Yamanouchi symbols.

A *Young tableau* is a Young diagram (see section (1.3) at page 8 for the definition of a Young diagram), of n boxes filled with n numbers in some well specified way.

A *standard Young tableau* is a Young diagram of n boxes which contains the numbers 1 to n in such a way that the numbers in each row increase from the left to the right, and in each column increase from the top to the bottom.

For S_3, using the result of table (1.5) at page 8, one finds four different possible standard Young tableaux, *i.e.*:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ ,\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\ \text{ and }\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ . \tag{3.1}$$

Notice that one Young diagram in general allows for several different standard Young tableaux.

Each standard tableau can be denoted in a compact and simple way by a *Yamanouchi symbol*, $M = M_1, M_2, \cdots, M_n$. This is a row of n numbers M_i $(i = 1, \cdots, n)$, where M_i is the number of the row in the standard Young tableau, counting from above, in which the number i appears. Below we show the Yamanouchi symbols corresponding to the standard Young tableaux for S_3 given in (3.1):

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\quad \left\{\begin{array}{l} 1 \text{ in the } 1^{\text{st}} \text{ row} \\ 2 \text{ in the } 2^{\text{nd}} \text{ row} \\ 3 \text{ in the } 3^{\text{rd}} \text{ row}\end{array}\right\}\quad M = 123\ ,$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\quad \left\{\begin{array}{l} 1 \text{ in the } 1^{\text{st}} \text{ row} \\ 2 \text{ in the } 1^{\text{st}} \text{ row} \\ 3 \text{ in the } 1^{\text{st}} \text{ row}\end{array}\right\}\quad M = 111\ ,$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \left\{ \begin{array}{l} 1 \text{ in the } 1^{\text{st}} \text{ row} \\ 2 \text{ in the } 1^{\text{st}} \text{ row} \\ 3 \text{ in the } 2^{\text{nd}} \text{ row} \end{array} \right\} \quad M = 112 \ \text{, and}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \left\{ \begin{array}{l} 1 \text{ in the } 1^{\text{st}} \text{ row} \\ 2 \text{ in the } 2^{\text{nd}} \text{ row} \\ 3 \text{ in the } 1^{\text{st}} \text{ row} \end{array} \right\} \quad M = 121 \ . \tag{3.2}$$

There exists moreover a certain order in the Yamanouchi symbols for the same Young diagram: First one orders these symbols with respect to increasing values of M_1, then with respect to increasing values of M_2, etc.

Let us, as an example, order the Yamanouchi symbols for the standard Young tableaux of the partition $[\mu] = [32]$ of the permutation group S_5. The possible Yamanouchi symbols, properly ordered, are given by:

$$11122 \ , \ 11212 \ , \ 11221 \ , \ 12112 \ \text{and} \ 12121 \ .$$

3.2 Hooklength, hookproduct and axial distance.

Other quantities related to Young diagrams, are the hooklength of a box, the hookproduct and the axial distance between boxes.

As the *hook* of a certain box j in a Young diagram, one defines a set of boxes given by the following receipt:

$$\begin{array}{ll} 1. & \text{The box } j \text{ itself.} \\ 2. & \text{All boxes to the right of } j \text{ in the same row.} \\ 3. & \text{All boxes below } j \text{ in the same column.} \end{array} \tag{3.3}$$

As an example, we show below the hook of the first box in the second row of the Young diagram belonging to the partition [433] of S_{10}:

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline \star & \star & \star & \multicolumn{1}{c}{} \\ \cline{1-3} \star & & & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

As the *hooklength* of a certain box j in a Young diagram, one defines the number of boxes in the hook of box j. In the above example one has a hooklength of 4.

The *hooktableau* of a Young diagram is obtained by placing in each box of the diagram, the hooklength of this box. For example, the hooktableau of the above diagram [433] of S_{10}, is given by:

$$\begin{array}{|c|c|c|c|}\hline 6 & 5 & 4 & 1 \\ \hline 4 & 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

The *hookproduct* $h[\mu]$ of the Young diagram belonging to the partition $[\mu]$, is the product of all numbers in its hooktableau. In the above example, $[\mu] = [433]$ of S_{10}, one finds $h[433] = 17280$.

For the group S_3, one has the following hooktableaux:

$$\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\ ,\quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}\quad\text{and}\quad \begin{array}{|c|c|}\hline 3&1\\\hline 1\\\cline{1-1}\end{array}\ . \tag{3.4}$$

The related hookproducts are $h[111] = h[3] = 6$ and $h[21] = 3$.

The *axial distance* $\rho(M; x, y)$ between two boxes x and y of a standard Young tableau M (M represents the Yamanouchi symbol of the tableau), is the number of steps (horizontal and/or vertical) to get from x to y, where steps contribute:

$$\left\{\begin{array}{ll} +1 & \text{when going } \textit{down} \text{ or to the } \textit{left}, \\ -1 & \text{when going } \textit{up} \text{ or to the } \textit{right}. \end{array}\right. \tag{3.5}$$

Notice that the axial distance does not depend on the path. Below we show as an example the axial distances for the Young tableaux which belong to the partition $[\mu] = [21]$ of the permutation group S_3.

$$\begin{array}{lll}
\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} & M = 112 & \left\{\begin{array}{l} \rho(112;1,2) = -\rho(112;2,1) = -1 \\ \rho(112;1,3) = -\rho(112;3,1) = +1 \\ \rho(112;2,3) = -\rho(112;3,2) = +2 \end{array}\right. \\
\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & M = 121 & \left\{\begin{array}{l} \rho(121;1,2) = -\rho(121;2,1) = +1 \\ \rho(121;1,3) = -\rho(121;3,1) = -1 \\ \rho(121;2,3) = -\rho(121;3,2) = -2 \end{array}\right.
\end{array} \tag{3.6}$$

3.3 The dimension of irreps for S_n.

In one of the following chapters we will see that the number of inequivalent irreducible representations for any finite group is equal to the number of equivalence classes of the group. Since, moreover, the different equivalence classes of S_n can be represented by Young diagrams, it will not be surprising that also the different irreps of S_n can be represented by Young diagrams. The dimension $f[\mu]$ of an irrep of S_n corresponding to a Young diagram $[\mu]$ (where $[\mu]$ represents the corresponding partition of n) can be found from:

$$f[\mu] = \text{ number of possible standard Young tableaux,} \tag{3.7}$$

or alternatively, from:

$$f[\mu] = \frac{n!}{h[\mu]}, \tag{3.8}$$

where $h[\mu]$ represents the hookproduct of the Young diagram $[\mu]$.

For S_3, using formula (3.7) in order to count the various possible standard Young tableaux of a partition (see (3.1)), or alternatively, using (3.8) for the hookproducts of the different Young diagrams of S_3 (see (3.4)), one finds:

$$f[111] = 1, \quad \text{or alternatively} \quad f[111] = \frac{3!}{6} = 1,$$

$$f[3] = 1, \quad \text{or alternatively} \quad f[3] = \frac{3!}{6} = 1, \quad \text{and}$$

$$f[21] = 2, \quad \text{or alternatively} \quad f[21] = \frac{3!}{3} = 2. \tag{3.9}$$

3.4 Young's orthogonal form for the irreps of S_n.

In section (1.2) we have seen at page 6 that the permutation group S_n can be generated by the group elements (12) and $(12\cdots n)$ (see formula 1.17), or, alternatively, by the transpositions (12), (23), ..., $(n-1,n)$. Consequently, it is more than sufficient to determine the matrix representations for the transpositions (12), (23), ..., $(n-1,n)$ of S_n. The matrix representation of the other elements of S_n can then be obtained by (repeated) multiplication.

As standard form we will use Young's orthogonal form. These matrices are real and unitary and therefore orthogonal.

The irrep $[\mu]$ of S_n is defined in an $f[\mu]$-dimensional vector space $V^{[\mu]}$. In this space we choose an orthonormal basis $\hat{e}_i^{[\mu]}$ $(i = 1, \cdots, f[\mu])$, with:

$$\left(\hat{e}_i^{[\mu]}, \hat{e}_j^{[\mu]}\right) = \delta_{ij} \quad \text{for} \quad (i,j = 1, \cdots, f[\mu]). \tag{3.10}$$

To the index i of the basisvector $\hat{e}_i^{[\mu]}$ we relate the Yamanouchi symbol M in the order which has been discussed in section (3.1) at page 27. So, alternatively we may write:

$$\hat{e}_i^{[\mu]} = \hat{e}_M^{[\mu]}, \tag{3.11}$$

where M corresponds to the i-th Yamanouchi symbol which belongs to the partition $[\mu]$.

For example, for the two-dimensional representation [21] of S_3, we have as a basis:

$$\hat{e}_1^{[21]} = \hat{e}_{112}^{[21]} \quad \text{and} \quad \hat{e}_2^{[21]} = \hat{e}_{121}^{[21]}. \tag{3.12}$$

In terms of the basis vectors $\hat{e}_M^{[\mu]}$ one obtains the Young's standard form for the matrix representation $D^{[\mu]}((k,k+1))$ of the transposition $(k,k+1)$ of the permutation group S_n by the following transformation rule:

$$\begin{aligned} D^{[\mu]}((k,k+1))\,\hat{e}_{M_1\cdots M_n}^{[\mu]} &= \left(\rho(M_1\cdots M_n;k+1,k)\right)^{-1}\ \hat{e}_{M_1\cdots M_n}^{[\mu]} + \\ &+\sqrt{1-\left(\rho(M_1\cdots M_n;k+1,k)\right)^{-2}}\ \hat{e}_{M_1\cdots M_{k+1}M_k\cdots M_n}^{[\mu]}, \end{aligned} \tag{3.13}$$

where $\rho(M;k+1,k)$ represents the axial distance between box $k+1$ and k in the standard Young diagram corresponding to the Yamanouchi symbol $M = M_1 \cdots M_n$.

3.5 The standard irreps of S_3.

For the two-dimensional representation [21] of S_3, one finds, using relation (3.13), the following:

$$D^{[21]}((12))\hat{e}^{[21]}_{112} = \hat{e}^{[21]}_{112}, \text{ and } D^{[21]}((12))\hat{e}^{[21]}_{121} = -\hat{e}^{[21]}_{121}. \tag{3.14}$$

So, when we represent the basis vector $\hat{e}^{[21]}_{112}$ by the column vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the basis vector $\hat{e}^{[21]}_{121}$ by the column vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then the corresponding matrix is given by:

$$D^{[21]}((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.15}$$

For the transposition (23) one finds:

$$D^{[21]}((23))\hat{e}^{[21]}_{112} = -\frac{1}{2}\hat{e}^{[21]}_{112} + \frac{\sqrt{3}}{2}\hat{e}^{[21]}_{121}, \text{ and } D^{[21]}((23))\hat{e}^{[21]}_{121} = \frac{1}{2}\hat{e}^{[21]}_{121} + \frac{\sqrt{3}}{2}\hat{e}^{[21]}_{112}. \tag{3.16}$$

Consequently, the corresponding matrix is given by:

$$D^{[21]}((23)) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}. \tag{3.17}$$

These results might be compared to the matrix representation for the invariance group of the triangle given in table (2.3) at page 19.

As far as the one-dimensional representations are concerned, one finds:

$$D^{[111]}((12))\hat{e}^{[111]}_{123} = -\hat{e}^{[111]}_{123}, \text{ and } D^{[111]}((23))\hat{e}^{[111]}_{123} = -\hat{e}^{[111]}_{123}. \tag{3.18}$$

Consequently, [111] corresponds to the irrep $D^{(1')}$, discussed in formula (2.17).

Also:

$$D^{[3]}((12))\hat{e}^{[3]}_{111} = \hat{e}^{[3]}_{111}, \text{ and } D^{[3]}((23))\hat{e}^{[3]}_{111} = \hat{e}^{[3]}_{111}. \tag{3.19}$$

So, [3] corresponds to the trivial irrep $D^{(1)}$ (see equation 2.16).

3.6 The standard irreps of S_4.

For the symmetric group S_4 we have five different partitions, *i.e.* [1111], [211], [22], [31] and [4] (see the table of equivalence classes of S_4, table (1.9) at page 12), and consequently five inequivalent irreducible representations. Let us first collect the necessary ingredients for the construction of those irreps, *i.e.* the resulting Young tableaux, Yamanouchi symbols and relevant axial distances, in the table below.

partition	Young tableaux	Yamanouchi symbols	axial distances
[1111]	$\begin{array}{c}1\\2\\3\\4\end{array}$	1234	$\rho(1234;2,1)=-1$ $\rho(1234;3,2)=-1$ $\rho(1234;4,3)=-1$
[211]	$\begin{array}{ll}1&2\\3&\\4&\end{array}$	1123	$\rho(1123;2,1)=+1$ $\rho(1123;3,2)=-2$ $\rho(1123;4,3)=-1$
	$\begin{array}{ll}1&3\\2&\\4&\end{array}$	1213	$\rho(1213;2,1)=-1$ $\rho(1213;3,2)=+2$ $\rho(1213;4,3)=-3$
	$\begin{array}{ll}1&4\\2&\\3&\end{array}$	1231	$\rho(1231;2,1)=-1$ $\rho(1231;3,2)=-1$ $\rho(1231;4,3)=+3$
[22]	$\begin{array}{ll}1&2\\3&4\end{array}$	1122	$\rho(1122;2,1)=+1$ $\rho(1122;3,2)=-2$ $\rho(1122;4,3)=+1$
	$\begin{array}{ll}1&3\\2&4\end{array}$	1212	$\rho(1212;2,1)=-1$ $\rho(1212;3,2)=+2$ $\rho(1212;4,3)=-1$
[31]	$\begin{array}{lll}1&2&3\\4&&\end{array}$	1112	$\rho(1112;2,1)=+1$ $\rho(1112;3,2)=+1$ $\rho(1112;4,3)=-3$
	$\begin{array}{lll}1&2&4\\3&&\end{array}$	1121	$\rho(1121;2,1)=+1$ $\rho(1121;3,2)=-2$ $\rho(1121;4,3)=+3$
	$\begin{array}{lll}1&3&4\\2&&\end{array}$	1211	$\rho(1211;2,1)=-1$ $\rho(1211;3,2)=+2$ $\rho(1211;4,3)=+1$
[4]	$\begin{array}{llll}1&2&3&4\end{array}$	1111	$\rho(1111;2,1)=+1$ $\rho(1111;3,2)=+1$ $\rho(1111;4,3)=+1$

Table 3.1: The Young tableaux, Yamanouchi symbols and axial distances for the various irreps of S_4.

Using the results of table (3.1) and formula (3.13) one obtains the matrices which are collected in table (3.2) below, for the three generator transpositions of S_4.

irrep	[1111]	[211]	[22]	[31]	[4]
basis	$\hat{e}^{[1111]}_{1234}$	$\hat{e}^{[211]}_{1123}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\hat{e}^{[211]}_{1213}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\hat{e}^{[211]}_{1231}=\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\hat{e}^{[22]}_{1122}=\begin{pmatrix}1\\0\end{pmatrix}$ $\hat{e}^{[22]}_{1212}=\begin{pmatrix}0\\1\end{pmatrix}$	$\hat{e}^{[31]}_{1112}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\hat{e}^{[31]}_{1121}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\hat{e}^{[31]}_{1211}=\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\hat{e}^{[4]}_{1111}$
$D((12))$	-1	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	+1
$D((23))$	-1	$\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}&0\\\frac{\sqrt{3}}{2}&\frac{1}{2}&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&\frac{\sqrt{3}}{2}\\0&\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$	+1
$D((34))$	-1	$\begin{pmatrix}-1&0&0\\0&-\frac{1}{3}&\frac{\sqrt{8}}{3}\\0&\frac{\sqrt{8}}{3}&\frac{1}{3}\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{3}&\frac{\sqrt{8}}{3}&0\\\frac{\sqrt{8}}{3}&\frac{1}{3}&0\\0&0&1\end{pmatrix}$	+1

Table 3.2: Young's orthogonal form for the irreps of the symmetric group S_4 for the three generator transpositions (12), (23) and (34).

For later use we will also determine the matrices for a representant of each of the equivalence classes of S_4 which are shown in table (1.9) at page 12. The only member of [1111] is the identity operator I, which is always represented by the unit matrix. For the equivalence class [211] one might select the transposition (12) as a representant. Its matrix representations for the five standard irreps are already shown in table (3.2). As a representant of the equivalence class [22] we select the operation (12)(34). Its matrix representations for the various standard irreps can be determined, using the property formulated in equation (2.5) for representations, *i.e.* $D((12)(34)) = D((12))D((34))$. The resulting matrices are collected in table (3.3). For [31] we select the operation $(123) = (12)(23)$ as a representant. Its matrix representations follow from $D((123)) = D((12))D((23))$ using the results of table (3.2). Finally, for the equivalence class [4] we select the operation $(1234) = (12)(23)(34)$. The resulting matrices are collected in table (3.3) below.

irrep	D((12)(34))	D((123))	D((1234))
[1111]	$+1$	$+1$	-1
[211]	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{\sqrt{8}}{3} \\ 0 & -\frac{\sqrt{8}}{3} & -\frac{1}{3} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{2} & \frac{1}{6} & -\frac{\sqrt{2}}{3} \\ 0 & \frac{\sqrt{8}}{3} & \frac{1}{3} \end{pmatrix}$
[22]	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
[31]	$\begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{1}{6} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{6} & -\frac{1}{2} \end{pmatrix}$
[4]	$+1$	$+1$	$+1$

Table 3.3: Young's orthogonal form for the irreps of the symmetric group S_4 for representants of the three equivalence classes [22], [31] and [4].

Chapter 4

The classification of irreps.

In this chapter we discuss some properties of irreducible representations and study the resulting classification of irreps for finite groups.

4.1 The character table of a representation.

In this section we introduce a powerful tool for the classification of irreps, *i.e.* the character table of a representation. The *character* $\chi(g)$ of the n-dimensional matrix representation $D(g)$ of an element g of group G is defined as the trace of the matrix $D(g)$, *i.e.*:

$$\chi(g) = Tr\left\{D(g)\right\} = \sum_{a=1}^{n} D_{aa}(g)\ . \tag{4.1}$$

g	$\chi^{(1)}(g)$	$\chi^{(1')}(g)$	$\chi^{(2)}(g)$	$\chi^{(w)}(g)$	$\chi^{(r)}(g)$	$\chi^{(wS)}(g)$
I	1	1	2	3	6	3
(132)	1	1	-1	0	0	0
(123)	1	1	-1	0	0	0
(23)	1	-1	0	1	0	1
(12)	1	-1	0	1	0	1
(13)	1	-1	0	1	0	1

Table 4.1: The character table of the permutation group S_3 for the representations discussed in chapter 2.

In table (4.1) are summarized the characters for the various representations of S_3, *i.e.* $D^{(1)}$ (see equation (2.16)), $D^{(1')}$ (see equation (2.17)), $D^{(2)}$ (see table (2.3) at page 19), $D^{(w)}$ (see table (2.1) at page 14), $D^{(r)}$ (see table (2.2) at page 16) and $D^{(wS)}$ (see table (2.4) at page 22).

In the following we will frequently use a property of the *trace* of a matrix, related to the trace of the product AB of two $n \times n$ matrices A and B, *i.e.*:

$$\begin{aligned} Tr\{AB\} &= \sum_{i=1}^{n} [AB]_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} B_{ji} A_{ij} = \sum_{j=1}^{n} [BA]_{jj} = Tr\{BA\} \ . \end{aligned} \tag{4.2}$$

Several properties of characters may be noticed from the character table (4.1) at page 35 of the permutation group S_3.

1. The characters of the word-representation $D^{(w)}$ and its equivalent representation $D^{(wS)}$ are identical.

This is not surprising, since for any two equivalent matrix representations $D^{(\beta)}$ and $D^{(\alpha)} = S^{-1} D^{(\beta)} S$ (see equation (2.20) for the definition of equivalent matrix representations), one has for each group element g of the group G, using the property (4.2) for the trace of a matrix, the identity:

$$\chi^{(\beta)}(g) = Tr\left\{D^{(\beta)}(g)\right\} = Tr\left\{S^{-1} D^{(\alpha)}(g) S\right\} = Tr\left\{D^{(\alpha)}(g)\right\} = \chi^{(\alpha)}(g) \ . \tag{4.3}$$

Consequently, any two equivalent representations have the same character table.

2. The character $\chi(I)$ of the identity operation I, indicates the dimension of the representation.

In the following we will indicate the dimension n of an n-dimensional representation $D^{(\alpha)}$ of the group G, by the symbol $f(\alpha) = n$. From the character table (4.1) at page 35 for S_3 we observe:

$$f(\alpha) = \chi^{(\alpha)}(I) \ . \tag{4.4}$$

3. Group elements which belong to the same equivalence class (see section (1.5) for the definition of equivalence classes), have the same characters.

This can easily be shown to be a general property of characters: Using formula (1.23) for the definition of two equivalent group elements a and $b = g^{-1} a g$, equation (2.5) for the representation of the product $D(a)D(b) = D(ab)$ of group elements and property (4.2) of the trace of a matrix, one finds:

$$\begin{aligned} \chi(b) &= Tr\{D(b)\} = Tr\left\{D(g^{-1} a g)\right\} = Tr\left\{D(g^{-1}) D(a) D(g)\right\} \\ &= Tr\left\{D(a) D(g) D(g^{-1})\right\} = Tr\{D(a) D(I)\} = Tr\{D(a)\} \\ &= \chi(a) \ . \end{aligned} \tag{4.5}$$

As a consequence, *characters are functions of equivalence classes.*

Given this property of representations, it is sufficient to know the character of one representant for each equivalence class of a group, in order to determine the character table of a group. Using the tables (3.2) and (3.3) we determine the character table of S_4, which is shown below. The number of elements for each equivalence class can be found in table (1.9) at page 12.

equivalence classes	number of elements	irreps [1111] χ	[211] χ	[22] χ	[31] χ	[4] χ
[1111]	1	+1	+3	+2	+3	+1
[211]	6	−1	−1	0	+1	+1
[22]	3	+1	−1	+2	−1	+1
[31]	8	+1	0	−1	0	+1
[4]	6	−1	+1	0	−1	+1

Table 4.2: The character table of the symmetry group S_4 for the standard irreducible representations discussed in the previous chapter.

4. The characters of the word-representation $D^{(w)}$ are the sums of the characters of the trivial representation $D^{(1)}$ and the two-dimensional representation $D^{(2)}$, *i.e.* for each group element g of S_3 we find:

$$\chi^{(w)}(g) = \chi^{(1)}(g) + \chi^{(2)}(g) \; . \tag{4.6}$$

The reason for the above property is, that the word-representation $D^{(w)}$ can be reduced to the direct sum of the trivial representation $D^{(1)}$ and the two-dimensional representation $D^{(2)}$ (see table (2.4) at page 22). So, the traces of the matrices $D^{(w)}(g)$ are the sums of the traces of the sub-matrices $D^{(1)}(g)$ and $D^{(2)}(g)$.

The property (4.6) can be used to discover how a representation can be reduced. For example, one might observe that for the regular representation $D^{(r)}$ one has for each group element g of S_3, the following:

$$\chi^{(r)}(g) = \chi^{(1)}(g) + \chi^{(1')}(g) + 2 \times \chi^{(2)}(g) \; . \tag{4.7}$$

This suggests that the 6×6 regular representation can be reduced into one time the trivial representation [3], one time the representation [111] and two times the two-dimensional representation [21], and that there exists a basis transformation S in the six-dimensional vector space V_6 defined in section (2.3), such that for each group element g of S_3 yields:

$$
S^{-1}D^{(r)}(g)S = \begin{pmatrix} D^{(1)}(g) & 0 & 0 & 0 & 0 & 0 \\ 0 & D^{(1')}(g) & 0 & 0 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$D^{(2)}(g)$}} & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$D^{(2)}(g)$}} \\ 0 & 0 & 0 & 0 & & \end{pmatrix} . \qquad (4.8)
$$

Indeed, such matrix S can be found for the regular representation of S_3, which shows that the regular representation is reducible.

For the 24×24 regular representation of S_4, using table (4.2), one finds similarly:

$$
\chi^{(r)}(g) = \chi^{[1111]}(g) + 3\chi^{[211]}(g) + 2\chi^{[22]}(g) + 3\chi^{[31]}(g) + \chi^{[4]}(g) \; , \qquad (4.9)
$$

where we also used the fact that the characters of the regular representation for any group vanish for all group elements except for the character of the identity operator.

One might observe from the expressions (4.7) and (4.9) that each representation appears as many times in the sum as the size of its dimension. This is generally true for the regular representation of any group, *i.e.*:

$$
\chi^{(r)}(g) = \sum_{\text{all irreps}} f(\text{irrep})\chi^{(\text{irrep})}(g) \; .
$$

For the identity operator I of a group of order p this has the following interesting consequence:

$$
p = \chi^{(r)}(I) = \sum_{\text{all irreps}} f(\text{irrep})\chi^{(\text{irrep})}(I) = \sum_{\text{all irreps}} \{f(\text{irrep})\}^2 \; . \qquad (4.10)
$$

For example, the permutation group S_3 has 6 elements. Its irreps are $D^{(1)}$ (see equation (2.16)), $D^{(1')}$ (see equation (2.17)) and $D^{(2)}$ (see table (2.3) at page 19), with respective dimensions 1, 1 and 2. So, we find for those irreps:

$$
1^2 + 1^2 + 2^2 = 6,
$$

which, using the above completeness relation (4.10), proofs that S_3 has no more inequivalent irreps.

4.2 The first lemma of Schur.

When, for an *irrep* D of a group $G = \{g_1, \ldots, g_p\}$ in a vector space V, a matrix A commutes with $D(g)$ (*i.e.* $AD(g) = D(g)A$) for all elements g of G, then the matrix A must be proportional to the unit matrix, *i.e.*:

$$A = \lambda \mathbf{1} \quad \text{for a complex (or real) constant } \lambda. \tag{4.11}$$

This is Schur's first lemma.

As an example, let us study the 2×2 irrep $D^{[21]}$ of S_3 (see section (3.5) at page 31). We define the matrix A by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ & \\ a_{21} & a_{22} \end{pmatrix} \quad .$$

First, we determine the matrix products $AD^{[21]}((12))$ and $D^{[21]}((12))A$, to find the following result:

$$AD^{[21]}((12)) = \begin{pmatrix} a_{11} & -a_{12} \\ & \\ a_{21} & -a_{22} \end{pmatrix} \quad \text{and} \quad D^{[21]}((12))A = \begin{pmatrix} a_{11} & a_{12} \\ & \\ -a_{21} & -a_{22} \end{pmatrix} \quad .$$

Those two matrices are equal under the condition that $a_{12} = a_{21} = 0$. So, we are left with the matrix:

$$A = \begin{pmatrix} a_{11} & 0 \\ & \\ 0 & a_{22} \end{pmatrix} \quad .$$

Next, we determine the matrix products $AD^{[21]}((23))$ and $D^{[21]}((23))A$, to obtain:

$$AD^{[21]}((23)) = \begin{pmatrix} -\frac{a_{11}}{2} & \frac{a_{11}\sqrt{3}}{2} \\ & \\ \frac{a_{22}\sqrt{3}}{2} & \frac{a_{22}}{2} \end{pmatrix} \quad \text{and} \quad D^{[21]}((23))A = \begin{pmatrix} -\frac{a_{11}}{2} & \frac{a_{22}\sqrt{3}}{2} \\ & \\ \frac{a_{11}\sqrt{3}}{2} & \frac{a_{22}}{2} \end{pmatrix} \quad .$$

Those two matrices are equal under the condition that $a_{11} = a_{22}$. So, we finally end up with the matrix:

$$A = a_{11}\mathbf{1}$$

in agreement with the above lemma of Schur.

The condition that the representation is irreducible can be shown to be a necessary condition by means of the following example: Let us consider the reducible word representation of S_3. From the matrices shown in table (2.1) at page 14 it is easy to understand that the matrix A given by:

$$A = \begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix}$$

commutes with all six matrices but is not proportional to the unit matrix.

Proof of Schur's first lemma:

Let $\vec{r}$ be an eigenvector in V of A with eigenvalue λ. Since D is an irrep of G, the set of vectors $\{\vec{r}_1 = D(g_1)\vec{r}, \ldots, \vec{r}_p = D(g_p)\vec{r}\}$ must span the whole vector space V, which means that each vector of V can be written as a linear combination of the vectors $\vec{r}_1$, ..., $\vec{r}_p$. But, using the condition that A commutes with $D(g)$ for all elements g of G, one has also the relation:

$$A\vec{r}_a = AD(g_a)\vec{r} = D(g_a)A\vec{r} = D(g_a)\lambda\vec{r} = \lambda\vec{r}_a \quad , \quad (a = 1, \ldots, p) \ .$$

Consequently, all vectors in V are eigenvectors of A with the same eigenvalue λ, which leads automatically to the conclusion (4.11).

4.3 The second lemma of Schur.

Let $D^{(1)}$ represent an $n_1 \times n_1$ irrep of a group $G = \{g_1, \ldots, g_p\}$ in a n_1-dimensional vector space V_1 and $D^{(2)}$ an $n_2 \times n_2$ irrep of G in a n_2-dimensional vector space V_2. Let moreover A represent an $n_2 \times n_1$ matrix describing transformations from V_1 on to V_2. When for all group elements g of G yields $AD^{(1)}(g) = D^{(2)}(g)A$, then:

$$\left\{ \begin{array}{l} \text{either} \;\; A = 0 \\ \\ \text{or} \;\; n_1 = n_2 \;\; \text{and} \;\; det(A) \neq 0 \ . \end{array} \right. \tag{4.12}$$

In the second case are moreover $D^{(1)}$ and $D^{(2)}$ equivalent.

For an example let us study the irreps $D^{[211]}$ and $D^{[22]}$ of the symmetric group S_4 (see table (3.2) at page 33) and the matrix A given by:

$$A = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \right) \quad .$$

First, we determine the matrix products $D^{[211]}((12))A$ and $AD^{[22]}((12))$, to find the following result:

$$D^{[211]}((12))A = \left(\begin{array}{cc} a_{11} & a_{12} \\ \\ -a_{21} & -a_{22} \\ \\ -a_{31} & -a_{32} \end{array} \right) \quad \text{and} \quad AD^{[22]}((12)) = \left(\begin{array}{cc} a_{11} & -a_{12} \\ \\ a_{21} & -a_{22} \\ \\ a_{31} & -a_{32} \end{array} \right) \quad .$$

Those two matrices are equal under the condition that $a_{12} = a_{21} = a_{31} = 0$. So, we are left with the matrix:

$$A = \left(\begin{array}{cc} a_{11} & 0 \\ 0 & a_{22} \\ 0 & a_{32} \end{array} \right) \quad .$$

Next, we determine the matrix products $D^{[211]}((23))A$ and $AD^{[22]}((23)$, to obtain:

$$D^{[211]}((23))A = \begin{pmatrix} -\frac{a_{11}}{2} & \frac{a_{22}\sqrt{3}}{2} \\ \\ \frac{a_{11}\sqrt{3}}{2} & \frac{a_{22}}{2} \\ \\ 0 & -a_{32} \end{pmatrix} \quad \text{and} \quad AD^{[22]}((23)) = \begin{pmatrix} -\frac{a_{11}}{2} & \frac{a_{11}\sqrt{3}}{2} \\ \\ \frac{a_{22}\sqrt{3}}{2} & \frac{a_{22}}{2} \\ \\ \frac{a_{32}\sqrt{3}}{2} & \frac{a_{32}}{2} \end{pmatrix} \quad .$$

Those two matrices are equal under the conditions that $a_{11} = a_{22}$ and $a_{32} = 0$. So, we are left with the matrix:

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \\ 0 & 0 \end{pmatrix} \quad .$$

Finally, we determine the matrix products $D^{[211]}((34))A$ and $AD^{[22]}((34)$, to get:

$$D^{[211]}((34))A = \begin{pmatrix} -a_{11} & 0 \\ \\ 0 & -\frac{a_{11}}{3} \\ \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad AD^{[22]}((34)) = \begin{pmatrix} a_{11} & 0 \\ \\ 0 & -a_{11} \\ \\ 0 & 0 \end{pmatrix} \quad .$$

Those two matrices are equal when $a_{11} = 0$. So, we end up $A = 0$ in accordance with the second lemma of Schur.

Proof of Schur's second lemma.

We consider separately the two cases $n_1 \leq n_2$ and $n_1 > n_2$:

(i) $\quad n_1 \leq n_2$

For $\vec{r}$ an arbitrary vector in the vector space V_1, $A\vec{r}$ is a vector in V_2. The subspace V_A of V_2 spanned by all vectors $A\vec{r}$ for all possible vectors in V_1, has a dimension n_A which is smaller or equal to n_1, *i.e.*:

$$n_A \leq n_1 \leq n_2 \quad .$$

Let us choose one vector $\vec{v} = A\vec{r}$ (for $\vec{r}$ in V_1) out of the subspace V_A of V_2. The transformed vector $\vec{v}_a$ under the transformation $D^{(2)}(g_a)$ satisfies:

$$\vec{v}_a = D^{(2)}(g_a)\vec{v} = D^{(2)}(g_a)A\vec{r} = AD^{(1)}(g_a)\vec{r} = A\vec{r}_a \quad ,$$

where $\vec{r}_a$ is a vector of V_1. Consequently, the vector $\vec{v}_a$ must be a vector of V_A.

So, the set of vectors $\{\vec{v}_1 = D^{(2)}(g_1)\vec{v}, \ldots, \vec{v}_p = D^{(2)}(g_p)\vec{v}\}$ spans the vector subspace V_A of V_2. However, since $D^{(2)}$ is an irrep, we must have $V_A = V_2$, unless $A = 0$. Consequently, we find:

$$n_A = n_1 = n_2 \quad \text{or} \quad A = 0 \quad .$$

In the case that $n_1 = n_2$ we may consider $V_1 = V_2$ for all practical purposes. When moreover, $A \neq 0$, the matrix A has an inverse, because then $V_A = V_2 = V_1$, which gives $D^{(1)}(g) = A^{-1}D^{(2)}(g)A$ for all group elements g of G and so $D^{(1)}$ and $D^{(2)}$ are equivalent.

(ii) $n_1 > n_2$

In that case there must exist vectors $\vec{r}$ in V_1 which are mapped on to zero in V_2 by A, *i.e.* $A\vec{r} = 0$. The subspace V_A' of vectors $\vec{r}$ in V_1 which are mapped on to zero in V_2 has a dimension n_A' smaller than or equal to n_1. Now, if $\vec{r}$ belongs to V_A', then $\vec{r}_a = D^{(1)}(g_a)\vec{r}$ also belongs to V_A', according to:

$$A\vec{r}_a = AD^{(1)}(g_a)\vec{r} = D^{(2)}(g_a)A\vec{r} = 0 \quad .$$

This contradicts the irreducibility of $D^{(1)}$, unless $n_A' = n_1$, in which case $A = 0$.

4.4 The orthogonality theorem.

Let $D^{(i)}$ represent an $n_i \times n_i$ irrep of a group $G = \{g_1, \ldots, g_p\}$ of order p, and $D^{(j)}$ an $n_j \times n_j$ irrep of G. Let moreover $D^{(i)}$ either not be equivalent to $D^{(j)}$ or be equal to $D^{(j)}$. Then:

$$\sum_{\text{all } g} \left[D^{(i)}(g)\right]^*_{ab} \left[D^{(j)}(g)\right]_{cd} = \frac{p}{n_i}\delta_{ij}\delta_{ac}\delta_{bd} \tag{4.13}$$

Proof

Let X be a $n_j \times n_i$ matrix, then we define the matrix A as follows:

$$A = \sum_{k=1}^{p} D^{(j)}(g_k)XD^{(i)}(g_k^{-1}) \quad . \tag{4.14}$$

This matrix has the property $D^{(j)}(g_a)A = AD^{(i)}(g_a)$ as can easily be verified, using the relation (2.5) and the fact that $D(g^{-1})D(g) = D(I) = \mathbf{1}$. Consequently, according to the second lemma of Schur (4.12) either $A = 0$ or $D^{(i)}$ and $D^{(j)}$ are equivalent. Remember that in case the two irreps are equivalent we assume that they are equal. When the two irreps are equal one has that according to the first lemma of Schur (4.11) $A = \lambda\mathbf{1}$. One might combine the two possibilities for A into:

$$A = \lambda\delta_{ij}\mathbf{1} = \begin{cases} 0 & \text{for } D^{(i)} \neq D^{(j)} \\ \\ \lambda\mathbf{1} & \text{for } D^{(i)} = D^{(j)} \end{cases} \tag{4.15}$$

The value of λ depends on the choice of the matrix X. Let us select matrices X which have zero matrix elements all but one. The nonzero matrix element equals 1 and is located at the intersection of the d-th row and the b-th column. For such matrices X one finds, using (4.14) and (4.15):

$$
\begin{aligned}
\lambda\delta_{ij}\delta_{ca} &= [A]_{ca} \\
&= \sum_{k=1}^{p}\sum_{\alpha=1}^{n_j}\sum_{\beta=1}^{n_i} [D^{(j)}(g_k)]_{c\alpha} X_{\alpha\beta} [D^{(i)}(g_k^{-1})]_{\beta a} \\
&= \sum_{k=1}^{p}\sum_{\alpha=1}^{n_j}\sum_{\beta=1}^{n_i} [D^{(j)}(g_k)]_{c\alpha} \delta_{\alpha d}\delta_{b\beta} [D^{(i)}(g_k^{-1})]_{\beta a} \\
&= \sum_{k=1}^{p} [D^{(j)}(g_k)]_{cd} [D^{(i)}(g_k^{-1})]_{ba}
\end{aligned}
\tag{4.16}
$$

The value of λ is only relevant when $i = j$ and $a = c$. In that case we take the sum over both sides of (4.16) in a, leading to:

$$
\begin{aligned}
\lambda n_i &= \sum_{a=1}^{n_i} \lambda\delta_{ii}\delta_{aa} \\
&= \sum_{a=1}^{n_i}\sum_{k=1}^{p} [D^{(i)}(g_k)]_{ad} [D^{(i)}(g_k^{-1})]_{ba} \\
&= \sum_{k=1}^{p} [D^{(i)}(I)]_{bd} \\
&= p\delta_{bd}
\end{aligned}
$$

Inserting this result for λ in (4.16) and using the fact that for a unitary representation $D(g^{-1}) = D^{-1}(g) = D^{\dagger}(g)$, one obtains the relation (4.13).

4.5 The orthogonality of characters for irreps.

Let us define for the representation $D^{(\alpha)}$ of a group G of order p and with elements $\{g_1, \cdots, g_p\}$, the following (complex) column vector $\vec{\chi}^{(\alpha)}$ of length p:

$$
\vec{\chi}^{(\alpha)} = \begin{pmatrix} \chi^{(\alpha)}(g_1) \\ \vdots \\ \chi^{(\alpha)}(g_p) \end{pmatrix} . \tag{4.17}
$$

In the following we will refer to such vectors as the *character* of the representation $D^{(\alpha)}$ of G.

We define moreover, the (complex) innerproduct of characters by:

$$\left(\vec{\chi}^{\,(\alpha)}\,,\,\vec{\chi}^{\,(\beta)}\right)=\sum_{i\,=\,1}^{p}\left\{\vec{\chi}^{\,(\alpha)}(g_i)\right\}^{*}\vec{\chi}^{\,(\beta)}(g_i)\;. \tag{4.18}$$

Let us concentrate on the characters of the irreps $D^{(1)}$, $D^{(1')}$ and $D^{(2)}$ of S_3 (see table (4.1) at page 35, given by the three following six-component vectors:

$$\vec{\chi}^{\,(1)}=\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}\;,\;\vec{\chi}^{\,(1')}=\begin{pmatrix}1\\1\\1\\-1\\-1\\-1\end{pmatrix}\;\text{and}\;\vec{\chi}^{\,(2)}=\begin{pmatrix}2\\-1\\-1\\0\\0\\0\end{pmatrix}\;. \tag{4.19}$$

First, we notice for those three characters, using the definition (4.18) of the innerproduct for characters, the property:

$$\left(\vec{\chi}^{\,(1)}\,,\,\vec{\chi}^{\,(1)}\right)=\left(\vec{\chi}^{\,(1')}\,,\,\vec{\chi}^{\,(1')}\right)=\left(\vec{\chi}^{\,(2)}\,,\,\vec{\chi}^{\,(2)}\right)=6\;,$$

which result is equal to the order of the permutation group S_3. Furthermore, we notice that:

$$\left(\vec{\chi}^{\,(1)}\,,\,\vec{\chi}^{\,(1')}\right)=\left(\vec{\chi}^{\,(1')}\,,\,\vec{\chi}^{\,(2)}\right)=\left(\vec{\chi}^{\,(2)}\,,\,\vec{\chi}^{\,(1)}\right)=0\;.$$

Or, more compact, one might formulate the above properties for S_3 as follows:

$$\left(\vec{\chi}^{\,(\alpha)}\,,\,\vec{\chi}^{\,(\beta)}\right)=6\delta_{\alpha\beta}\;\text{ for }\;\alpha,\beta=1,1'\;\text{ and }\;2. \tag{4.20}$$

Consequently, the six-component vectors (4.19) form an orthogonal basis for a three-dimensional sub-space of all possible vectors in six dimensions.

This can be shown to be true for any finite group. Using the property formulated in equation (4.13), we obtain for two non-equivalent irreps (α) of dimension $f(\alpha)$ and (β) of dimension $f(\beta)$ of a group G of order p , the following:

$$\begin{aligned}
\left(\vec{\chi}^{\,(\alpha)}\,,\,\vec{\chi}^{\,(\beta)}\right) &= \sum_{g\,\in\,G}\left\{\chi^{(\alpha)}(g)\right\}^{*}\chi^{(\beta)}(g)\\
&= \sum_{g\,\in\,G} Tr\left\{D^{(\alpha)}(g)\right\}^{*} Tr\left\{D^{(\beta)}(g)\right\}\\
&= \sum_{g\,\in\,G}\sum_{a\,=\,1}^{f(\alpha)}\left\{D^{(\alpha)}_{aa}(g)\right\}^{*}\sum_{b\,=\,1}^{f(\beta)}D^{(\beta)}_{bb}(g)\\
&= \sum_{a\,=\,1}^{f(\alpha)}\sum_{b\,=\,1}^{f(\beta)}\sum_{g\,\in\,G}\left\{D^{(\alpha)}_{aa}(g)\right\}^{*}D^{(\beta)}_{bb}(g)\\
&= \sum_{a\,=\,1}^{f(\alpha)}\sum_{b\,=\,1}^{f(\beta)}\frac{p}{f(\alpha)}\delta_{\alpha\beta}\,\delta_{ab}\,\delta_{ab}\;=\;p\delta_{\alpha\beta}\;.
\end{aligned} \tag{4.21}$$

Consequently, the characters for different (non-equivalent) irreps of a finite group are orthogonal.

Next, we remember that the characters for group elements which belong to the same equivalence class of S_3, are equal (see formula (4.5)). So, all possible characters of different representations (α) of S_3 have the following form:

$$\vec{\chi}^{\,(\alpha)} = \begin{pmatrix} a \\ b \\ b \\ c \\ c \\ c \end{pmatrix} ,$$

where $a = f(\alpha)$ represents the dimension of the representation (α) of S_3, where b represents the trace of the matrix representation for either of the two equivalent symmetric permutations of S_3 and where c represents the trace of any of the three equivalent antisymmetric permutations of S_3.

However, such six-component vectors can be decomposed as linear combinations of the three basis vectors (4.19), according to:

$$\vec{\chi}^{\,(\alpha)} = \begin{pmatrix} a \\ b \\ b \\ c \\ c \\ c \end{pmatrix} = \frac{1}{6}(a+2b+3c)\vec{\chi}^{\,(1)} + \frac{1}{6}(a+2b-3c)\vec{\chi}^{\,(1')} + \frac{1}{3}(a-b)\vec{\chi}^{\,(2)} , \quad (4.22)$$

which shows that the characters $\vec{\chi}^{\,(1)}$, $\vec{\chi}^{\,(1')}$ and $\vec{\chi}^{\,(2)}$ form a basis for the space of all possible characters of S_3.

4.6 Irreps and equivalence classes.

In the previous section, we found that the characters for the irreps of S_3 as defined in equation (4.19) form the basis for the space of all possible characters for representations of this group.

This is true for any group: The characters of all non-equivalent irreps of a group form a complete and orthogonal basis for the vector space of characters of all possible representations of that group.

Moreover, the number of independent characters which can be formed for the representations of a group, equals the number of equivalence classes of the group. Consequently, for a finite group G we have the following property:

$$\text{number of irreps } G = \text{number of equivalence classes } G. \quad (4.23)$$

4.7 Abelian groups.

For Abelian groups, the above relation (4.23) has a particularly interesting consequence, because the number of equivalence classes of an Abelian group G is equal to its order p. This latter property, using the definition (1.23) for equivalent group elements, can be understood as follows: Suppose that the elements a and b of the Abelian group G are equivalent. Then there exists a third element g of G, such that:

$$b = g^{-1}ag.$$

But since the group is Abelian, we obtain:

$$b = g^{-1}ag = g^{-1}ga = Ia = a.$$

So, each group element is only equivalent to itself for an Abelian group, which proofs that the number of equivalence classes is equal to the order p of G. As a consequence of (4.23) also the number of irreps of an Abelian group G equals the order p of G.

We then find, using also relation (4.10), for the dimensions $f(i)$ $(i = 1, \cdots, p)$ of the p irreps of G that:

$$\{f(1)\}^2 + \{f(2)\}^2 + \cdots + \{f(p)\}^2 = p,$$

which has only one possible solution:

$$\{f(1)\} = \{f(2)\} = \cdots = \{f(p)\} = 1. \tag{4.24}$$

We find as a conclusion for a finite Abelian group, that all its irreps are one-dimensional.

Chapter 5

The triangular harmonic oscillator.

In this chapter we study a system of three identical springs connected to three identical masses, such that the whole system forms an equilateral triangle. We assume that the springs have no flexibility in the lateral directions and consequently, only provide for motion in their longitudinal directions. This way, the masses can just perform oscillations in the plane of the triangle. For that reason, we will assume here that the masses can only move "freely" in the (x, y)-plane.

5.1 The equations of motion.

The system of the masses and the springs in its equilibrium position, is shown in figure (5.1) below.

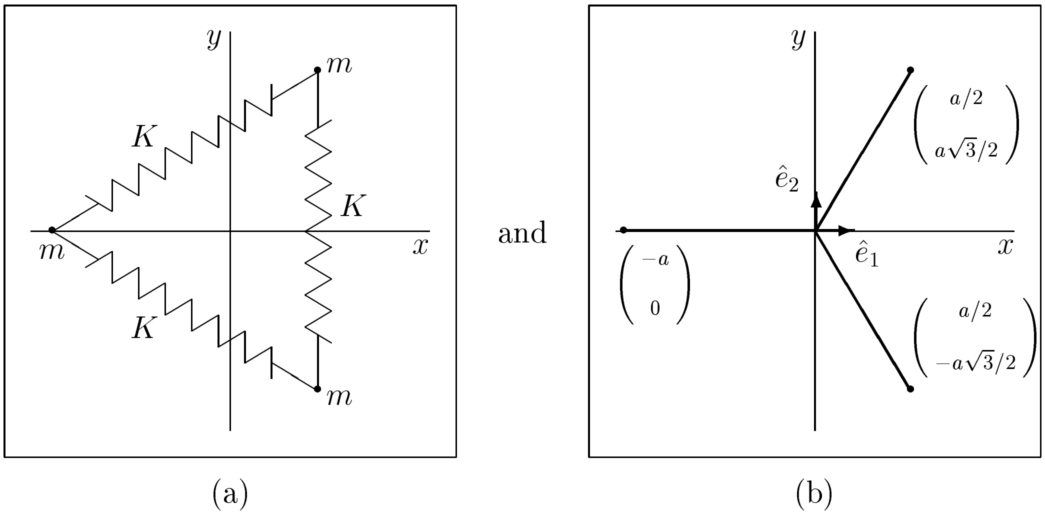


Figure 5.1: The equilateral triangle of harmonic oscillators. The centre of the triangle (a) coincides with the origin of the coordinate system. In (b) are indicated the coordinates of the corners of the triangle and the unit vectors in the plane.

The equilibrium positions of the masses are indicated by:

$$\vec{r}_{10} = \begin{pmatrix} a/2 \\ \\ a\sqrt{3}/2 \end{pmatrix} \ , \ \vec{r}_{20} = \begin{pmatrix} a/2 \\ \\ -a\sqrt{3}/2 \end{pmatrix} \ , \ \vec{r}_{30} = \begin{pmatrix} -a \\ \\ 0 \end{pmatrix} \ . \tag{5.1}$$

In fact, we have numbered the three indistinguishable masses by the following receipt:

$$\begin{aligned} \text{number 1} &= \text{mass with coordinates indicated by } \vec{r}_{10} \\ \text{number 2} &= \text{mass with coordinates indicated by } \vec{r}_{20} \\ \text{number 3} &= \text{mass with coordinates indicated by } \vec{r}_{30} \end{aligned} \tag{5.2}$$

Furthermore, we study only (infinitesimally) small oscillations of the masses, which guarantees that we allways can tell unambigiously which number is associated to which of the masses.

From figure (5.1) at page 47 one might deduce that the equilibrium length of each of the springs equals $a\sqrt{3}$. When the length of one of the springs is stretched or shrunk to a length l, then its potential energy is given by:

$$U_{\text{pot}} = \frac{K}{2}(l - a\sqrt{3})^2 \ , \tag{5.3}$$

where K is the elasticity constant of the springs.

For an arbitrary displacement away from their equilibrium position of the three masses, we indicate their position vectors by $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$, as indicated in figure (5.2) below.

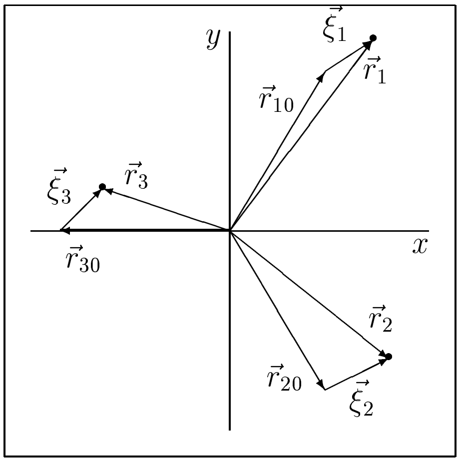


Figure 5.2: The position vectors, $\vec{r}_i$ $(i = 1, 2, 3)$, of the three masses for arbitrary displacement, $\vec{\xi}_i$ $(i = 1, 2, 3)$, away from their equilibrium positions, $\vec{r}_{i0}$ $(i = 1, 2, 3)$.

The total potential energy stored in the system of the three springs, due to an arbitrary displacement of the three masses, is given by:

$$\begin{aligned} U_{pot}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \frac{K}{2} \sum_{i=1}^{3} \sum_{j>i} \left(|\vec{r}_i - \vec{r}_j| - a\sqrt{3} \right)^2 \\ &= \frac{K}{2} \left\{ \left(|\vec{r}_1 - \vec{r}_2| - a\sqrt{3} \right)^2 + \right. \\ &\quad \left. + \left(|\vec{r}_1 - \vec{r}_3| - a\sqrt{3} \right)^2 + \left(|\vec{r}_2 - \vec{r}_3| - a\sqrt{3} \right)^2 \right\} \quad . \end{aligned} \tag{5.4}$$

The equations of motion of each of the masses, *i.e.* the time-dependence of the components of their coordinates (x_i, y_i) $(i = 1, 2, 3)$, can be determined by the following set of coupled differential equations:

$$\begin{aligned} m \frac{d^2 x_i}{dt^2} &= -\frac{\partial U_{pot}}{\partial x_i} \quad \text{and} \\ m \frac{d^2 y_i}{dt^2} &= -\frac{\partial U_{pot}}{\partial y_i} \quad (i = 1, 2, 3) \ . \end{aligned} \tag{5.5}$$

This set of equations can be solved for small deviations from the equilibrium positions. But, a more elegant and especially more simple set of equations can be found, using the symmetry of the problem.

5.2 The symmetry of the problem.

We study in this chapter (infinitesimally) small oscillations of the triangle, *i.e.* oscillations which do not change the form of the triangle. Picturial representations of the displacements of the three masses, like the one given in figure (5.2) at page 48, are only meant to define the relevant quantities of the problem. The actual displacements are thought to be orders of magnitude smaller than represented in that figure. In the refered figure we show for an arbitrary situation the three displacement vectors, $\vec{\xi}_1$, $\vec{\xi}_2$ and $\vec{\xi}_3$, which indicate the positions of the three masses with respect to their equilibrium positions, *i.e.*:

$$\vec{r}_i(t) = \vec{r}_{i0} + \vec{\xi}_i(t) \quad , \quad \text{for} \quad (i = 1, 2, 3). \tag{5.6}$$

So, the six degrees of freedom of the system of the three oscillating masses in two dimensions, are indicated by the six components of the three displacement vectors $\vec{\xi}_i$ for $(i = 1, 2, 3)$, for example according to:

$$\vec{\xi}(t) = \begin{pmatrix} \xi_{1x}(t) \\ \xi_{2x}(t) \\ \xi_{3x}(t) \\ \xi_{1y}(t) \\ \xi_{2y}(t) \\ \xi_{3y}(t) \end{pmatrix} \quad . \tag{5.7}$$

Evidently, the oscillations of the masses are independent of our choice of coordinate systems. Following the boundary condition that the masses oscillate at well defined positions, *i.e.* the orientation of the axes of the (x, y)-coordinate system must be such that the masses are located at the positions indicated in formula (5.2), we find that the allowed basis transformations are the operations of the invariance group of the triangle, which is discussed in section (2.4). Consequently, there must exist a set of basis solutions of the system which transform as a representation of the symmetry group of the triangle, *i.e.* the permutation group S_3. So, we would like to know with which representation we are dealing:

First, instead of the x- and y-coordinates of the plane one might alternatively introduce as a basis for the six independent degrees of freedom of the system, the following set $\{\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3, \hat{\sigma}_4, \hat{\sigma}_5, \hat{\sigma}_6\}$ of unit vectors:

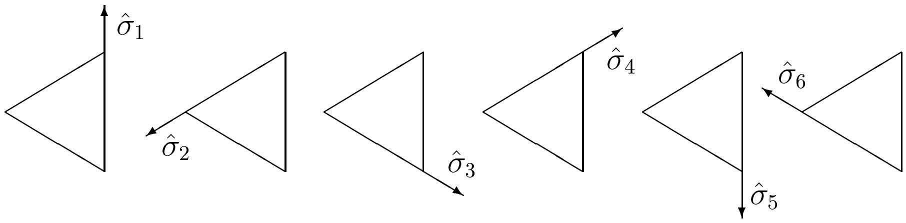


Figure 5.3: Six independent displacements for the three masses.

With the help of the above basis one can describe an arbitrary displacement of for instance only the mass numbered by 1, in case the other two masses remain in their equilibrium positions, by a linear combination of $\hat{\sigma}_1$ and $\hat{\sigma}_4$. A general arbitrary displacement might be given by:

$$
\begin{aligned}
\vec{\xi}_1(t) &= \Sigma_1(t)\hat{\sigma}_1 + \Sigma_4(t)\hat{\sigma}_4 \ , \\
\vec{\xi}_2(t) &= \Sigma_3(t)\hat{\sigma}_3 + \Sigma_5(t)\hat{\sigma}_5 \ \text{ and} \\
\vec{\xi}_3(t) &= \Sigma_2(t)\hat{\sigma}_2 + \Sigma_6(t)\hat{\sigma}_6 \ .
\end{aligned}
\tag{5.8}
$$

The six equations of motion (5.5) might eventually be written as a set of six coupled differential equations for the components Σ_i (for $i = 1, \ldots, 6$). However, we prefer to stick to the equations of motion for the components of $\vec{\xi}(t)$, since their basis is orthonormal which implies that those components are independent.

Second, we notice that the representation must be six-dimensional, because of the six degrees of freedom of the system. Remember that such representation is reducible (see formula (4.7)).

Third, when we perform a group operation on the triangle, then the numbering of the masses changes. Let us for example perform the group operation $g_3 = (123)$. According to table (2.3) at page 19, the group element (123) of S_3 is represented by a rotation of 240° of the coordinate system. The new numbering of the masses as derived from the receipt (5.2) for the "new" coordinate system induced by $\hat{e}'_1$ and $\hat{e}'_2$ and the consequently new numbering of the six displacement basis vectors $\hat{\sigma}_i$ $(i = 1, \ldots, 6)$, is shown in figure (5.4) below.

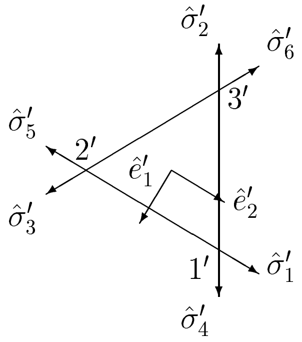


Figure 5.4: The new numbering of the three masses.

But, this is just a change of names, nothing has changed in the displacements of the various masses, only the basis vectors obtained another name. For example, the original basis displacement $\hat{\sigma}_1$ has after the operation $g_3 = (123)$ obtained the new name $\hat{\sigma}'_2$. We might, however, also view this as a transformation $D((123))$ of the basisvector $\hat{\sigma}_2$, *i.e.*

$$D((123))\hat{\sigma}_2 = \hat{\sigma}'_2 = \hat{\sigma}_1.$$

In table (5.1) below we show the resulting transformation for all basis displacements.

new name	$\hat{\sigma}'_1$	$\hat{\sigma}'_2$	$\hat{\sigma}'_3$	$\hat{\sigma}'_4$	$\hat{\sigma}'_5$	$\hat{\sigma}'_6$
original name	$\hat{\sigma}_3$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\sigma}_5$	$\hat{\sigma}_6$	$\hat{\sigma}_4$

Table 5.1: The transformation $D((123))$ for the basis displacement vectors.

One might compare this table to the transformations given in formula (2.8), in order to find that the above transformation is equivalent to the regular transformation. However, we must find the transformation for the components of an arbitrary solution under the group operation (123).

Suppose that in the original basis the movements of the three masses for a certain solution of the equations of motion (5.5), are described by:

$$\vec{\Sigma}(t) = \begin{pmatrix} \Sigma_1(t) \\ \Sigma_2(t) \\ \Sigma_3(t) \\ \Sigma_4(t) \\ \Sigma_5(t) \\ \Sigma_6(t) \end{pmatrix} ,$$

where the first component refers to the coefficient of $\hat{\sigma}_1$, the second to the coefficient of $\hat{\sigma}_2$, etc.

Then, in the new coordinate system, the position vector of the three masses as a function of time is given by:

$$\vec{\Sigma}(t) = \begin{pmatrix} \Sigma_3(t) \\ \Sigma_1(t) \\ \Sigma_2(t) \\ \Sigma_5(t) \\ \Sigma_6(t) \\ \Sigma_4(t) \end{pmatrix} \quad ,$$

where the first component refers to the coefficient of $\hat{\sigma}_1'$, the second to the coefficient of $\hat{\sigma}_2'$, etc.

The transformation matrix for the components of this six-column vectors can easily be found by comparing the two above expressions. One obtains the following 6×6 matrix:

$$D(g_3 = (123)) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \left[D^{(r)}((123))\right]^{-1} . \qquad (5.9)$$

So, we found that the represention of (123) in the function space is given by the inverse of the regular representation of (123) (compare formula (2.9)).

One may repeat the above procedure for the other group elements of S_3 in order to discover that indeed all group elements in the six-dimensional function space are represented by the inverses of their regular representations as given in table (2.2) at page 16.

Fourth, for this representation we know (see formula (4.7)) that it can be reduced to the direct sum of the two one-dimensional irreps $D^{(1)}$ and $D^{(1')}$ and two times the two-dimensional irrep $D^{(2)}$.

Without solving any system of coupled differential equations, we obtained already a lot of information on the structure of the solution space by just some symmetry arguments. In the following we will see that it is also not very difficult to find the various sub-spaces of the solution space which transform according to one of the irreps of S_3.

5.3 The inner product of the Σ-space.

Before continuing to exploit the symmetry properties of the triangular harmonic oscillator, we have to pay the price of introducing as a basis for the solution space of the Σ-components, a set of three pairs of unit vectors $\hat{\sigma}$ which pair-wise are not orthogonal. This has its consequences for the choice of inner product in the Σ-space.

Let us first establish the components of the $\hat{\sigma}$-basis shown in figure (5.3) with respect to the x-axis and the y-axis of the plane, *i.e.*:

$$\begin{aligned}
&\hat{\sigma}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \hat{\sigma}_2 = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix} , \quad \hat{\sigma}_3 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix} , \\
&\hat{\sigma}_4 = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} , \quad \hat{\sigma}_5 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ and } \quad \hat{\sigma}_6 = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} .
\end{aligned} \tag{5.10}$$

The pairs of unit vectors of the above set $(\hat{\sigma}_1,\ \hat{\sigma}_4)$, $(\hat{\sigma}_3,\ \hat{\sigma}_5)$ and $(\hat{\sigma}_2,\ \hat{\sigma}_6)$ which correspond to the local bases for the description of displacements of the masses numbered by respectively 1, 2 and 3, are not pair-wise orthogonal. In fact one has:

$$\hat{\sigma}_1 \cdot \hat{\sigma}_4 = \hat{\sigma}_3 \cdot \hat{\sigma}_5 = \hat{\sigma}_2 \cdot \hat{\sigma}_6 = \frac{1}{2} \quad .$$

As a consequence, displacement vectors which are linear combinations of the two unit vectors out of a pair, do not have a simple inner product in terms of their coefficients. For example, let us consider the two displacement vectors $\vec{\xi}_{1a}$ and $\vec{\xi}_{1b}$ both describing arbitrary displacements of the mass numbered by 1. According to the relations (5.8) such displacements are described by the first and fourth components of a six dimensional column vector in Σ-space. Let us, in agreement with formula (5.8), define:

$$\vec{\xi}_{1a} = \Sigma_{1a}\hat{\sigma}_1 + \Sigma_{4a}\hat{\sigma}_4 \ \text{ and } \ \vec{\xi}_{1b} = \Sigma_{1b}\hat{\sigma}_1 + \Sigma_{4b}\hat{\sigma}_4 \quad .$$

The inner product of those two vectors is given by:

$$\begin{aligned}
\vec{\xi}_{1a} \cdot \vec{\xi}_{1b} &= \Sigma_{1a}\Sigma_{1b}\hat{\sigma}_1 \cdot \hat{\sigma}_1 + \Sigma_{1a}\Sigma_{4b}\hat{\sigma}_1 \cdot \hat{\sigma}_4 + \Sigma_{4a}\Sigma_{1b}\hat{\sigma}_4 \cdot \hat{\sigma}_1 + \Sigma_{4a}\Sigma_{4b}\hat{\sigma}_4 \cdot \hat{\sigma}_4 \\
&= \Sigma_{1a}\Sigma_{1b} + \frac{1}{2}\Sigma_{1a}\Sigma_{4b} + \frac{1}{2}\Sigma_{4a}\Sigma_{1b} + \Sigma_{4a}\Sigma_{4b} \quad .
\end{aligned}$$

This can be written in a more convenient form, after we define the metrical matrix g given by:

$$g = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \quad .$$

Using this definition, we obtain for the expression of the inner product of the two displacement vectors the following:

$$\begin{aligned}
\vec{\xi}_{1a} \cdot \vec{\xi}_{1b} &= (\Sigma_{1a},\ \Sigma_{4a}) \begin{pmatrix} \Sigma_{1b} + \frac{1}{2}\Sigma_{4b} \\ \\ \frac{1}{2}\Sigma_{1b} + \Sigma_{4b} \end{pmatrix} \\
&= (\Sigma_{1a},\ \Sigma_{4a})\ g \begin{pmatrix} \Sigma_{1b} \\ \\ \Sigma_{4b} \end{pmatrix} \quad .
\end{aligned}$$

For linear combinations of the other pairs of unit vectors $\hat{\sigma}$ one finds similar expressions for their inner products. This leads us to the introduction of a 6×6 metrical matrix in the Σ-space, defined by:

$$G = \begin{pmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{5.11}$$

Notice that the positions of the matrix elements which equal $1/2$ correspond to the combination of the $\hat{\sigma}$-basis into pairs, *i.e.* (1,4) and (4,1), (3,5) and (5,3) and (2,6) and (6,2).

Using the above definition (5.11) of the metric G in Σ-space, we choose for the inner product of two six-component vectors $\vec{\Sigma}_1$ and $\vec{\Sigma}_2$ the following expression:

$$\vec{\Sigma}_1 \cdot \vec{\Sigma}_2 = \vec{\Sigma}_1^T\ G\,\vec{\Sigma}_2 \quad . \tag{5.12}$$

This choice will show a suitable inner product for Σ-space as we will see in the following.

5.4 The reduction of the regular representation of S_3.

In section (5.2) we found that the six-component displacement vectors $\vec{\Sigma}(t)$ of the triangular harmonic oscillator transform according to the inverse of the regular representation under the operations of the symmetric group S_3. Moreover, are the 6×6 matrices for this representation equal to the matrices shown in table (2.2) at page 16. Now, the inverse of the regular representation of S_3 is almost equal to the regular representation itself, except that the matrices for the group elements (132) and (123) are interchanged. Furthermore, one might observe the following general structure for those 6×6 matrices depending on whether they represent even permutations or odd permutations of three objects, *i.e.*

$$D(\text{even}) = \begin{pmatrix} A & 0 \\ & \\ 0 & A^T \end{pmatrix} \quad \text{and} \quad D(\text{odd}) = \begin{pmatrix} 0 & A \\ & \\ A^T & 0 \end{pmatrix} \quad , \tag{5.13}$$

where the symbol 0 represents a 3×3 matrix with only zero matrix elements and where A represents a 3×3 matrix which contains in each row and in each column two zeroes and one 1. The various matrices A are shown in table (5.2) below.

g	A
I and (23)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(123) and (12)	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
(132) and (13)	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Table 5.2: The relevant part A as defined in formula (5.13) for the six matrices $D(g)$ of the inverse of the regular representation of S_3.

In the following we will exploit the symmetric and extremely simple form of the matrices $D(g)$ given in formula (5.13) in order to guess the basis vectors in Σ-space for the various subspaces related to the reduction of D into irreducible representations of S_3.

5.4.1 The trivial representation.

It is rather easy to guess that the six-component vector $\hat{A}_1$, given by:

$$\hat{A}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , \tag{5.14}$$

is invariant under the transformations $D(g)$ of formula (5.13). Consequently, any displacement vector $\vec{\Sigma}(t) = \Sigma(t)\hat{A}_1$ transforms into itself under those transformations, *i.e.*

$$D(g)\vec{\Sigma}(t) = \vec{\Sigma}(t) \quad . \tag{5.15}$$

So, in the one-dimensional subspace spanned by $\hat{A}_1$, the transformations $D(g)$ are represented by the number 1, which is the trivial representation of S_3.

Notice that $\hat{A}_1$ is properly normalized according to the definition (5.12) for the inner product in Σ-space.

5.4.2 The representation $D^{[111]}$.

It is also not difficult to guess that the six-component vector $\hat{A}_2$, given by:

$$\hat{A}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad , \tag{5.16}$$

transforms into itself under the even permutations and into minus itself under the odd permutations for the transformations $D(g)$ of formula (5.13). Consequently, in the one-dimensional subspace spanned by $\hat{A}_2$, the transformations $D(g)$ are represented by $+1$ for the even permutations and by -1 for the odd permutations. This corresponds to the irrep [111] of S_3.

Notice that, using the definition (5.12) for the inner product in Σ-space, and using the definitions (5.14) and (5.16) for respectively $\hat{A}_1$ and $\hat{A}_2$, one finds the following orthonormality relations:

$$\hat{A}_i \cdot \hat{A}_j = \hat{A}_i^T \, G \, \hat{A}_j = \delta_{ij} \quad , \quad \text{for} \quad i,j = 1,2 \quad .$$

5.4.3 One of the representations $D^{[21]}$.

It is also not very difficult to find that the two six-component vectors $\hat{A}_3$ and $\hat{A}_4$, given by:

$$\hat{A}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \hat{A}_4 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} \quad , \tag{5.17}$$

remain in the two-dimensional subspace they span under the transformations $D(g)$ of formula (5.13). For an example, let us study the transformation $D((123))$. Using table (5.2) one finds:

$$D((123))\hat{A}_3 = -\frac{1}{2}\hat{A}_3 + \frac{\sqrt{3}}{2}\hat{A}_4 \quad \text{and} \quad D((123))\hat{A}_4 = -\frac{\sqrt{3}}{2}\hat{A}_3 - \frac{1}{2}\hat{A}_4 \quad .$$

For the other group elements similar relations hold as can be verified by the reader.

Consequently, $\hat{A}_3$ and $\hat{A}_4$ span a two-dimensional subspace of the Σ-space which serves as a vector space for the irrep $D^{[21]}$ of S_3.

Notice that, using the definition (5.12) for the inner product in Σ-space, and using the definitions (5.14), (5.16) and (5.17) for respectively $\hat{A}_1$, $\hat{A}_2$ and $\hat{A}_3$ and $\hat{A}_4$, one finds the following orthonormality relations:

$$\hat{A}_i \cdot \hat{A}_j = \hat{A}_i^T \, G \, \hat{A}_j = \delta_{ij} \quad , \quad \text{for } \; i,j = 1,\ldots,4 \quad .$$

5.4.4 The other representation $D^{[21]}$.

In the preceding we obtained three subspaces of the Σ-space and four orthonormal basis vectors. The remaining two-dimensional subspace must have basis vectors which are orthogonal to those four. After some algebra one finds as a possible basis for the remaining subspace the following two vectors:

$$\hat{A}_5 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \hat{A}_6 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad , \tag{5.18}$$

Let us, as an example, perform the transformation $D((123))$ in order to see whether the subspace is invariant under that transformation, *i.e.*

$$D((123))\hat{A}_5 = -\frac{1}{2}\hat{A}_5 + \frac{\sqrt{3}}{2}\hat{A}_6 \quad \text{and} \quad D((123))\hat{A}_6 = -\frac{\sqrt{3}}{2}\hat{A}_5 - \frac{1}{2}\hat{A}_6 \quad .$$

We find that for $D((123))$ the transformed vectors can indeed be written as linear combinations of the basis vectors $\hat{A}_5$ and $\hat{A}_6$. Similar relations hold for the other group elements of S_3, as can be verified by the reader. This shows that we found the other two-dimensional subspace for the irrep $D^{[21]}$.

So, we obtained in this section a full reduction of the 6×6 regular representation of S_3 into two one-dimensional and two two-dimensional irreps. Next, we will return to the physics problem of finding the normal modes for infinitesimal oscillations of the triangular harmonic oscillator in two dimensions.

5.5 Construction of the solution space.

Using the information of the previous sections, we intend here to mix mathematics with "physics" intuition in order to find the equations of motion in the six-dimensional solution space of equation (5.5).

5.5.1 Symmetric oscillations.

The most symmetric situation which one might imagine for the oscillations of the three masses is immediatly suggested by the symmetry of the geometric figure of the system, *i.e.* the vibrations along the directions shown below.

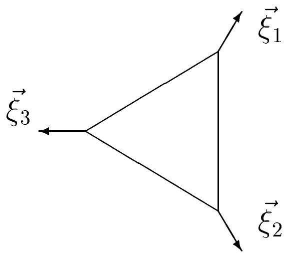


Figure 5.5: The displacement vectors of the three masses for a symmetric vibrational mode of the triangle.

When the three masses oscillate such that their displacements at each instant of time are equal in length and directed toward or pointing away from the center of the triangle which is represented by the origin of the coordinate system (see figure above), then the six-component displacement vector may be given by:

$$\vec{\xi}(t) = \begin{pmatrix} \xi_{1x}(t) \\ \xi_{2x}(t) \\ \xi_{3x}(t) \\ \xi_{1y}(t) \\ \xi_{2y}(t) \\ \xi_{3y}(t) \end{pmatrix} = \epsilon_1(t)\hat{a}_1 \ \text{, where } \ \hat{a}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ \sqrt{3}/2 \\ -\sqrt{3}/2 \\ 0 \end{pmatrix} \ . \tag{5.19}$$

From the symmetry of the situation as depicted in figure (5.5) it is clear that S_3 operations do not have any effect on the form of the solution in the various transformed coordinate systems. Consequently, the vector $\hat{a}_1$ transforms always into itself and thus $\hat{a}_1$ spans the one-dimensional sub-space which transforms according to the irreducible trivial matrix representation $D^{[4]}$ under S_3.

The equation of motion for ϵ_1 will be discussed in the following.

Let us next establish the relation of $\hat{a}_1$ in ξ-space with $\hat{A}_1$ (see formula 5.14 for its definition) in Σ-space. We define an arbitrary displacement $\vec{\Sigma}(t)$ in the subspace spanned by $\hat{A}_1$, according to:

$$\vec{\Sigma}(t) = \epsilon_1(t)\ \hat{A}_1 \quad .$$

Using the relations (5.8) and (5.10), we find for the three displacement vectors of the three masses, the expressions:

$$\begin{aligned}
\vec{\xi}_1(t) = \epsilon_1(t)\frac{1}{3}\left(\hat{\sigma}_1 + \hat{\sigma}_4\right) &= \epsilon_1\frac{1}{\sqrt{3}}\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \quad , \\
\vec{\xi}_2(t) = \epsilon_1(t)\frac{1}{3}\left(\hat{\sigma}_3 + \hat{\sigma}_5\right) &= \epsilon_1\frac{1}{\sqrt{3}}\begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} \quad \text{and} \\
\vec{\xi}_3(t) = \epsilon_1(t)\frac{1}{3}\left(\hat{\sigma}_2 + \hat{\sigma}_6\right) &= \epsilon_1\frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad .
\end{aligned}$$

When one puts the six components of the above displacement vectors in one six-component column vector, one obtains exactly the column vector of formula (5.19). Consequently, the basis vector $\hat{a}_1$ in ξ-space corresponds directly to the basis vector $\hat{A}_1$ in Σ-space.

5.5.2 Rotations.

The basis vector $\hat{A}_2$ in Σ-space shown in formula (5.16) spans the one-dimensional subspace for the irrep $D^{[111]}$ of S_3. Let an arbitrary (but infinitesimal) displacement $\vec{\Sigma}(t)$ in this subspace be given by:

$$\vec{\Sigma}(t) = \epsilon_2(t)\ \hat{A}_2 \quad .$$

Using the relations (5.8) and (5.10), we find for the three displacement vectors of the three masses, the expressions:

$$\begin{aligned}
\vec{\xi}_1(t) = \epsilon_2(t)\frac{1}{\sqrt{3}}\left(\hat{\sigma}_1 - \hat{\sigma}_4\right) &= \epsilon_2(t)\frac{1}{\sqrt{3}}\begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad , \\
\vec{\xi}_2(t) = \epsilon_2(t)\frac{1}{\sqrt{3}}\left(\hat{\sigma}_3 - \hat{\sigma}_5\right) &= \epsilon_2(t)\frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad \text{and} \\
\vec{\xi}_3(t) = \epsilon_2(t)\frac{1}{\sqrt{3}}\left(\hat{\sigma}_2 - \hat{\sigma}_6\right) &= \epsilon_2(t)\frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad .
\end{aligned} \tag{5.20}$$

The three resulting displacement vectors are displayed in figure (5.6) below. As this figure suggests, the subspace for $D^{[111]}$ of S_3 corresponds to the subspace of infinitesimal rotations.

An infinitesimal rotation of the triangle as a whole with an angle ϕ, starting from the equilibrium position, gives as a result for the position vector of mass number 1 :

$$\vec{r}_1 = \begin{pmatrix} 1 & -\phi \\ \phi & 1 \end{pmatrix}\begin{pmatrix} a/2 \\ a\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} a/2 \\ a\sqrt{3}/2 \end{pmatrix} + \begin{pmatrix} -a\phi\sqrt{3}/2 \\ a\phi/2 \end{pmatrix} \quad .$$

From the above result one deduces that the displacement vector for the infinitesimal rotation is for mass number 1 given by:

$$\vec{\xi}_1 = \begin{pmatrix} -a\phi\sqrt{3}/2 \\ \\ a\phi/2 \end{pmatrix} \quad . \tag{5.21}$$

For the other two masses one finds similarly:

$$\vec{\xi}_2 = \begin{pmatrix} a\phi\sqrt{3}/2 \\ \\ a\phi/2 \end{pmatrix} \text{ and } \vec{\xi}_3 = \begin{pmatrix} 0 \\ \\ -a\phi \end{pmatrix} \quad . \tag{5.22}$$

Substituting $a\phi$ in (5.21) and (5.22) by $\epsilon_2/\sqrt{3}$ one obtains for the infinitesimal rotations exactly the same expressions for the three displacement vectors as shown in formula (5.20).

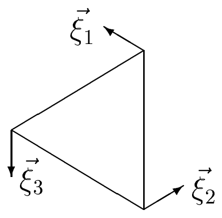


Figure 5.6: The displacement vectors of the three masses for an infinitesimal rotation of the triangle as a whole.

The combination of the three displacement vectors into one six-component column vector, leads to the expression:

$$\vec{\xi}(t) = \begin{pmatrix} \xi_{1x}(t) \\ \xi_{2x}(t) \\ \xi_{3x}(t) \\ \xi_{1y}(t) \\ \xi_{2y}(t) \\ \xi_{3y}(t) \end{pmatrix} = \epsilon_2(t)\hat{a}_2 \text{ , where } \hat{a}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{3}/2 \\ \sqrt{3}/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1 \end{pmatrix} \quad . \tag{5.23}$$

Under the operations of the group elements of S_3 the three vectors of figure (5.6) remain either unaltered, or turn into their inverses as far as their components with respect to the transformed coordinate systems are concerned. The total potential energy vanishes when the triangle as a whole performs a rotation around its center. This gives then for $\epsilon_2(t)$ the equation of motion:

$$m\ddot{\epsilon}_2(t) = 0 \quad . \tag{5.24}$$

Such motions, where no oscillations occur ($\omega = 0$), are refered to as *zero mode* solutions.

5.5.3 Translations.

The basis vectors $\hat{A}_3$ and $\hat{A}_4$ in Σ-space (see formula 5.17) span a two-dimensional subspace for the irrep $D^{[21]}$ of S_3. Let an arbitrary (but infinitesimal) displacement $\vec{\Sigma}(t)$ in this subspace be given by:

$$\vec{\Sigma}(t) = \tilde{\epsilon}_3(t)\ \hat{A}_3 + \tilde{\epsilon}_4(t)\ \hat{A}_4 \quad .$$

Using the relations (5.8) and (5.10), we find for the displacement vector of the three masses in ξ-space, the expression:

$$\vec{\xi}(t) = \tilde{\epsilon}_3(t)\ \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{3}/2 \\ -\sqrt{3}/2 \\ -\sqrt{3}/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \tilde{\epsilon}_4(t)\ \frac{1}{3} \begin{pmatrix} \sqrt{3}/2 \\ \sqrt{3}/2 \\ \sqrt{3}/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{pmatrix} \quad . \tag{5.25}$$

For obvious reasons, we prefer to select a different basis in the two-dimensional subspace of the ξ-space, given by the following unit vectors:

$$\hat{a}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{a}_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad . \tag{5.26}$$

When one inserts this basis in the expression (5.25) for an arbitrary displacement in the two-dimensional subspace under consideration, then one obtains:

$$\vec{\xi}(t) = \epsilon_3(t)\ \hat{a}_3 + \epsilon_4(t)\ \hat{a}_4 \quad , \tag{5.27}$$

where we have introduced $\epsilon_3 = (-\tilde{\epsilon}_3\sqrt{3} + \tilde{\epsilon}_4)/2$ and $\epsilon_4 = (\tilde{\epsilon}_3 + \tilde{\epsilon}_4\sqrt{3})/2$. Such displacements represent translations of the triangle as a whole in the (x, y)-plane.

When all three masses have at the same instant of time the same displacements $\vec{\xi}_i$ $(i = 1, 2, 3)$, then the total potential energy of the system (see formula (5.4)) vanishes. This situation represents a translation of the triangle as a whole, whithout deformation.

If, for example, the motion of the three masses at each instant of time is given by a translation in the x-direction of the triangle as a whole, then its motion can be described by $\vec{\xi}(t) = \epsilon_3(t)\ \hat{a}_3$. Similarly, a translation in the y-direction, can be described by $\vec{\xi}(t) = \epsilon_4(t)\ \hat{a}_4$.

The potential energy, U_{pot}, vanishes in both cases at all instants of time. So, the equations of motion are given by:

$$m\ddot{\epsilon}_3(t) = 0 \quad \text{and} \quad m\ddot{\epsilon}_4(t) = 0 \quad . \tag{5.28}$$

These differential equations are solved by the movement with constant velocity in the x-direction, respectively y-direction, of the triangle as a whole. Consequently, we found two more *zero mode* solutions of the equations of motion (5.5).

A translation in an arbitrary direction is described by the expression in formula (5.27), which implies for the displacement vector of each mass the general form:

$$\vec{\xi}_i(t) = \frac{1}{\sqrt{3}}\epsilon_3(t)\hat{e}_1 + \frac{1}{\sqrt{3}}\epsilon_4(t)\hat{e}_2 \;\; (i = 1, 2, 3) \ . \tag{5.29}$$

Under the group transformation (123) of S_3 which is shown in figure (5.4) at page 51, one finds for the same displacement (5.29) the expression:

$$\begin{aligned} \vec{\xi}_i(t) &= \frac{1}{\sqrt{3}}\left[\epsilon_3(t)\left\{-\frac{1}{2}\hat{e}'_1 + \frac{\sqrt{3}}{2}\hat{e}'_2\right\} + \epsilon_4(t)\left\{-\frac{\sqrt{3}}{2}\hat{e}'_1 - \frac{1}{2}\hat{e}'_2\right\}\right] \\ &= \frac{1}{\sqrt{3}}\left[\left\{-\frac{1}{2}\epsilon_3(t) - \frac{\sqrt{3}}{2}\epsilon_4(t)\right\}\hat{e}'_1 + \left\{\frac{\sqrt{3}}{2}\epsilon_3(t) - \frac{1}{2}\epsilon_4(t)\right\}\hat{e}'_2\right] \\ &\quad \text{for } \ (i = 1, 2, 3) \ . \end{aligned} \tag{5.30}$$

Consequently, for the components of each of the displacement vectors one finds the transformation rule:

$$\left(\begin{array}{c} \epsilon'_3(t) \\ \\ \epsilon'_4(t) \end{array}\right) = \left(\begin{array}{cc} -1/2 & -\sqrt{3}/2 \\ \\ \sqrt{3}/2 & -1/2 \end{array}\right)\left(\begin{array}{c} \epsilon_3(t) \\ \\ \epsilon_4(t) \end{array}\right) = \left[D^{[21]}((123))\right]^{-1}\left(\begin{array}{c} \epsilon_3(t) \\ \\ \epsilon_4(t) \end{array}\right) \ . \tag{5.31}$$

So, the components of each of the displacement vectors of the three masses transform corresponding the inverse of the two-dimensional matrix representation of element (123) of S_3 (compare to the corresponding matrix of table (2.3) at page 19). In this case it is not important that also the numbering of the masses changes due to the group operation, since the three displacement vectors contribute in exactly the same way to the six-component vector (5.27). Therefore, we obtain for the space spanned by the basis vectors $\hat{a}_3$ and $\hat{a}_4$ (5.26) the transformation:

$$\vec{\xi}(t) = \left\{-\frac{1}{2}\epsilon_3(t) - \frac{\sqrt{3}}{2}\epsilon_4(t)\right\}\hat{a}'_3 + \left\{\frac{\sqrt{3}}{2}\epsilon_3(t) - \frac{1}{2}\epsilon_4(t)\right\}\hat{a}'_4 \ . \tag{5.32}$$

We find thus that the six-component vectors of the two-dimensional sub-space spanned by $\hat{a}_3$ and $\hat{a}_4$ transform in exactly the same way as the two-component vectors in equation (5.31) under the operation of the element (123) of S_3. One might verify the same for the other group elements. Consequently, on this sub-space is S_3 represented by the irrep $D^{[21]}$ as we expected.

5.5.4 The remaining two-dimensional subspace.

The basis vectors $\hat{A}_5$ and $\hat{A}_6$ in Σ-space (see formula 5.18) also span a two-dimensional subspace for the irrep $D^{[21]}$ of S_3. Let an arbitrary displacement $\vec{\Sigma}(t)$ in this subspace be described by:

$$\vec{\Sigma}(t) = \epsilon_5(t)\ \hat{A}_5 + \epsilon_6(t)\ \hat{A}_6 \quad .$$

Using the relations (5.8) and (5.10), we find for the displacement vector of the three masses in ξ-space, the expression:

$$\vec{\xi}(t) = \epsilon_5(t)\,\frac{1}{\sqrt{3}}\begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ -\sqrt{3}/2 \\ \sqrt{3}/2 \\ 0 \end{pmatrix} + \epsilon_6(t)\,\frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{3}/2 \\ -\sqrt{3}/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1 \end{pmatrix} \quad . \tag{5.33}$$

The reader may verify that $\vec{\xi}(t)$ transforms according to the irrep $D^{[21]}$ of S_3.

5.6 The complete solution space.

Using the relations (5.19) and (5.33) we find for an arbitrary oscillation (disregarding translations and rotations) the general expression:

$$\vec{\xi}(t) = \begin{pmatrix} \xi_{1x}(t) \\ \xi_{2x}(t) \\ \xi_{3x}(t) \\ \xi_{1y}(t) \\ \xi_{2y}(t) \\ \xi_{3y}(t) \end{pmatrix} = \epsilon_1(t)\hat{a}_1 + \epsilon_5(t)\hat{a}_5 + \epsilon_6(t)\hat{a}_6 \ \ ,$$

which gives in an explicit form, the expression:

$$\vec{\xi}(t) = \begin{pmatrix} \epsilon_1(t)/\sqrt{12} & + & \epsilon_5(t)/\sqrt{12} & + & \epsilon_6(t)/2 \\ \epsilon_1(t)/\sqrt{12} & + & \epsilon_5(t)/\sqrt{12} & + & -\epsilon_6(t)/2 \\ -\epsilon_1(t)/\sqrt{3} & + & -\epsilon_5(t)/\sqrt{3} & + & 0 \\ \epsilon_1(t)/2 & + & -\epsilon_5(t)/2 & + & \epsilon_6(t)/\sqrt{12} \\ -\epsilon_1(t)/2 & + & \epsilon_5(t)/2 & + & \epsilon_6(t)/\sqrt{12} \\ 0 & + & 0 & + & -\epsilon_6(t)/\sqrt{3} \end{pmatrix} \quad . \tag{5.34}$$

The associated relation of the total potential energy stored in the system for an arbitrary displacement $\vec{\xi}$, can now be determined in terms of the functions $\epsilon_1(t)$, $\epsilon_5(t)$ and $\epsilon_6(t)$. Using the formulas (5.6), (5.1) and (5.34) we determine first:

$$\vec{r}_1 - \vec{r}_2 = \vec{r}_{10} + \vec{\xi}_1 - \vec{r}_{20} - \vec{\xi}_2 = \begin{pmatrix} \epsilon_6 \\ a\sqrt{3} + \epsilon_1 - \epsilon_5 \end{pmatrix} \ ,$$

$$\vec{r}_2 - \vec{r}_3 = \vec{r}_{20} + \vec{\xi}_2 - \vec{r}_{30} - \vec{\xi}_3 = \frac{1}{2}\begin{pmatrix} 3a + \epsilon_1\sqrt{3} + \epsilon_5\sqrt{3} - \epsilon_6 \\ -a\sqrt{3} - \epsilon_1 + \epsilon_5 + \epsilon_6\sqrt{3} \end{pmatrix} \ \text{and}$$

$$\vec{r}_3 - \vec{r}_1 = \vec{r}_{30} + \vec{\xi}_3 - \vec{r}_{10} - \vec{\xi}_1 = \frac{1}{2}\begin{pmatrix} -3a - \epsilon_1\sqrt{3} - \epsilon_5\sqrt{3} - \epsilon_6 \\ -a\sqrt{3} - \epsilon_1 + \epsilon_5 - \epsilon_6\sqrt{3} \end{pmatrix} \ .$$

Then, using formula (5.4) and the above results, we obtain:

$$U_{pot} = \frac{K}{2}\left[\left\{\sqrt{(\epsilon_6)^2 + (a\sqrt{3} + \epsilon_1 - \epsilon_5)^2} - a\sqrt{3}\right\}^2 + \right.$$

$$+\left\{\sqrt{(3a + \epsilon_1\sqrt{3} + \epsilon_5\sqrt{3} - \epsilon_6)^2/4 + (-a\sqrt{3} - \epsilon_1 + \epsilon_5 + \epsilon_6\sqrt{3})^2/4} - a\sqrt{3}\right\}^2 +$$

$$\left. + \left\{\sqrt{(-3a - \epsilon_1\sqrt{3} - \epsilon_5\sqrt{3} - \epsilon_6)^2/4 + (-a\sqrt{3} - \epsilon_1 + \epsilon_5 - \epsilon_6\sqrt{3})^2/4} - a\sqrt{3}\right\}^2\right] . \tag{5.35}$$

For this potential we will determine the time dependence of the functions ϵ_1, ϵ_5 and ϵ_6, using the equations of motion (5.5), or equivalently, the Lagrange equations. The total kinetic energy of the three masses is given by:

$$T_{kin}(\dot{\vec{r}}_1, \dot{\vec{r}}_2, \dot{\vec{r}}_3) = \frac{1}{2}m\left\{\dot{\vec{r}}_1^{\,2} + \dot{\vec{r}}_2^{\,2} + \dot{\vec{r}}_3^{\,2}\right\} \quad , \tag{5.36}$$

which, disregarding possible translations and rotations of the triangle as a whole, using the definitions (5.6) and (5.34), can be denoted by:

$$T_{kin}(\dot{\epsilon}_1, \dot{\epsilon}_5, \dot{\epsilon}_6) = \frac{1}{2}m\left\{\dot{\epsilon}_1^2 + \dot{\epsilon}_5^2 + \dot{\epsilon}_6^2\right\} \quad . \tag{5.37}$$

Consequently, using the Lagrange equations of classical mechanics, we find:

$$m\ddot{\epsilon}_i = \frac{d}{dt}\left(\frac{\partial(T-U)}{\partial\dot{\epsilon}_i}\right) = \frac{\partial(T-U)}{\partial\epsilon_i} = -\frac{\partial U_{pot}}{\partial\epsilon_i} \quad . \tag{5.38}$$

for $(i = 1, 5, 6)$.

The potential (5.35) might first be expanded in a Taylor series around the point given by: $\epsilon_1 = \epsilon_5 = \epsilon_6 = 0$, *i.e.*:

$$\begin{aligned} U_{pot}(\epsilon_1, \epsilon_5, \epsilon_6) &= U_{pot}(0,0,0) + \\ & + \sum_{i=1,5,6}\left\{\left.\frac{\partial U_{pot}}{\partial\epsilon_i}\right|_{\epsilon_1 = \epsilon_5 = \epsilon_6 = 0}\right\}\epsilon_i + \\ & + \frac{1}{2!}\sum_{i,j=1,5,6}\left\{\left.\frac{\partial^2 U_{pot}}{\partial\epsilon_i\partial\epsilon_j}\right|_{\epsilon_1 = \epsilon_5 = \epsilon_6 = 0}\right\}\epsilon_i\epsilon_j + \cdots \end{aligned}$$

The zeroeth order term in this series vanishes, as can easily be seen from the expression (5.35). Also the first order terms vanish. We end up with a series which starts out at the second order terms, according to:

$$U_{pot} = \frac{1}{2!}\frac{K}{2}\left(6\epsilon_1^2 + 3\epsilon_5^2 + 3\epsilon_6^2\right) + \text{ higher order terms.} \tag{5.39}$$

For small oscillations we restrict ourselves to the second order terms in U_{pot}. The Lagrange equations (5.38) lead then to the following set of differential equations:

$$m\ddot{\epsilon}_1 = -3K\epsilon_1 \text{ and } m\ddot{\epsilon}_j = -\frac{3K}{2}\epsilon_j \quad (j = 5, 6). \tag{5.40}$$

The solutions of those equations are in general given by:

$$\epsilon_i(t) = A_i \cos(\omega_i t + \phi_{0i}) \quad (i = 1, 5, 6), \tag{5.41}$$

where the frequencies of the various basis oscillations (refered to as the *normal modes* of the system), are determined by:

$$\omega_1 = \sqrt{\frac{3K}{m}} \ , \ \omega_5 = \sqrt{\frac{3K}{2m}} \text{ and } \omega_6 = \sqrt{\frac{3K}{2m}} \ , \tag{5.42}$$

and where A_i $(i = 1, 5, 6)$ represent the amplitudes of the various normal modes of the system and where ϕ_{0i} $(i = 1, 5, 6)$ represent the various relative phases.

Chapter 6

Series of matrices and direct products.

Since it will be frequently used in the following, we discuss here some properties of the matrix series expansion given by:

$$M = \mathbf{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \quad = \exp(A), \tag{6.1}$$

where M and A represent $n \times n$ matrices.

Example 1

For an example, let us take for A the following matrix:

$$A = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad .$$

This matrix has the following properties:

$$A^2 = -\alpha^2 \mathbf{1} \ , \quad A^3 = -\alpha^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \quad A^4 = \alpha^4 \mathbf{1} \ , \quad \ldots$$

Using those properties, we determine the series (6.1) for the matrix A, *i.e.*

$$\begin{aligned}
M &= \exp(A) \\
&= \mathbf{1} + \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\alpha^2}{2!}\mathbf{1} + \frac{\alpha^3}{3!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\alpha^4}{4!}\mathbf{1} + \cdots \\
&= \mathbf{1} \left\{ 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots \right\} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left\{ \alpha - \frac{\alpha^3}{3!} + \cdots \right\} \\
&= \mathbf{1} \cos(\alpha) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin(\alpha) \\
&= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \ ,
\end{aligned} \tag{6.2}$$

which matrix represents a rotation in two dimensions with a rotation angle α.

6.1 Det(Exp)=Exp(Trace).

Let us assume that there exists a similarity transformation S, for which A in formula (6.1) obtains the triangular form, *i.e.*:

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & \text{"non} & \text{zero"} \\ \text{zero} & & \ddots & \\ & & & \lambda_n \end{pmatrix} . \tag{6.3}$$

The diagonal elements in the triangular form of $S^{-1}AS$ are indicated by λ_1, λ_2, ..., λ_n. In the triangle of "non-zero" matrix elements incidentally may appear zeroes of course. When all matrix elements in that part vanish, the matrix $S^{-1}AS$ has the diagonal form.

An $n \times n$ matrix B which has the triangular form (6.3) can in general be given by the following expression for its matrix elements:

$$B_{ij} = \begin{cases} \text{any (complex) value} & \text{for} \quad j \geq i \, , \\ \\ 0 & \text{for} \quad j < i \, . \end{cases} \tag{6.4}$$

For the matrix elements of the product of two such $n \times n$ matrices, B and C, we have:

$$[BC]_{ij} \; = \; \sum_{k=1}^{n} B_{ik}\, C_{kj} \quad \text{for } (i, j = 1, \ldots, n).$$

Using expression (6.4) for the matrix elements of matrices which are in the triangular form (6.3), we find:

$$[BC]_{ij} \; = \; \sum_{\substack{k \geq i \\ j \geq k}} B_{ik}\, C_{kj} \; = \begin{cases} \text{any (complex) value} & \text{for} \quad j \geq i \, , \\ \\ 0 & \text{for} \quad j < i \, . \end{cases} \tag{6.5}$$

and also:

$$[BC]_{ii} \; = \; B_{ii}\, C_{ii} \quad \text{for } (i = 1, \ldots, n). \tag{6.6}$$

Consequently, the product BC of two triangular matrices B and C is again triangular (see formula (6.5)), and its diagonal matrix elements are just the products of the corresponding diagonal matrix elements of B and C (see formula (6.6)).

So, when $S^{-1}AS$ has the triangular form as given in (6.3), then the matrices defined by:

$$\left\{S^{-1}AS\right\}^2 = S^{-1}A^2S \quad , \quad \left\{S^{-1}AS\right\}^3 = S^{-1}A^3S \;\; \text{etc.} \tag{6.7}$$

also have the triangular form, and, using (6.6) we find for the related diagonal matrix elements:

$$
\begin{aligned}
\left[\left\{S^{-1}AS\right\}^2\right]_{ii} &= \left[\left\{S^{-1}AS\right\}_{ii}\right]^2 = \left(\lambda_i\right)^2 \\
\left[\left\{S^{-1}AS\right\}^3\right]_{ii} &= \left[\left\{S^{-1}AS\right\}_{ii}\right]^3 = \left(\lambda_i\right)^3 \\
&\text{etc.}
\end{aligned}
\tag{6.8}
$$

When we apply the similarity transformation S to the series (6.1) using also (6.7), then we obtain:

$$
\begin{aligned}
S^{-1}MS &= S^{-1}\mathbf{1}S + S^{-1}AS + S^{-1}\frac{A^2}{2!}S + S^{-1}\frac{A^3}{3!}S + \cdots \\
&= \mathbf{1} + S^{-1}AS + \frac{\{S^{-1}AS\}^2}{2!} + \frac{\{S^{-1}AS\}^3}{3!} + \cdots \\
&= \exp\left(S^{-1}AS\right) \quad .
\end{aligned}
$$

Each term in the sum is triangular, so $S^{-1}MS$ is triangular, and also:

$$
\begin{aligned}
\left[S^{-1}MS\right]_{ii} &= 1 + \left\{S^{-1}AS\right\}_{ii} + \frac{\left[\{S^{-1}AS\}^2\right]_{ii}}{2!} + \frac{\left[\{S^{-1}AS\}^3\right]_{ii}}{3!} + \cdots \\
&= 1 + \left(\lambda_i\right) + \frac{\left(\lambda_i\right)^2}{2!} + \frac{\left(\lambda_i\right)^3}{3!} + \cdots \\
&= \exp\left(\lambda_i\right) \quad \text{for } (i = 1, \ldots, n).
\end{aligned}
\tag{6.9}
$$

Now, the determinant of a triangular matrix is just the product of its diagonal matrix elements. So, using the above formula (6.9), we find:

$$
\begin{aligned}
\det\left\{S^{-1}MS\right\} &= \exp\left(\lambda_1\right)\exp\left(\lambda_2\right)\cdots\exp\left(\lambda_n\right) \\
&= \exp\left(\lambda_1 + \lambda_2 + \lambda_n\right) = \exp\left(\mathrm{Tr}\left\{S^{-1}AS\right\}\right) \quad .
\end{aligned}
\tag{6.10}
$$

Furthermore, using the property that the determinant of the product of matrices equals the product of the determinants of each of those matrices separately, and the property formulated in equation (4.2) for the trace of a product of matrices, one has:

$$
\det(M) = \det\left\{S^{-1}MS\right\} \quad \text{and} \quad \mathrm{Tr}(A) = \mathrm{Tr}\left\{S^{-1}AS\right\} \quad .
$$

Consequently, inserting those properties in formula (6.10), we end up with the final result:

$$\det(M) = \exp\left\{\mathrm{Tr}(A)\right\} \quad , \tag{6.11}$$

i.e. The exponent of the trace of matrix A equals the determinant of the exponent of A.

One might verify this property for the example given in formula (6.2), *i.e.*

$$Tr(A) = 0 \ \text{ and } \ det(M) = 1 = \exp(0) = \exp(Tr(A)) \quad .$$

6.2 The Baker-Campbell-Hausdorff formula.

For the exponents of the $n \times n$ matrices A and B it is in general not true that the product of their exponents equals the exponent of their sum, as it is true for complex numbers. However, there exists an $n \times n$ matrix C, such that:

$$e^{A}\, e^{B} \;=\; e^{C} \quad . \tag{6.12}$$

In the following we will derive a relation for the matrix C in terms of the matrices A and B. For that purpose we define a complex parameter λ and a set of $n \times n$ matrices P_1, P_2, $\ldots$, and introduce the expression:

$$\exp(\lambda A)\ \exp(\lambda B) \;=\; \exp\left(\sum_{k\,=\,1}^{\infty} \lambda^{k}\, P_{k}\right) \quad . \tag{6.13}$$

For $\lambda = 1$ in (6.13) one obtains for the matrix C of (6.12) the result:

$$C \;=\; P_1 + P_2 + \cdots \quad . \tag{6.14}$$

Using the expansion (6.1) for $\exp(\lambda A)$ and for $\exp(\lambda B)$, we find for their product the following expansion in increasing powers of λ:

$$\begin{aligned} e^{\lambda A}\, e^{\lambda B} \;&=\; \mathbf{1} + \lambda(A+B) + \lambda^2\left(\frac{A^2}{2!} + AB + \frac{B^2}{2!}\right) + \\ &\quad + \lambda^3\left(\frac{A^3}{3!} + \frac{A^2B}{2!} + \frac{AB^2}{2!} + \frac{B^3}{3!}\right) + \cdots \end{aligned} \tag{6.15}$$

Similarly, for the matrices P_k we find:

$$\begin{aligned} \exp\left(\sum_{k\,=\,1}^{\infty} \lambda^{k}\, P_{k}\right) \;&=\; \mathbf{1} + \left(\sum_{k\,=\,1}^{\infty} \lambda^{k}\, P_{k}\right) + \frac{1}{2!}\left(\sum_{k\,=\,1}^{\infty} \lambda^{k}\, P_{k}\right)^2 + \cdots \\ &=\; \mathbf{1} + \lambda P_1 + \lambda^2\left(P_2 + \frac{P_1^2}{2!}\right) + \\ &\quad + \lambda^3\left(P_3 + \frac{P_1P_2}{2!} + \frac{P_2P_1}{2!} + \frac{P_1^3}{3!}\right) + \cdots \end{aligned} \tag{6.16}$$

The parameter λ in relation (6.13) can take any (complex) value, so, the "coefficients" for equal powers in λ in the expansions given by the formulas (6.15) and (6.16) must be equal. This gives as a result:

$$\begin{aligned} P_1 &= A+B \\ P_2 &= \frac{1}{2}(AB-BA) \\ P_3 &= \frac{1}{12}(A^2B-2ABA+BA^2+AB^2-2BAB+B^2A) \\ & \quad \textit{etc.} \end{aligned}$$

With the help of the relation (6.14) between the matrix C and the matrices P_k, one finds then:

$$C = A+B+\frac{1}{2}[A,B]+\frac{1}{12}\{[A,[A,B]]+[[A,B],B]\}+\cdots \quad . \tag{6.17}$$

This is the expression which should be inserted in formula (6.12) in order to give the matrix C in the exponent equivalent to the product of the two exponentiated matrices A and B. The result (6.17) is known as the *Baker-Campbell-Hausdorff* formula.

Notice that $C = A+B$ only and only if $[A,B]=0$.

Example 2

For the case $[A,B]=0$, we take as an example the matrices A and B given by:

$$A = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \quad .$$

Those matrices evidently commute and have moreover the property $A^2 = B^2 = 0$. Consequently, for the series (6.1) for A and B we obtain:

$$\exp(A) = \mathbf{1}+A = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \quad \text{and} \quad \exp(B) = \mathbf{1}+B = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \quad .$$

The product of those two matrices gives:

$$\exp(A)\exp(B) = \begin{pmatrix} 1 & 0 \\ \alpha+\beta & 1 \end{pmatrix} \quad .$$

It is now rather simple to find the exponent which yields this latter matrix as a result, *i.e.*

$$\begin{pmatrix} 1 & 0 \\ \alpha+\beta & 1 \end{pmatrix} = \exp\{\begin{pmatrix} 0 & 0 \\ \alpha+\beta & 0 \end{pmatrix}\} \quad ,$$

which result agrees with (6.17) for commuting matrices A and B.

Example 3

For the case $[A,B]\neq 0$, we take as an example the matrices A and B given by:

$$A=\left(\begin{array}{cc}0&0\\ \alpha&0\end{array}\right)\quad\text{and}\quad B=\left(\begin{array}{cc}0&\beta\\ 0&0\end{array}\right)\quad .$$

Using the results of the previous example 2, we find for the product of the exponents of those two matrices the following result:

$$\exp(A)\exp(B)=\left(\begin{array}{cc}1&0\\ \alpha&1\end{array}\right)\left(\begin{array}{cc}1&\beta\\ 0&1\end{array}\right)=\left(\begin{array}{cc}1&\beta\\ \alpha&1+\alpha\beta\end{array}\right)\quad .$$

In order to find the matrix C which exponentiated yields as a result the latter matrix, we follow the receipt of formula (6.17). Let us first collect the various ingredients of this formula, *i.e.*

$$\begin{aligned}
[A,B] &= \alpha\beta\left(\begin{array}{cc}-1&0\\ 0&1\end{array}\right)\quad ,\\
[A,[A,B]] &= -2\alpha^2\beta\left(\begin{array}{cc}0&0\\ 1&0\end{array}\right)\quad ,\\
[[A,B],B] &= -2\alpha\beta^2\left(\begin{array}{cc}0&1\\ 0&0\end{array}\right)\quad ,\ldots
\end{aligned}$$

In the following expansion we will only take into account terms up to the third order in α and β. Using formula (6.17), one obtains for the matrix C the result:

$$C=\left(\begin{array}{cc}-\frac{1}{2}\alpha\beta+\cdots & \beta-\frac{1}{6}\alpha\beta^2+\cdots\\ \\ \alpha-\frac{1}{6}\alpha^2\beta+\cdots & \frac{1}{2}\alpha\beta+\cdots\end{array}\right)\quad .$$

Next, we determine the higher order powers of C as far as terms up to the third order in α and β are involved, *i.e.*

$$C^2=\left(\begin{array}{cc}\alpha\beta+\cdots & 0+\cdots\\ \\ 0+\cdots & \alpha\beta+\cdots\end{array}\right)\quad ,\quad C^3=\left(\begin{array}{cc}0+\cdots & \alpha\beta^2+\cdots\\ \\ \alpha^2\beta+\cdots & 0+\cdots\end{array}\right)\quad ,\ldots$$

When we put the pieces together, then we obtain the result:

$$\begin{aligned}
\exp(C) &= \mathbf{1}+C+\frac{1}{2}C^2+\frac{1}{6}C^3+\cdots\\
&= \left(\begin{array}{cc}1&0\\ 0&1\end{array}\right)+\left(\begin{array}{cc}-\frac{1}{2}\alpha\beta+\cdots & \beta-\frac{1}{6}\alpha\beta^2+\cdots\\ \\ \alpha-\frac{1}{6}\alpha^2\beta+\cdots & \frac{1}{2}\alpha\beta+\cdots\end{array}\right)+\\
&\quad +\frac{1}{2}\left(\begin{array}{cc}\alpha\beta+\cdots & 0+\cdots\\ \\ 0+\cdots & \alpha\beta+\cdots\end{array}\right)+\frac{1}{6}\left(\begin{array}{cc}0+\cdots & \alpha\beta^2+\cdots\\ \\ \alpha^2\beta+\cdots & 0+\cdots\end{array}\right)+\cdots\\
&= \left(\begin{array}{cc}1&\beta\\ \alpha&1+\alpha\beta\end{array}\right)+\text{ higher order terms.}
\end{aligned}$$

So, we have shown that the terms in $\alpha^2\beta$ and $\alpha\beta^2$ vanish in the sum of all terms in the series of $\exp(C)$. When one continues the expansion, the higher order terms also cancel. Consequently, the complete result yields:

$$\exp(C) = \left(\begin{array}{cc} 1 & \beta \\ \alpha & 1+\alpha\beta \end{array} \right) \quad .$$

However, the matrix C is a rather complicated expression, even for this simple example.

6.3 Orthogonal transformations.

Let us consider an orthogonal coordinate system $\mathcal{S}$ in n dimensions characterized by coordinates x_i $(i = 1, \ldots, n)$ and by the orthonormal basis vectors $\hat{e}_i$ $(i = 1, \ldots, n)$. And let us furthermore consider in the same n-dimensional space a different orthogonal coordinate system $\mathcal{S}'$ characterized by coordinates x'_i $(i = 1, \ldots, n)$ and by the orthonormal basis vectors $\hat{e}'_i$ $(i = 1, \ldots, n)$. The orthonormality of the basis vectors is expressed by:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \quad \text{and} \quad \hat{e}'_i \cdot \hat{e}'_j = \delta_{ij} \tag{6.18}$$

The basis vectors of $\mathcal{S}$ and $\mathcal{S}'$ are related via linear transformation rules, given by:

$$\hat{e}'_j = R_{ij}\hat{e}_i \quad \text{and} \quad \hat{e}_j = (R^{-1})_{ij}\hat{e}'_i . \tag{6.19}$$

Using the orthonormality relations (6.18), we find for the matrix elements R_{ij} of the transformation matrix R, the following property:

$$\begin{aligned} \delta_{ij} &= \hat{e}'_i \cdot \hat{e}'_j = \left(R_{ki}\hat{e}_k\right) \cdot \left(R_{lj}\hat{e}_l\right) = R_{ki}R_{lj}\hat{e}_k \cdot \hat{e}_l \\ &= R_{ki}R_{lj}\delta_{kl} = R_{ki}R_{kj} = \left(R^T\right)_{ik} R_{kj} \\ &= \left(R^T R\right)_{ij} , \end{aligned}$$

or equivalently:

$$R^T R = \mathbf{1} \quad \text{or} \quad R^T = R^{-1} . \tag{6.20}$$

Linear transformations for which the transposed of the matrix equals the inverse of the matrix, are said to be *orthogonal*. The determinant of such matrices equals ± 1, as can be seen from:

$$\{det(R)\}^2 = det(R)det(R) = det(R)det(R^T) = det(RR^T) = det(\mathbf{1}) = 1. \tag{6.21}$$

Rotations are *unimodular* which means that they have determinant $+1$, anti-orthogonal transformations have determinant -1.

Rotations.

When the matrix $M = \exp(A)$ in (6.1) represents a *rotation*, then the matrix A is *traceless* and *anti-symmetric*.

1. Traceless.

This is a direct consequence of the relations (6.11) and (6.21), *i.e.*

$$\exp(Tr(A)) = det(M) = 1 = \exp(0) \quad \text{consequently} \quad Tr(A) = 0 \ . \tag{6.22}$$

2. Anti-symmetric.

The inverse of (6.1) is equal to $M = \exp(-A)$. This follows from the fact that A commutes with $-A$ and thus, according to formulas (6.12) and (6.17), one finds

$$\exp(A)\exp(-A) = \exp(A - A) = \mathbf{1} \quad .$$

For the transposed M^T of M, using

$$[A^2]^T = (A^T)^2 \ , \quad [A^3]^T = (A^T)^3 \ , \ldots$$

we obtain:

$$M^T = \exp(A^T) \quad .$$

At this stage it is convenient to introduce a real parameter λ and define $M_\lambda = \exp(\lambda A)$. Then also:

$$[M_\lambda]^{-1} = \exp(-\lambda A) \quad \text{and} \quad [M_\lambda]^T = \exp(\lambda A^T) \quad . \tag{6.23}$$

So, if M_λ is orthogonal, we find, expanding both expressions in the above formula (6.23), the following equation:

$$\mathbf{1} - \lambda A + \cdots = [M_\lambda]^{-1} = [M_\lambda]^T = \mathbf{1} + \lambda A^T + \cdots$$

which is valid for arbitrary values of the real parameter λ and consequently yields the solution $A^T = -A$. In particular for $\lambda = 1$, we end up with:

$$[\exp(A)]^{-1} = [\exp(A)]^T \ \text{ if and only if } \ A^T = -A \quad . \tag{6.24}$$

In example 1 (see formula 6.2) A is anti-symmetric and therefor M orthogonal.

6.4 Unitary transformations.

Complex $n \times n$ matrices $M = \exp(A)$ which satisfy the property:

$$M^\dagger = M^{-1} \quad , \tag{6.25}$$

are said to be unitary. Similar arguments as used above to show that for orthogonal matrices M, A is anti-symmetric (*i.e.* formula 6.24), can be used here to show that for unitary matrices M, A is anti-Hermitean, *i.e.*

$$[\exp(A)]^{-1} = [\exp(A)]^\dagger \ \text{ if and only if } \ A^\dagger = -A \quad . \tag{6.26}$$

6.5 The direct product of two matrices.

We will use the following definition for the direct product of two matrices: Let $\mathcal{A}$ represent an $n_1 \times n_1$ matrix and $\mathcal{B}$ an $n_2 \times n_2$ matrix, then for the direct product of $\mathcal{A}$ and $\mathcal{B}$ we define:

$$\mathcal{A} \otimes \mathcal{B} = \begin{pmatrix} A_{11}\mathcal{B} & \dots & A_{1n_1}\mathcal{B} \\ \vdots & & \vdots \\ A_{n_1 1}\mathcal{B} & \dots & A_{n_1 n_1}\mathcal{B} \end{pmatrix} \tag{6.27}$$

$$= \begin{pmatrix} A_{11}B_{11} & \dots & A_{11}B_{1n_2} & A_{12}B_{11} & \dots & \dots & A_{1n_1}B_{1n_2} \\ \vdots & & \vdots & \vdots & & & \vdots \\ A_{11}B_{n_2 1} & \dots & A_{11}B_{n_2 n_2} & A_{12}B_{n_2 1} & \dots & \dots & A_{1n_1}B_{n_2 n_2} \\ A_{21}B_{11} & \dots & A_{21}B_{1n_2} & A_{22}B_{11} & \dots & \dots & A_{2n_1}B_{1n_2} \\ \vdots & & \vdots & \vdots & & & \vdots \\ A_{n_1 1}B_{n_2 1} & \dots & A_{n_1 1}B_{n_2 n_2} & A_{n_1 2}B_{n_2 1} & \dots & \dots & A_{n_1 n_1}B_{n_2 n_2} \end{pmatrix} .$$

In terms of the matrix elements for the direct product of $\mathcal{A}$ and $\mathcal{B}$, one might alternatively define:

$$[\mathcal{A} \otimes \mathcal{B}]_{(k-1)n_2 + i\ ,\ (\ell - 1)n_2 + j} = A_{k\ell}\, B_{ij} \quad \begin{cases} k, \ell = 1, \dots, n_1 \\ i, j = 1, \dots, n_2 \end{cases} \tag{6.28}$$

For the product of two such matrices one has the following expression:

$$(\mathcal{A} \otimes \mathcal{B})\,(\mathcal{C} \otimes \mathcal{D}) = (\mathcal{A}\mathcal{C}) \otimes (\mathcal{B}\mathcal{D}) \quad \begin{cases} \mathcal{A}, \mathcal{C} \quad n_1 \times n_1 \quad \text{matrices} \\ \mathcal{B}, \mathcal{D} \quad n_2 \times n_2 \quad \text{matrices} \end{cases} \tag{6.29}$$

proof:

We demonstrate below that identity (6.29) holds for an arbitrary matrix element, using definition (6.27) for the direct product of two matrices, or, alternatively, definition (6.28) for the matrix elements of a direct product:

$$[(\mathcal{A} \otimes \mathcal{B})\,(\mathcal{C} \otimes \mathcal{D})]_{(a-1)n_2 + b\ ,\ (c-1)n_2 + d} =$$

$$= \sum_{n=1}^{n_1 n_2} [\mathcal{A} \otimes \mathcal{B}]_{(a-1)n_2 + b\ ,\ n}\ [\mathcal{C} \otimes \mathcal{D}]_{n\ ,\ (c-1)n_2 + d}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n_1} \sum_{i=1}^{n_2} [\mathcal{A} \otimes \mathcal{B}]_{(a-1)n_2+b\ ,\ (k-1)n_2+i} \\
&\qquad\qquad [\mathcal{C} \otimes \mathcal{D}]_{(k-1)n_2+i\ ,\ (c-1)n_2+d} \\
&= \sum_{k=1}^{n_1} \sum_{i=1}^{n_2} A_{ak} B_{bi} C_{kc} D_{id} \\
&= \left(\sum_{k=1}^{n_1} A_{ak} C_{kc} \right) \left(\sum_{i=1}^{n_2} B_{bi} D_{id} \right) \\
&= [\mathcal{A}\mathcal{C}]_{ac}\ [\mathcal{B}\mathcal{D}]_{bd} = [(\mathcal{A}\mathcal{C}) \otimes (\mathcal{B}\mathcal{D})]_{(a-1)n_2+b\ ,\ (c-1)n_2+d} \quad .
\end{aligned}
$$

Similarly, one may for the inverse of a direct product matrix show that:

$$
[(\mathcal{A} \otimes \mathcal{B})]^{-1} = \left(\mathcal{A}^{-1}\right) \otimes \left(\mathcal{B}^{-1}\right) \quad , \tag{6.30}
$$

assuming that $\mathbf{1} \otimes \mathbf{1}$ represents the identity matrix.

Chapter 7

The special orthogonal group in two dimensions.

In this chapter we study the representations of the group of rotations in two dimensions. This group is the most easy example of a group which has an infinite number of elements.

7.1 The group SO(2).

The *special orthogonal group in two dimensions*, $SO(2)$, is defined by the set of unimodular (*i.e. with unit determinant*), real and orthogonal 2×2 matrices.

In order to find the most general form of such matrices, let us assume that a, b, c and d represent four real numbers which satisfy the property that $ad - bc = 1$. With those numbers we construct a 2×2 matrix A, according to:

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{with} \quad \det(A) = ad - bc = 1. \tag{7.1}$$

When we assume that A is orthogonal, then $A^{-1} = A^T$. This leads to the following relations for its components:

$$d = a \quad \text{and} \quad c = -b. \tag{7.2}$$

Using also the condition (7.1) for the components of A, we find the following relation for a and b:

$$a^2 + b^2 = 1 \quad . \tag{7.3}$$

Now, recalling that a and b are real numbers, we obtain as a consequence of this relation that:

$$-1 \leq a \leq +1 \quad \text{and} \quad -1 \leq b \leq +1 \quad . \tag{7.4}$$

At this stage it is opportune to introduce a real parameter α such that:

$$a = \cos(\alpha) \quad \text{and} \quad b = \sin(\alpha) \quad , \tag{7.5}$$

in which case a and b satisfy simultaneously the conditions (7.3) and (7.4).

So, we obtain for a unimodular, real and orthogonal 2×2 matrix $R(\alpha)$, the following general form:

$$R(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{for} \quad \alpha \in (-\pi, +\pi] \quad . \tag{7.6}$$

When α takes values running from $-\pi$ to $+\pi$, then one obtains all possible (infinitely many!) unimodular, real and orthogonal 2×2 matrices.

Next we must establish that the matrices (7.6) form a group:

1. Product.

Normal matrix multiplication defines a suitable group product for the set of matrices (7.6), *i.e.*

$$R(\alpha_2)R(\alpha_1) = R(\alpha_1 + \alpha_2). \tag{7.7}$$

Strictly speaking might $\alpha_1 + \alpha_2$ be outside the parameter space $(-\pi, +\pi]$ when α_1 and α_2 are elements of this parameter space. But, we assume always an angle modulus 2π if necessary.

2. Associativity.

Since matrix multiplication is associative, the above defined group product is automatically endowed with the property of associativity.

3. Identity operator.

The identity operator, I, of the group product (7.7), which has the property $IR(\alpha) = R(\alpha)I = R(\alpha)$, is defined by the unit matrix:

$$I = R(\alpha = 0) = \mathbf{1} \quad . \tag{7.8}$$

4. Inverse.

For each group element $R(\alpha)$ exists an inverse group element $R(-\alpha)$ in $SO(2)$, such that:

$$R(-\alpha)R(\alpha) = R(\alpha)R(-\alpha) = \mathbf{1} \quad . \tag{7.9}$$

As a consequence of the above properties 1, 2, 3 and 4 we find that the special orthogonal transformations in two dimensions, *i.e.* $SO(2)$, form a group. This group is moreover Abelian, because of the property (7.7), *i.e.*:

$$R(\alpha_2)R(\alpha_1) = R(\alpha_1 + \alpha_2) = R(\alpha_2 + \alpha_1) = R(\alpha_1)R(\alpha_2) \quad . \tag{7.10}$$

7.2 Irreps of SO(2).

Since $SO(2)$ is an Abelian group, its irreps are one-dimensional. This property of Abelian groups has been shown for finite groups in (4.24). Consequently, the representation (7.6) of the group elements $R(\alpha)$ of $SO(2)$ is *reducible*.

Following the definition of a representation, the 2×2 matrices (7.6) might represent transformations $D(R(\alpha))$ in a complex two-dimensional vector space. It is easy to verify that the unit vectors $\hat{u}_1$ and $\hat{u}_{-1}$, given by:

$$\hat{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \hat{u}_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad ,$$

are eigenvectors of the matrices (7.6) for all values of α. The eigenvalues are respectively given by:

$$c_1(\alpha) = e^{-i\alpha} \quad \text{and} \quad c_{-1}(\alpha) = e^{+i\alpha} \quad .$$

So, at the one-dimensional subspace spanned by the unit vector $\hat{u}_1$, $R(\alpha)$ is represented by the complex number $\exp(-i\alpha)$, and at the one-dimensional subspace spanned by $\hat{u}_{-1}$, by $\exp(+i\alpha)$. The similarity transformation which transforms the matrices $D(R(\alpha))$ into a diagonal equivalent representation, is thus found to be:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} D(R(\alpha)) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} \exp(-i\alpha) & 0 \\ 0 & \exp(+i\alpha) \end{pmatrix} \quad .$$

The two resulting non-equivalent one-dimensional irreps of $SO(2)$ are given by:

$$D^{(1)}(R(\alpha)) = \exp(-i\alpha) \quad \text{and} \quad D^{(-1)}(R(\alpha)) = \exp(+i\alpha) \quad .$$

In the following we will simplify the notation for D and write $D(\alpha)$ instead of $D(R(\alpha))$. As we will see below, there are infinitely many non-equivalent irreps for $SO(2)$.

An one-dimensional irrep is a set of linear transformations $\{D(\alpha);\ \alpha \in (-\pi, +\pi]\}$ of a one-dimensional complex vector space V into itself, *i.e.*:

$$D(\alpha) \ : \ V \longrightarrow V \ .$$

When $\hat{u}$ represents the only basis vector of V, then the transformation $D(\alpha)$ is fully characterized by:

$$D(\alpha)\hat{u} = c(\alpha)\hat{u} \ , \ \text{for a complex number } \ c(\alpha) \ . \tag{7.11}$$

Moreover, representations reflect the properties of the group. When, for example, for rotations yields the property:

$$[R(\alpha)]^n = R(n\alpha) \ , \ n = 1, 2, \ldots$$

then for the representation $D(\alpha)$ must yield the same property, *i.e.*:

$$[c(\alpha)]^n \hat{u} = [D(\alpha)]^n \hat{u} = D(n\alpha)\hat{u} = c(n\alpha)\hat{u} \ . \tag{7.12}$$

The identity operator $R(\alpha = 0)$ is represented by the complex number 1, *i.e.*:

$$D(\alpha = 0) = 1 \ . \tag{7.13}$$

Next, let us study the possible representations for other values of $\alpha \neq 0$. Let us for example take $\alpha = 45°$. Using the relations (7.12) and (7.13), we obtain:

$$[c(\pi/4)]^8 = c(2\pi) = c(0) = 1 \ .$$

Consequently, for $R(\pi/4)$ one finds the following possible representations:

$$D(R(\pi/4)) = \exp\{-ik\pi/4\} \ \ , \ \ k = 0, \pm 1, \pm 2, \ldots$$

We may repeat this procedure for values of α of the form $\alpha = n\pi/m$ (where $|n| < m = 1, 2, \ldots$), in order to find that for the representations of such transformations one has the possibilities given by:

$$D(R(n\pi/m)) = \exp\{-ikn\pi/m\} \ \ , \ \ k = 0, \pm 1, \pm 2, \ldots$$

Now, an arbitrary value for α can to any degree of accuracy be approximated by $\alpha = n\pi/m$ for some integer numbers n and $m > 0$. Consequently, we are lead to the conclusion that the possible representations for $R(\alpha)$ of $SO(2)$ are given by:

$$D(R(\alpha)) = \exp\{-ik\alpha\} \ \ , \ \ k = 0, \pm 1, \pm 2, \ldots \tag{7.14}$$

Each value of k gives an irrep which is not equivalent to any of the irreps for other values of k. This can easily be shown, since for any complex number s one has:

$$s^{-1} \exp\{-ik_1\alpha\} s = \exp\{-ik_2\alpha\} \ \text{ if and only if } \ k_1 = k_2 \ .$$

Consequently, we may indicate the various one-dimensional vector spaces by an index k in order to distinguish them, as well as the corresponding irrep, according to:

$$D^{(k)}(\alpha) \ \ : \ \ V^{(k)} \longrightarrow V^{(k)} \ .$$

We might moreover indicate the only basis vector of $V^{(k)}$ by $\hat{u}_k$, in order to obtain:

$$D^{(k)}(\alpha)\hat{u}_k = e^{-ik\alpha}\hat{u}_k \ \ , \ \ k = 0, \pm 1, \pm 2, \ldots \tag{7.15}$$

The representations $D^{(k)}$ are called the *standard irreps* of $SO(2)$.

7.3 Active rotations in two dimensions.

An *active* rotation $R(\alpha)$ of a two-dimensional plane, over an angle α around the origin of the coordinate system in the plane, is defined as the linear transformation of the plane into itself which rotates each vector of the plane over an angle α. The coordinate system though, remains in its place. The linear transformation $R(\alpha)$ is fully characterized once the images, $\vec{u}\,'_1$ and $\vec{u}\,'_2$, are given of the vectors, $\vec{u}_1 = \hat{e}_1$ and $\vec{u}_2 = \hat{e}_2$, which originally are at the same positions as the two basis vectors of the coordinate system, *i.e.*:

$$\begin{aligned}\vec{u}\,'_1 &= \hat{e}_1\cos(\alpha) + \hat{e}_2\sin(\alpha)\\ &\text{and}\\ \vec{u}\,'_2 &= -\hat{e}_1\sin(\alpha) + \hat{e}_2\cos(\alpha)\ .\end{aligned} \tag{7.16}$$

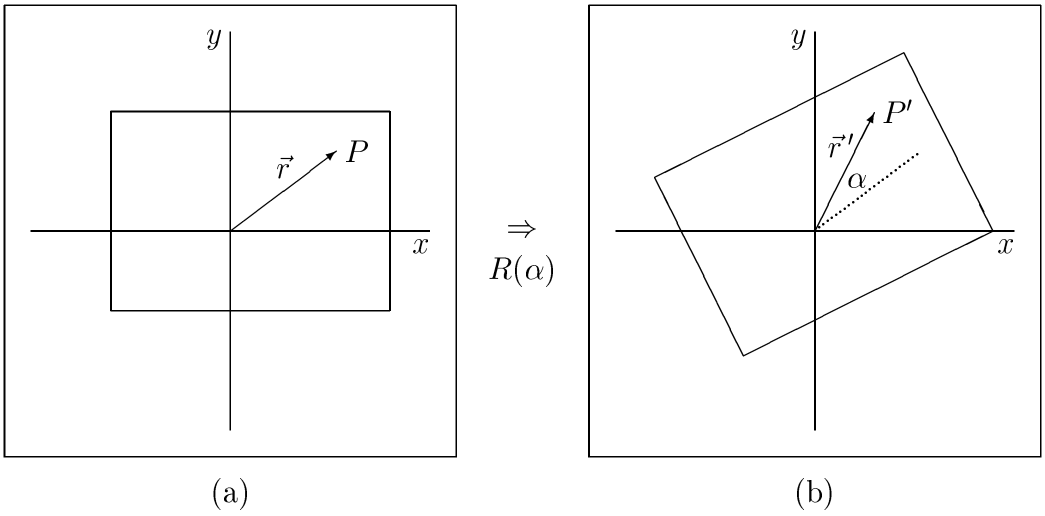


Figure 7.1: An active rotation of the plane: Physical objects are rotated around the origin; the coordinate system remains in its place. Indicated are the situations before (a) and after (b) the rotation $R(\alpha)$.

The image, $\vec{w} = w_1\hat{e}_1 + w_2\hat{e}_2$, under the rotation $R(\alpha)$ of an arbitrary vector, $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2$, is then given by:

$$\begin{aligned}\vec{w} &= v_1\vec{u}\,'_1 + v_2\vec{u}\,'_2\\ &= \{v_1\cos(\alpha) - v_2\sin(\alpha)\}\,\hat{e}_1 + \{v_1\sin(\alpha) + v_2\cos(\alpha)\}\,\hat{e}_2\ ,\end{aligned}$$

from which expression we deduce that the relation between the components (w_1, w_2) of the rotated vector and the components (v_1, v_2) of the original vector, is as follows:

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = R(\alpha)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\ . \tag{7.17}$$

Consequently, the active rotation $R(\alpha)$ is in the space of the components of two-dimensional vectors represented by the matrix (7.6). So, *an active rotation in two dimensions is a rotation of the position vectors of the plane with respect to a fixed coordinate system.* Equation (7.17) relates the components of the original vector to the components of the rotated vector in the fixed basis.

A possible way to concretize this transformation, is to imagine the x-axis and the y-axis of the coordinate system to be drawn at your desk and a sheet of paper, representing the plane, on top of it. A certain point at the sheet of paper, indicated by P in figure (7.1), is characterized by the position vector $\vec{r}$ before the rotation of the sheet. After rotating the sheet over an angle α around the origin, the same point at the sheet moves to a different position, indicated by P' in figure (7.1). The point P' is characterized by the position vector $\vec{r}\,' = R(\alpha)\vec{r}$.

Now, imagine the sheet of paper to be replaced by a sheet of metal which is heated in one corner such that the temperature at the metal sheet is a function of position, say represented by the function $T(\vec{r})$. After rotating the sheet, its temperature distribution remains the same. However, in the new situation the function which describes this distribution, is a different function of position, say $T'(\vec{r})$. But, we assume that the rotation of the sheet does not change the temperature of a given point P at the metal sheet. Consequently, one has the following relation between the functions T and T':

$$T'(R(\alpha)\vec{r}) = T(\vec{r}) \ \text{ or equivalently } \ T'(\vec{r}) = T([R(\alpha)]^{-1}\vec{r}) \ \ .$$

So, in function space (*i.e.* the space of functions $f(\vec{r})$ defined on the plane), the active rotation $R(\alpha)$ induces the transformation:

$$R(\alpha) \ : \ f \longrightarrow D(R(\alpha))f \ \ , \tag{7.18}$$

$$\text{where } \ D(R(\alpha))f(\vec{r}) = f'(\vec{r}) = f([R(\alpha)]^{-1}\vec{r}) \ \ .$$

7.4 Passive rotations in two dimensions.

A *passive* rotation $R(\alpha)$ of the plane is a rotation of the coordinate system of the plane. The vectors of the plane remain in their place, but the coordinate system rotates over an angle indicated by α. The basis vectors, $\hat{e}'_1$ and $\hat{e}'_2$, of the new coordinate system are related to the original basis vectors, $\hat{e}_1$ and $\hat{e}_2$, as follows:

$$\begin{aligned} \hat{e}'_1 &= \hat{e}_1\cos(\alpha) + \hat{e}_2\sin(\alpha) \\ &\text{and} \\ \hat{e}'_2 &= -\hat{e}_1\sin(\alpha) + \hat{e}_2\cos(\alpha) \ \ . \end{aligned} \tag{7.19}$$

which is the same expression as given in formula (7.16), but with a completely different meaning: In (7.16), the vectors $\vec{u}\,'_1$ and $\vec{u}\,'_2$ represent the images of the vectors which originally have the same positions as the basic vectors $\hat{e}_1$ and $\hat{e}_2$ of the fixed coordinate system. Here, $\hat{e}'_1$ and $\hat{e}'_2$ represent the basis vectors of the new coordinate system.

A vector $\vec{v}$ can be characterized by its components (v_1, v_2) with respect to the original basis vectors, as well as by its components (v_1', v_2') in the new coordinate system, *i.e.*:

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 = v_1'\hat{e}_1' + v_2'\hat{e}_2' \ . \tag{7.20}$$

The relation between (v_1, v_2) and (v_1', v_2'), which can be found by substituting the expressions (7.19) into formula (7.20), is given by:

$$\begin{aligned} v_1' &= v_1\cos(\alpha) + v_2\sin(\alpha) \\ &\text{and} \\ v_2' &= -v_1\sin(\alpha) + v_2\cos(\alpha) \ , \end{aligned}$$

or, using formula (7.6), in a more compact notation:

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = [R(\alpha)]^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ . \tag{7.21}$$

We find that in the case of a passive or coordinate transformation, the components v_1 and v_2, which describe the vector $\vec{v}$ in the original coordinate system, transform according to the inverse of the matrix $R(\alpha)$ into the components v_1' and v_2', which describe the same vector $\vec{v}$ in the new coordinate system $\hat{e}_1'$ and $\hat{e}_2'$. Because of this property, vectors are called *contra-variant.*

7.5 The standard irreps of SO(2).

From Fourier-analysis we know that the space of "well-behaved" functions at the interval $(-\pi, +\pi]$ has as a basis the functions:

$$\psi_m(\varphi) = e^{im\varphi} \ , \ \ m = 0, \pm 1, \pm 2, \ldots \tag{7.22}$$

Any "well-behaved" function $f(\varphi)$ ($\varphi \in (-\pi, +\pi]$) can be expanded in a linear combination of the basis (7.22), *i.e.*:

$$f(\varphi) = \sum_{m=-\infty}^{\infty} a_m\psi_m(\varphi) \ , \tag{7.23}$$

where the coefficients a_m are given by the innerproduct of the basis vector ψ_m with the "vector" f.

The innerproduct of the orthonormal set of basis vectors ψ_m of this complex vector space of "well-behaved" functions at the interval $(-\pi, +\pi]$, is defined by:

$$(\psi_m \ , \ \psi_n) = \int_{-\pi}^{+\pi} \frac{d\varphi}{2\pi}\psi_m^\star(\varphi)\psi_n(\varphi) = \int_{-\pi}^{+\pi} \frac{d\varphi}{2\pi} e^{i(n-m)\varphi} = \delta_{mn} \ . \tag{7.24}$$

Consequently, the coefficients a_m of the expansion (7.23) are given by:

$$a_m = (\psi_m \ , \ f) = \int_{-\pi}^{+\pi} \frac{d\varphi}{2\pi} \psi_m^\star(\varphi) f(\varphi) = \int_{-\pi}^{+\pi} \frac{d\varphi}{2\pi} f(\varphi) e^{-im\varphi} \quad . \tag{7.25}$$

At the Fourier basis ψ_m one might view the function f as an infinitely long column vector with components a_m, *i.e.*

$$f = \begin{pmatrix} \vdots \\ a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} \quad . \tag{7.26}$$

In the following we study the effect of a passive rotation $R(\alpha)$ in the (x, y)-plane on the space of "well-behaved" functions. The situation is shown in the figure (7.2) below.

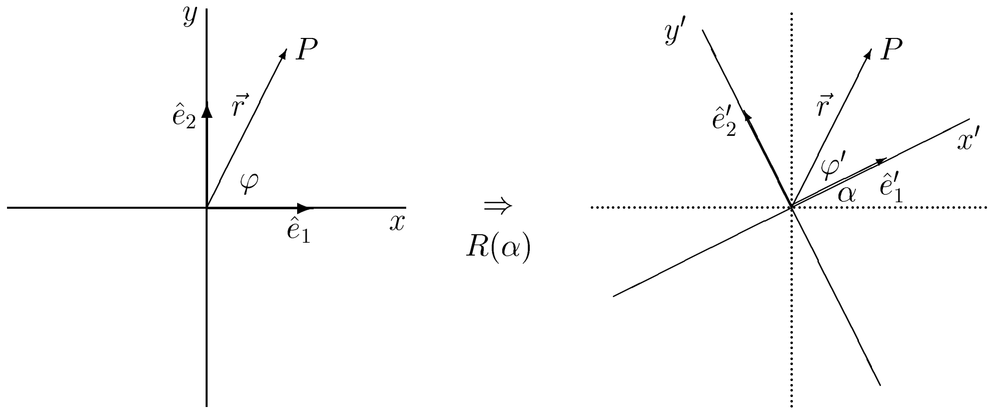


Figure 7.2: A passive rotation of the plane: The coordinate system is rotated around the origin; physical objects remain in their place.

Let us associate to each point at the unit circle in the plane a complex number such that the resulting function is "well-behaved". In the coordinate system $(\hat{e}_1, \hat{e}_2)$ this is described by a function a function $f(\varphi)$ of the parameter φ. Whereas in the coordinate system $(\hat{e}'_1, \hat{e}'_2)$ the same complex numbers are described by a function $f'(\varphi')$ of the angle φ'. Now, in a certain point P of the unit circle is the function value independent of the choice of coordinates. Consequently, there exists a relation between f and f', given by:

$$f(\varphi) = f'(\varphi') = f'(\varphi - \alpha) \quad . \tag{7.27}$$

The Fourier basis at a given coordinate system is defined in formula (7.22). The relevant variable is the azimuthal angle with respect to the basis vectors of the coordinate system. Consequently, in the unprimed coordinate system are the values of the Fourier basis in the point P (see figure 7.2) determined by:

$$\psi_m(\varphi) = e^{im\varphi} \quad , \quad m = 0, \pm 1, \pm 2, \ldots$$

and in the primed coordinate system by:

$$\psi'_m(\varphi') = e^{im\varphi'} = e^{im(\varphi - \alpha)} \quad , \quad m = 0, \pm 1, \pm 2, \ldots$$

So, we obtain the following relation between the two different Fourier bases:

$$\psi'_m(\varphi') = \psi_m(\varphi - \alpha) \ .$$

This relation defines a representation $D^{(F)}(\alpha)$ of $R(\alpha)$ in the space of "well-behaved" functions:

$$D^{(F)}(\alpha)\psi_m(\varphi) = \psi'_m(\varphi') = e^{-im\alpha}e^{im\varphi} = D^{(m)}(\alpha)\psi_m(\varphi) \ . \tag{7.28}$$

In the vector space defined by vectors of the form (7.26), the representation $D^{(F)}(\alpha)$ of $R(\alpha)$ becomes a matrix representation. The matrix $D^{(F)}(\alpha)$ might be viewed here as a diagonal matrix at the Fourier basis ψ_m, with diagonal elements $\exp\{-im\alpha\}$, *i.e.*:

$$D^{(F)}(\alpha) = \begin{pmatrix} \ddots & & & & & & \\ & e^{2i\alpha} & & & & 0 & \\ & & e^{i\alpha} & & & & \\ & & & 1 & & & \\ & & & & e^{-i\alpha} & & \\ & 0 & & & & e^{-2i\alpha} & \\ & & & & & & \ddots \end{pmatrix} \ . \tag{7.29}$$

In terms of the one-dimensional standard irreps defined in (7.15) the above matrix can be written as:

$$D^{(F)}(\alpha) = \begin{pmatrix} \ddots & & & & & & \\ & D^{(-2)}(\alpha) & & & & 0 & \\ & & D^{(-1)}(\alpha) & & & & \\ & & & D^{(0)}(\alpha) & & & \\ & & & & D^{(1)}(\alpha) & & \\ & 0 & & & & D^{(2)}(\alpha) & \\ & & & & & & \ddots \end{pmatrix} \ . \tag{7.30}$$

As a consequence we find that the representation $D^{(F)}$ is completely reducible into a direct sum the standard irreps $D^{(m)}$. For example, consider the one-dimensional subspace of "well-behaved" functions, which have the form:

$$f(\varphi) = a_1\psi_1(\varphi),$$

where a_1 may take any complex value. At this subspace is $D^F(\alpha)$ given by:

$$D^F(\alpha)f(\varphi) = D^F(\alpha)\begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ a_1 \\ 0 \\ \vdots \end{pmatrix}(\varphi) = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ D^{(1)}(\alpha)a_1 \\ 0 \\ \vdots \end{pmatrix}(\varphi) = D^{(1)}(\alpha))f(\varphi) \ . \quad (7.31)$$

The Fourier components of the function $f(\varphi)$ are given by a_m in the unprimed coordinate system and by a'_m in the primed system, *i.e.*:

$$f(\varphi) = \sum_{m=-\infty}^{\infty} a_m\psi_m(\varphi) \quad \text{and} \quad f'(\varphi') = \sum_{m=-\infty}^{\infty} a'_m\psi'_m(\varphi') \ . \qquad (7.32)$$

So, using (7.27), (7.22) and (7.28), we obtain:

$$\begin{aligned} \sum_{m=-\infty}^{\infty} a'_m\psi'_m(\varphi') &= f'(\varphi') = f(\varphi) = \sum_{m=-\infty}^{\infty} a_m\psi_m(\varphi) \\ &= \sum_{m=-\infty}^{\infty} a_m e^{im\varphi} = \sum_{m=-\infty}^{\infty} a_m e^{im\alpha} e^{im(\varphi-\alpha)} \\ &= \sum_{m=-\infty}^{\infty} \left(a_m e^{im\alpha}\right)\psi'_m(\varphi') \qquad (7.33) \end{aligned}$$

From which equality, using (7.29) and (7.30), we may conclude that:

$$a'_m = a_m e^{im\alpha} = \left\{\left[D^{(F)}(\alpha)\right]^{-1}\right\}_{mm} a_m \ . \qquad (7.34)$$

As a result, we find that in the case of a passive rotation $R(\alpha)$, the Fourier basis transforms with $D^{(F)}(\alpha)$ given in (7.29) and the Fourier components of a "well-behaved" function with the inverse of that matrix. It is in a way similar to the transformation of the basis vectors of the coordinate system as given in formula (7.19) in relation to the transformation of the components of a contra-variant vector as shown in formula (7.21).

7.6 The generator of $SO(2)$.

The group elements of $SO(2)$ can be characterized by one parameter, the rotation angle; that is that all possible rotations in two dimensions are given by:

$$R(\alpha) \quad , \quad -\pi < \alpha \leq +\pi. \tag{7.35}$$

In the neighbourhood of the unit operation ($\alpha = 0$), the above defined matrices form a one dimensional continuous matrix field. So, we might define the matrix A which is given by the derivative of the matrix field at $\alpha = 0$, as follows:

$$A = \left.\frac{d}{d\alpha}R(\alpha)\right|_{\alpha = 0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.36}$$

The matrix A is called the *generator* of rotations in two dimensions. In the following it will become clear why: For angles different from zero one has similarly:

$$\begin{aligned} \frac{d}{d\alpha}R(\alpha) &= \begin{pmatrix} -\sin(\alpha) & -\cos(\alpha) \\ \cos(\alpha) & -\sin(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \\ &= A\, R(\alpha), \end{aligned}$$

which differential equation can be solved by:

$$R(\alpha) = \exp\{\alpha A\}. \tag{7.37}$$

The expansion of the exponent for the matrix A of formula (7.36) is shown in detail in (6.2) and it confirms the above equality.

Because of the above representation of a rotation in terms of a parameter, α, and the matrix A, it is that this matrix is called the generator of rotations in two dimensions.

A representation $D(\alpha)$ for the group elements $R(\alpha)$ of $SO(2)$, can be translated into a representation $d(A)$ of the generator A of the group, according to:

$$D(\alpha) = \exp\{\alpha d(A)\} \quad . \tag{7.38}$$

In the cae of the standard irreps (7.15) of $SO(2)$, we find for the representations $d^{(k)}(A)$ of the generator A of $SO(2)$ the result:

$$d^{(k)}(A) = -ik \quad , \quad k = 0, \pm 1, \pm 2, \ldots \tag{7.39}$$

7.7 A differential operator for the generator.

In the space of "well-behaved" functions $f(\vec{r}) = f(x, y)$ of the two real parameters x and y, we found in formula (7.18) the following representation for an active rotation $R(\alpha)$ of the parameter space (*i.e.* the (x, y)-plane):

$$D(\alpha)f\left(\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}\right) = f\left([R(\alpha)]^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) \quad . \tag{7.40}$$

The lefthand side of this relation can be rewritten in terms of the corresponding representation $d(A)$ of the generator of $SO(2)$, using the definition given in (7.38), *i.e.*:

$$D(\alpha)f(\vec{r}) = \exp\{\alpha d(A)\}f(\vec{r}) \quad . \tag{7.41}$$

Moreover, might the righthand side of equation (7.40) be expanded in a Taylor series of the form:

$$\begin{aligned} f\left(\vec{r}+\left(\begin{array}{c}\Delta x\\ \Delta y\end{array}\right)\right) &= f(\vec{r}) + \left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)f(\vec{r}) + \\ &+ \frac{1}{2}\left((\Delta x)^2\frac{\partial^2}{\partial x^2} + 2\Delta x\Delta y\frac{\partial^2}{\partial x\partial y} + (\Delta y)^2\frac{\partial^2}{\partial y^2}\right)f(\vec{r}) + \cdots \end{aligned} \tag{7.42}$$

Let us expand the series (7.41) and (7.42) in α. For (7.41) we obtain:

$$D(\alpha)f(\vec{r}) = [1 + \alpha d(A) + \frac{\alpha^2}{2}\{d(A)\}^2 + \cdots]f(\vec{r}) \quad . \tag{7.43}$$

For (7.42) we need the expansions for Δx and Δy in α, *i.e.*:

$$\left(\begin{array}{c}\Delta x\\ \Delta y\end{array}\right) = R(-\alpha)\vec{r} - \vec{r} = \left(\begin{array}{c}\alpha y - \frac{\alpha^2}{2}x + \cdots\\ \\ -\alpha x - \frac{\alpha^2}{2}y + \cdots\end{array}\right) \quad . \tag{7.44}$$

Consequently, for (7.42) we obtain to first order in α, the following:

$$f([R(\alpha)]^{-1}\vec{r}) = f(\vec{r}) + \alpha\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)f(\vec{r}) \quad . \tag{7.45}$$

So, when we compare formula (7.43) with formula (7.45), then we find for the representation $d(A)$ of the generator A of $SO(2)$ in the space of "well-behaved" functions $f(x,y)$, to first order in α, the following differential operator:

$$d(A) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \quad . \tag{7.46}$$

Now, one might like to inspect the higher order terms in the expansions of the formulas (7.41) and (7.42). For (7.42), the second order in α term equals:

$$\frac{\alpha^2}{2}\left(-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial x^2} - 2xy\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial y^2}\right)f(\vec{r}) = \frac{\alpha^2}{2}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)^2 f(\vec{r}) \quad . \tag{7.47}$$

So, in comparing formula (7.43) with formula (7.47), we find that also to second order in α, the solution is given by the differential operator (7.46). In fact, this result is consistent with expansions up to any order in α.

Moreover, when we introduce the azimuthal angle φ in the (x,y)-plane, according to:

$$x = |\vec{r}|\cos(\varphi) \quad \text{and} \quad y = |\vec{r}|\sin(\varphi) \quad ,$$

then we get for $d(A)$, using formula (7.46), the differential operator given by:

$$d(A) = -\frac{\partial}{\partial\varphi} \quad . \tag{7.48}$$

Because of this result, $\partial/\partial\varphi$ is sometimes refered to as the generator of rotations in the (x, y)-plane.

It is easy to demonstrate that relation (7.48) is the correct representation $d(A)$ in function space. The modulus of $\vec{r}$ remains invariant under a rotation, so the only relevant variable of $f(\vec{r})$ is the azimuthal angle φ. Consequently, instead of formula (7.40), one might take the more simple expression:

$$D(\alpha)f(\vec{r}) = f(\varphi - \alpha) = f(\vec{r}) - \alpha\frac{\partial}{\partial\varphi}f(\vec{r}) + \frac{\alpha^2}{2}\left(\frac{\partial}{\partial\varphi}\right)^2 f(\vec{r}) + \cdots$$

Comparing this expansion with (7.43), we are unambiguously lead to the identification (7.48).

7.8 The generator for physicists.

Physicists prefer operators like x and $-i\partial/\partial x$. So, instead of the operator $d(A)$ of formula (7.46), they prefer:

$$-ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} \quad .$$

In order to suit that need, we take for the generator of $SO(2)$ the operator L, defined by:

$$L = iA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad . \tag{7.49}$$

The group elements $R(\alpha)$ of $SO(2)$ can then be written in the form:

$$R(\alpha) = exp\{-i\alpha L\} \quad . \tag{7.50}$$

And, using the expressions (7.46) and (7.48), the representation $d(L)$ of the generator L turns out to be:

$$d(L) = -ix\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} = -i\frac{\partial}{\partial \varphi} \quad . \tag{7.51}$$

Clearly, there exists no essential difference between the generator L and the generator A. It is just a matter of taste. However, notice that whereas A of formula (7.36) is anti-symmetric, L of formula (7.49) is Hermitean. Both, A and L are traceless, since the group elements $R(\alpha)$ of $SO(2)$ are unimodular (see formula 6.22).

In the case of the standard irreps (7.39), we find for the representations of $d^{(k)}(L)$ of the generator L the form:

$$d^{(k)}(L) = k \quad , \quad k = 0, \pm 1, \pm 2, \ldots \tag{7.52}$$

7.9 The Lie-algebra.

The special orthogonal group in two dimensions, $SO(2)$, is generated by the operator L. This means that each element $R(\alpha)$ of the group can be written in the form $R(\alpha) = \exp(-i\alpha L)$. The reason that we only need one generator for the group $SO(2)$ is the fact that all group elements $R(\alpha)$ can be parametrized by one parameter α, representing the rotation angle.

Since α is a continuous parameter we may define the derivative of of the matrix field $R(\alpha)$ with respect to α. When this derivative and all higher derivatives exist, then a group is called a *Lie-group*. $SO(2)$ is a Lie-group.

The operator L spans an *algebra*, which is a vector space endowed with a product. A vector space is clear, because linear combinations, like $\alpha_1 L + \alpha_2 L$, generate group elements, like $R(\alpha_1 + \alpha_2)$. However the discussion on the precise details of the product will be postponed to the next chapter.

Since this algebra generates a Lie-group it is called a Lie-algbra.

Chapter 8

The orthogonal group in two dimensions.

In this chapter we study the representations of the group of orthogonal transformations in two dimensions. But, before doing so, let us first recapitulate what we have learned so far.

8.1 Representations and the reduction procedure

Since, as we concluded from property (7.10) for its group elements, $SO(2)$ is an Abelian group, its irreps are one-dimensional. This property of Abelian groups has been shown for finite groups in (4.24). The arguments are, loosely speaking, as follows:

The equivalence classes of an Abelian group consist all of only one group element. Hence, there are as many equivalence classes as group elements. Moreover, the number of non-equivalent irreducible representations of any group equals the number of equivalence classes of the group. Consequently, in the case of an Abelian group one has as many non-equivalent irreducible representations as there are group elements in the group.

Furthermore, one has that the sum of the square of the dimensions of the non-equivalent irreducible representations also equals the number of group elements, as is shown in formula (4.10). This has for an Abelian group then as a consequence that those dimensions can only be equal to one.

So, for any irreducible representation $D^{(\text{irrep})}$, a group element $R(\alpha)$ of $SO(2)$ is represented by a transformation of the complex numbers into the complex numbers, *i.e.*

$$D^{(\text{irrep})}(R(\alpha)) \quad : \quad \mathcal{C} \longrightarrow \mathcal{C} \quad . \tag{8.1}$$

An arbitrary representation of $SO(2)$ in a vector space of a higher dimension can be reduced to one-dimensional irreducible representations at the basis of invariant one-dimensional subspaces. We have discussed the procedure for the defining representation of $SO(2)$ in section (7.2) and for the space of complex functions of one variable at the interval $(-\pi, +\pi)$ at the Fourier basis in section (7.5), culminating in formula (7.30).

8.2 The group O(2)

The *orthogonal group in two dimensions*, $O(2)$, is defined by the set of real and orthogonal (*i.e.* $A^{-1} = A^T$) 2×2 matrices. For the determinant of such matrices one has the following relation

$$\begin{aligned}
[\det(A)]^2 &= \det(A)\det(A) = \det(A)\det\left(A^T\right) = \det(A)\det\left(A^{-1}\right) = \\
&= \det\left(AA^{-1}\right) = \det\left(\mathbf{1}\right) = 1 \quad ,
\end{aligned}$$

which leads to

$$\det(A) = \pm 1 \quad . \tag{8.2}$$

For the unimodular, real and orthogonal 2×2 matrices, $R(\alpha)$, we found in section (7.1) the general form

$$R(\alpha) = \left(\begin{array}{cc} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{array}\right) \quad \text{for} \quad \alpha \in (-\pi, +\pi] \quad . \tag{8.3}$$

Those transformations of the two-dimensional plane onto itself are called *proper rotations*.

Following a similar reasoning as given in section (7.1), we find for the real and orthogonal 2×2 matrices with determinant -1, $\bar{R}(\beta)$, the general form

$$\bar{R}(\beta) = \left(\begin{array}{cc} \sin(\beta) & \cos(\beta) \\ \cos(\beta) & -\sin(\beta) \end{array}\right) \quad \text{for} \quad \beta \in (-\pi, +\pi] \quad . \tag{8.4}$$

The latter type of matrices, which are referred to as the *improper rotations*, might also be written in the following form

$$\bar{R}(\beta) \ = \ PR(\beta) \quad \text{with} \quad P \ = \ \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \quad . \tag{8.5}$$

From expression (8.3) one learns moreover that the proper rotations form a subgroup of $O(2)$, which is isomorphous to $SO(2)$. Hence representations of $O(2)$ are also representations of $SO(2)$. But this does not imply that irreducible representations of $O(2)$ are irreducible representations of $SO(2)$. The latter statement can easily be verified for the defining representation of $O(2)$, given in formulas (8.3) and (8.4), since the unit vectors $\hat{u}_1$ and $\hat{u}_{-1}$, given by:

$$\hat{u}_1 = \frac{1}{\sqrt{2}}\left(\begin{array}{c} 1 \\ i \end{array}\right) \quad \text{and} \quad \hat{u}_{-1} = \frac{1}{\sqrt{2}}\left(\begin{array}{c} i \\ 1 \end{array}\right) \quad ,$$

which are eigenvectors of the matrices (8.3) for all values of α, as discussed in section (7.2), are not eigenvectors of the matrices (8.4). Consequently, the two-dimensional vector space has no invariant subspaces under the defining representation of $O(2)$, which representation is hence irreducible. However, its restriction to the subgroup $SO(2)$ is reducible as we have seen in section (7.2).

The group product is conveniently studied once the square of the operation P, defined in formula (8.5), is known and the product of $R(\alpha)$ and P is expressed in terms of proper and improper rotations. Hence, we determine

$$P^2 = \mathbf{1} \quad \text{and} \quad R(\alpha)P = \begin{pmatrix} -\sin(\alpha) & \cos(\alpha) \\ \cos(\alpha) & \sin(\alpha) \end{pmatrix} = \bar{R}(-\alpha) \quad . \tag{8.6}$$

Using the definition (8.5) and the result (8.6), one may easily deduce that all possible products of group elements of $O(2)$ yield group elements of $O(2)$, *i.e.*

$$\begin{aligned} &\text{(a)} \quad R(\alpha)R(\beta) = R(\alpha+\beta) \quad , \\ &\text{(b)} \quad R(\alpha)\bar{R}(\beta) = \bar{R}(-\alpha+\beta) \quad , \\ &\text{(c)} \quad \bar{R}(\alpha)R(\beta) = \bar{R}(\alpha+\beta) \quad , \quad \text{and} \\ &\text{(d)} \quad \bar{R}(\alpha)\bar{R}(\beta) = R(-\alpha+\beta) \quad . \end{aligned} \tag{8.7}$$

The group structure follows from formula (8.7). In particular we find for the inverses of the two types of group elements

$$\begin{aligned} &\text{(a)} \quad [R(\alpha)]^{-1} = R(-\alpha) \quad , \quad \text{and} \\ &\text{(b)} \quad \left[\bar{R}(\alpha)\right]^{-1} = \bar{R}(\alpha) \quad . \end{aligned} \tag{8.8}$$

Furthermore, we obtain the following equivalence relations for proper and improper rotations.

$$\begin{aligned} &\text{(a)} \quad R(\beta)R(\alpha)R(-\beta) = R(\alpha) \quad , \\ &\text{(b)} \quad \bar{R}(\beta)R(\alpha)\bar{R}(\beta) = R(-\alpha) \quad , \\ &\text{(c)} \quad R(\beta)\bar{R}(\alpha)R(-\beta) = \bar{R}(\alpha-2\beta) \quad , \quad \text{and} \\ &\text{(d)} \quad \bar{R}(\beta)\bar{R}(\alpha)\bar{R}(\beta) = \bar{R}(-\alpha+2\beta) \quad . \end{aligned} \tag{8.9}$$

From relations (8.9c and d) we conclude that all improper rotations are equivalent and from relation (8.9b) that all proper rotations are pairwise equivalent, except for the unit element. Hence we obtain the following equivalence classes:

$$\begin{aligned} &\text{(type a)} \quad \{R(0) = \mathbf{1}\} \quad , \\ &\text{(type b)} \quad \{R(\alpha)\ ,\ R(-\alpha)\} \quad , \quad \text{for each } \alpha \in (0, +\pi] \quad , \quad \text{and} \\ &\text{(type c)} \quad \left\{\bar{R}(\beta)\ ,\ \beta \in (-\pi, +\pi]\ \right\} \quad . \end{aligned} \tag{8.10}$$

There is an infinity of type (b) equivalence classes, but half as many as group elements in $SO(2)$, which subgroup itself contains half as much group elements as

$O(2)$, as can be seen from the construction given in formulas (8.3) and (8.4). The other two equivalence classes are related to the trivial and the almost trivial irreducible representations. Ignoring those two equivalence classes, we come to the conclusion that there are four times more group elements in $O(2)$ than equivalence classes and hence than non-equivalent irreducible representations. As a consequence all irreps of $O(2)$ must be two-dimensional, with the exception of the two already mentioned. The defining representation of $O(2)$, given in formulas (8.3) and (8.4), is an example of such representation.

8.3 Irreps of O(2).

An arbitrary representation, D, of $O(2)$ has been defined in the foregoing as a set of unitary transformations of a vector space, V, into itself, *i.e.*

$$D(g) \;:\; V \longrightarrow V \quad .$$

For the vector space V one might select the space of functions of one variable at the interval $(-\pi, +\pi)$, as in chapter (7). Now, in section (8.2) we have come to the conclusion that D also forms a representation when restricted to the subgroup of proper rotations, which is isomorphous to $SO(2)$, albeit possibly not irreducible. Moreover, in chapter (7) we have studied the irreducible representations of $SO(2)$, hence we know that we may select a basis, $\{\ldots\psi_{-2}, \psi_{-1}, \psi_0, \psi_1, \psi_2, \ldots\}$, in the vector space V at which the representation of a proper rotation, $D\left(R(\alpha)\right)$ acts as follows:

$$D\left(R(\alpha)\right)\psi_m \;=\; e^{-im\alpha}\psi_m \quad . \tag{8.11}$$

Consequently, once the action of the representation of the improper rotation P, defined in formula (8.5), is known, the whole representation is characterized. For that purpose we define

$$|\varphi\rangle \;=\; D\left(P\right)\psi_m \quad . \tag{8.12}$$

We want to determine the relation of $|\varphi\rangle$ with the elements of the basis of the vector space V. Hence, also using formula (8.11), we do

$$\begin{aligned} D\left(R(\alpha)\right)|\varphi\rangle \;&=\; D\left(R(\alpha)\right)D\left(P\right)\psi_m \;=\; D\left(R(\alpha)P\right)\psi_m \;= \\ &=\; D\left(PR(-\alpha)\right)\psi_m \;=\; D\left(P\right)D\left(R(-\alpha)\right)\psi_m \;= \\ &=\; D\left(P\right)e^{im\alpha}\psi_m \;=\; e^{im\alpha}D\left(P\right)\psi_m \;=\; e^{im\alpha}|\varphi\rangle \quad , \end{aligned}$$

from which we conclude that $|\varphi\rangle$ is proportional to ψ_{-m}, *i.e.*

$$D\left(P\right)\psi_m \;=\; |\varphi\rangle \;=\; a_m\psi_{-m} \quad . \tag{8.13}$$

Notice that because of this result one has two-dimensional invariant subspaces of the vector space V which are spanned by the basis elements ψ_{-m} and ψ_m, except for the cases where $m = 0$. Consequently, when one organizes the basis of V in the following order:

$$\{\psi_0,\ \psi_{-1}, \psi_{+1},\ \psi_{-2}, \psi_{+2},\ \psi_{-3}, \psi_{+3}, \ldots\} \quad , \tag{8.14}$$

then $D(g)$ takes the form

$$D(g) = \begin{pmatrix} D^{(0)}(g) & & & & \\ & D^{(1)}(g) & & 0 & \\ & & D^{(2)}(g) & & \\ & 0 & & D^{(3)}(g) & \\ & & & & \ddots \end{pmatrix} \quad , \tag{8.15}$$

where $D^{(0)}$ is a number and where $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, ... are 2×2 matrices, for all elements g of $SO(2)$.

What is left now, is to figure out the value of the constant of proportionality a_m of formula (8.13). First we settle that its absolute value must equal one, since we want an unitary representation, *i.e.*

$$a_m \; = \; e^{i\alpha_m} \quad .$$

Next, also using formulas (8.6) and (8.13), we determine

$$\begin{aligned} \psi_m \; &= \; \mathbf{1}\psi_m \; = \; D(\mathbf{1})\,\psi_m \; = \; D\left(P^2\right)\psi_m \; = \; D(P)\,D(P)\,\psi_m \; = \\ &= \; D(P)\,e^{i\alpha_m}\psi_{-m} \; = \; e^{i\alpha_m}D(P)\,\psi_{-m} \; = \; e^{i\alpha_m}e^{i\alpha_{-m}}\psi_m \quad , \end{aligned}$$

from which we conclude that

$$\alpha_{-m} + \alpha_m \; = \; 0 \;,\; \pm 2\pi \;,\; \pm 4\pi \;,\; \ldots \quad . \tag{8.16}$$

For $m \neq 0$ one might even get rid of the whole phase factor α_m, by choosing an equivalent representation for which the new bases of the two-dimensional subspaces are given by

$$|m\rangle \; = \; \psi_m \quad \text{and} \quad |-m\rangle \; = \; e^{i\alpha_m}\psi_{-m} \quad , \tag{8.17}$$

because then one has

$$\begin{aligned} D(P)\,|m\rangle \; &= \; D(P)\,\psi_m e^{i\alpha_m}\psi_{-m} \; = \; |-m\rangle \\ \text{and} \quad D(P)\,|-m\rangle \; &= \; D(P)\,e^{i\alpha_m}\psi_{-m} \; = \; e^{i\alpha_m}e^{i\alpha_{-m}}\psi_m \; = \; |m\rangle \quad . \end{aligned}$$

We thus find that, besides the two trivial representations, the irreducible representations of $SO(2)$ are two-dimensional and characterized by positive integers $m = 1, 2, 3, \ldots$. The representations of the proper and the improper rotations are given by

$$
\begin{aligned}
D^{(m)}\left(R(\alpha)\right)|m\rangle &= e^{-im\alpha}|m\rangle \quad , \\
D^{(m)}\left(R(\alpha)\right)|-m\rangle &= e^{im\alpha}|-m\rangle \quad , \\
D^{(m)}\left(\bar{R}(\beta)\right)|m\rangle &= e^{-im\beta}|-m\rangle \quad , \\
D^{(m)}\left(\bar{R}(\beta)\right)|-m\rangle &= e^{im\beta}|m\rangle \quad .
\end{aligned} \tag{8.18}
$$

The two trivial irreducible representations are given by

$$
D^{(1)}(R) = D^{(1)}(\bar{R}) = 1 \quad \text{and} \quad D^{(1')}(R) = -D^{(1')}(\bar{R}) = 1 \quad . \tag{8.19}
$$

Chapter 9

The special orthogonal group in three dimensions.

In this chapter we will study the group of special orthogonal 3×3 matrices, $SO(3)$. Those matrices represent rotations in three dimensions as we have seen in section (6.3) at page 73.

9.1 The rotation group $SO(3)$.

Similar to the group of rotations around the origin in two dimensions, we have the group of rotations around the origin in three dimensions. An important difference with rotations in two dimensions is that in three dimensions rotations do not commute. Consequently, the rotation group in three dimensions is not Abelian. The three rotations around the principal axes of the orthogonal coordinate system (x, y, z) are given by:

$$R(\hat{x}, \alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad R(\hat{y}, \vartheta) = \begin{pmatrix} \cos(\vartheta) & 0 & \sin(\vartheta) \\ 0 & 1 & 0 \\ -\sin(\vartheta) & 0 & \cos(\vartheta) \end{pmatrix},$$

$$\text{and} \quad R(\hat{z}, \varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9.1}$$

Those matrices are unimodular (*i.e.* have unit determinant) and orthogonal.

As an example that rotations in general do not commute, let us take a rotation of 90° around the x-axis and a rotation of 90° around the y-axis. It is, using the above definitions (9.1), easy to show that:

$$R(\hat{x}, 90^\circ)R(\hat{y}, 90^\circ) \neq R(\hat{y}, 90^\circ)R(\hat{x}, 90^\circ) \quad . \tag{9.2}$$

An arbitrary rotation can be characterized in various different ways. One way is as follows: Let $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ represent the orthonormal basis vectors of the coordinate system and let $\vec{u}_1 = \hat{e}_1$, $\vec{u}_2 = \hat{e}_2$ and $\vec{u}_3 = \hat{e}_3$ be three vectors in three dimensions which before the rotation R are at the positions of the three basis vectors. The images of the three vectors are after an active rotation R given by:

$$\vec{u}_i' = R\,\vec{u}_i \quad \text{for} \quad i = 1, 2, 3.$$

The rotation matrix for R at the above defined basis $\hat{e}_i$ ($i = 1, 2, 3$), consists then of the components of those image vectors, *i.e.*

$$R = \begin{pmatrix} (\vec{u}_1')_1 & (\vec{u}_2')_1 & (\vec{u}_3')_1 \\ \\ (\vec{u}_1')_2 & (\vec{u}_2')_2 & (\vec{u}_3')_2 \\ \\ (\vec{u}_1')_3 & (\vec{u}_2')_3 & (\vec{u}_3')_3 \end{pmatrix} \quad . \tag{9.3}$$

A second way to characterize a rotation is by means of its rotation axis, which is the one-dimensional subspace of the three dimensional space which remains invariant under the rotation, and by its rotation angle.

In both cases are three free parameters involved: The direction of the rotation axis in the second case, needs two parameters and the rotation angle gives the third. In the case of the rotation R of formula (9.3) we have nine different matrix elements. Now, if $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ form a righthanded set of unit vectors, *i.e.* $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, then consequently form the rotated vectors $\vec{u}_1'$, $\vec{u}_2'$ and $\vec{u}_3'$ also a righthanded orthonormal system, *i.e.* $\vec{u}_3' = \vec{u}_1' \times \vec{u}_2'$. This leaves us with the six components of $\vec{u}_1'$ and $\vec{u}_2'$ as parameters. But there are three more conditions, $|\vec{u}_1'| = |\vec{u}_2'| = 1$ and $\vec{u}_1' \cdot \vec{u}_2' = 0$. So, only three of the nine components of R are free.

The matrix R of formula (9.3) is unimodular and orthogonal. In order to proof those properties of R, we first introduce, for now and for later use, the Levi-Civita tensor ϵ_{ijk}, given by:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for} \quad ijk = \ 123,\ 312 \ \text{and} \ 231. \\ -1 & \text{for} \quad ijk = \ 132,\ 213 \ \text{and} \ 321. \\ \ \ 0 & \text{for} \quad \text{all other combinations.} \end{cases} \tag{9.4}$$

This tensor has the following properties:

(i) For symmetric permutations of the indices:

$$\epsilon_{jki} = \epsilon_{kij} = \epsilon_{ijk}. \tag{9.5}$$

(ii) For antisymmetric permutations of indices:

$$\epsilon_{ikj} = \epsilon_{jik} = \epsilon_{kji} = -\epsilon_{ijk}. \tag{9.6}$$

Now, using the definition of the Levi-Civita tensor and the fact that the rotated vectors $\vec{u}_1'$, $\vec{u}_2'$ and $\vec{u}_3'$ form a righthanded orthonormal system, we find for the determinant of the rotation matrix R of formula (9.3) the following:

$$\begin{aligned} \det(R) &= \epsilon_{ijk} R_{i1} R_{j2} R_{k3} = \epsilon_{jik} (\vec{u}_1')_i (\vec{u}_2')_j (\vec{u}_3')_k \\ \\ &= (\vec{u}_1' \times \vec{u}_2')_k (\vec{u}_3')_k = \vec{u}_3' \cdot \vec{u}_3' = 1 \quad . \end{aligned}$$

And for the transposed of the rotation matrix R we obtain moreover:

$$\left(R^T R\right)_{ij} = \left(R^T\right)_{ik} R_{kj} = R_{ki} R_{kj} = (\vec{u}_i')_k (\vec{u}_j')_k = \delta_{ij} \quad .$$

Consequently, the rotation matrix R is unimodular and orthogonal.

9.2 The Euler angles.

So, rotations in three dimensions are characterized by three parameters. Here we consider the rotation which rotates a point $\vec{a}$, defined by:

$$\vec{a} = (\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta)) \,, \tag{9.7}$$

to the position $\vec{b}$, defined by

$$\vec{b} = (\sin(\vartheta')\cos(\varphi'), \sin(\vartheta')\sin(\varphi'), \cos(\vartheta')) \,. \tag{9.8}$$

Notice that there exists various different rotations which perform this operation. Here, we just select one. Using the definitions (9.1), (9.7) and (9.8), it is not very difficult to show that:

$$R(\hat{y}, -\vartheta)R(\hat{z}, -\varphi)\vec{a} = \hat{z}, \;\; R(\hat{z}, \varphi')R(\hat{y}, \vartheta')\hat{z} = \vec{b} \;\text{ and }\; R(\hat{y}, \vartheta')R(\hat{y}, -\vartheta) = R(\hat{y}, \vartheta' - \vartheta).$$

As a consequence of this results, we may conclude that a possible rotation which transforms $\vec{a}$ (9.7) into $\vec{b}$ (9.8), is given by:

$$R(\varphi', \vartheta' - \vartheta, \varphi) = R(\hat{z}, \varphi')R(\hat{y}, \vartheta' - \vartheta)R(\hat{z}, -\varphi). \tag{9.9}$$

This parametrization of an arbitrary rotation in three dimensions is due to Euler. The three independent angles φ', $\vartheta' - \vartheta$ and φ are called the *Euler angles*.

9.3 The generators.

A second parametrization involves the *generators* of rotations in three dimensions, as, similar to the matrix A (7.36) for two dimensions, are called the following three matrices which result from the three basic rotations defined in (9.1):

$$A_1 = \frac{d}{d\alpha} R(\hat{x}, \alpha)\bigg|_{\alpha = 0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \quad A_2 = \frac{d}{d\vartheta} R(\hat{y}, \vartheta)\bigg|_{\vartheta = 0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad A_3 = \frac{d}{d\varphi} R(\hat{z}, \varphi)\bigg|_{\varphi = 0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{9.10}$$

In terms of the Levi-Civita tensor, defined in formula (9.4), we can express the matrix representation (9.10) for the generators of $SO(3)$, by:

$$(A_i)_{jk} = -\epsilon_{ijk} . \tag{9.11}$$

The introduction of the Levi-Civita tensor is very useful for the various derivations in the following, since it allows a compact way of formulating matrix multiplications, as we will see. However, one more property of this tensor should be given here, *i.e.* the contraction of one index in the product of two Levi-Civita tensors:

$$\begin{aligned}\epsilon_{ijk}\epsilon_{ilm} &= \epsilon_{1jk}\epsilon_{1lm}+\epsilon_{2jk}\epsilon_{2lm}+\epsilon_{3jk}\epsilon_{3lm} \\ &= \delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}.\end{aligned} \tag{9.12}$$

Equiped with this knowledge, let us determine the commutator of two generators (9.10), using the above properties (9.5), (9.6) and (9.12). First we concentrate on one matrix element (9.11) of the commutator:

$$\begin{aligned}\{[A_i,A_j]\}_{kl} &= (A_iA_j)_{kl}-(A_jA_i)_{kl}=(A_i)_{km}(A_j)_{ml}-(A_j)_{km}(A_i)_{ml} \\ &= \epsilon_{ikm}\epsilon_{jml}-\epsilon_{jkm}\epsilon_{iml}=\epsilon_{mik}\epsilon_{mlj}-\epsilon_{mjk}\epsilon_{mli} \\ &= \delta_{il}\delta_{kj}-\delta_{ij}\delta_{kl}-(\delta_{jl}\delta_{ki}-\delta_{ji}\delta_{kl})=\delta_{il}\delta_{kj}-\delta_{jl}\delta_{ki} \\ &= \epsilon_{mij}\epsilon_{mlk}=-\epsilon_{ijm}\epsilon_{mkl}=\epsilon_{ijm}(A_m)_{kl} \\ &= (\epsilon_{ijm}A_m)_{kl}.\end{aligned}$$

So, for the commutator of the generators (9.10) we find:

$$[A_i,A_j]=\epsilon_{ijm}A_m. \tag{9.13}$$

Why commutation relations are important for the construction of representations of Lie-groups, will become clear in the following.

9.4 The rotation axis.

In order to determine a second parametrization of a rotation in three dimensions, we define an arbitrary vector $\vec{n}$ by:

$$\vec{n}=(n_1,n_2,n_3), \tag{9.14}$$

as well as its "innerproduct" with the three generators (9.10), given by the expression:

$$\vec{n}\cdot\vec{A}=n_iA_i=n_1A_1+n_2A_2+n_3A_3. \tag{9.15}$$

In the following we need the higher order powers of this "innerproduct". Actually, it is sufficient to determine the third power of (9.15), *i.e.*:

$$(\vec{n}\cdot\vec{A})^3 = (n_iA_i)(n_jA_j)(n_kA_k) = n_in_jn_kA_iA_jA_k.$$

We proceed by determining one matrix element of the resulting matrix. Using the above property (9.12) of the Levi-Civita tensor, we find:

$$\begin{aligned}
\{(\vec{n}\cdot\vec{A})^3\}_{ab} &= n_in_jn_k\{A_iA_jA_k\}_{ab} = n_in_jn_k(A_i)ac(A_j)_{cd}(A_k)_{db} \\
&= -n_in_jn_k\epsilon_{iac}\epsilon_{jcd}\epsilon_{kdb} = -n_in_jn_k\{\delta_{id}\delta_{aj}-\delta_{ij}\delta_{ad}\}\epsilon_{kdb} \\
&= -n_dn_an_k\epsilon_{kdb} + n^2n_k\epsilon_{kab} = 0 - n^2n_k(A_k)_{ab} \\
&= \{-n^2\vec{n}\cdot\vec{A}\}_{ab}.
\end{aligned}$$

The zero in the forelast step of the above derivation, comes from the deliberation that using the antisymmetry property (9.6) of the Levi-Civita tensor, we have the following result for the contraction of two indices with a symmetric expression:

$$\epsilon_{ijk}n_jn_k = -\epsilon_{ikj}n_jn_k = -\epsilon_{ikj}n_kn_j = -\epsilon_{ijk}n_jn_k, \tag{9.16}$$

where in the last step we used the fact that contracted indices are dummy and can consequently be represented by any symbol.

So, we have obtained for the third power of the "innerproduct" (9.15) the following:

$$(\vec{n}\cdot\vec{A})^3 = -n^2\vec{n}\cdot\vec{A}. \tag{9.17}$$

Using this relation repeatedly for the higher order powers of $\vec{n}\cdot\vec{A}$, we may also determine its exponential, *i.e.*

$$\begin{aligned}
\exp\{\vec{n}\cdot\vec{A}\} &= \mathbf{1} + \vec{n}\cdot\vec{A} + \frac{1}{2!}(\vec{n}\cdot\vec{A})^2 + \frac{1}{3!}(\vec{n}\cdot\vec{A})^3 + \frac{1}{4!}(\vec{n}\cdot\vec{A})^4 + \cdots \\
&= \mathbf{1} + \vec{n}\cdot\vec{A} + \frac{1}{2!}(\vec{n}\cdot\vec{A})^2 + \frac{1}{3!}(-n^2\vec{n}\cdot\vec{A}) + \frac{1}{4!}(-n^2(\vec{n}\cdot\vec{A})^2) + \cdots \\
&= \mathbf{1} + \{1 - \frac{n^2}{3!} + \frac{n^4}{5!} - \frac{n^6}{7!} + \cdots\}(\vec{n}\cdot\vec{A}) + \\
&\quad + \{\frac{1}{2!} - \frac{n^2}{4!} + \frac{n^4}{6!} - \frac{n^6}{8!} + \cdots\}(\vec{n}\cdot\vec{A})^2 \\
&= \mathbf{1} + \{n - \frac{n^3}{3!} + \frac{n^5}{5!} - \frac{n^7}{7!} + \cdots\}(\hat{n}\cdot\vec{A}) + \\
&\quad + \{\frac{n^2}{2!} - \frac{n^4}{4!} + \frac{n^6}{6!} - \frac{n^8}{8!} + \cdots\}(\hat{n}\cdot\vec{A})^2.
\end{aligned}$$

We recognize here the Taylor expansions for the cosine and sine functions. So, substituting these goniometric functions for their expansions, we obtain the following result:

$$\exp\{\vec{n}\cdot\vec{A}\} = \mathbf{1} + \sin(n)(\hat{n}\cdot\vec{A}) + (1-\cos(n))(\hat{n}\cdot\vec{A})^2. \tag{9.18}$$

Next, we will show that this exponential operator leaves the vector $\vec{n}$ invariant. For that purpose we proof, using formula (9.16), the following:

$$\{(\vec{n}\cdot\vec{A})\vec{n}\}_i = (\vec{n}\cdot\vec{A})_{ij}n_j = (n_k A_k)_{ij}n_j = n_k(A_k)_{ij}n_j = -n_k\epsilon_{kij}n_j = 0,$$

or equivalently:

$$(\vec{n}\cdot\vec{A})\vec{n} = 0. \tag{9.19}$$

Consequently, the exponential operator (9.18) acting at the vector $\vec{n}$, gives the following result:

$$\exp\{\vec{n}\cdot\vec{A}\}\vec{n} = \left[\mathbf{1} + \vec{n}\cdot\vec{A} + \cdots\right]\vec{n} = \mathbf{1}\vec{n} = \vec{n} \tag{9.20}$$

So, the exponential operator (9.18) leaves the vector $\vec{n}$ invariant and of course also the vectors $a\vec{n}$, where a represents an arbitrary real constant. Consequently, the axis through the vector $\vec{n}$ is invariant, which implies that it is the rotation axis when the exponential operator represents a rotation, *i.e.* when this operator represents an unimodular, orthogonal transformation. Now, the matrix $\vec{n}\cdot\vec{A}$ of formula (9.15) is explicitly given by:

$$\vec{n}\cdot\vec{A} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} , \tag{9.21}$$

which clearly is a traceless and anti-symmetric matrix. So, using the formulas (6.22) and (6.24), we are lead to the conclusion that $\exp\{\vec{n}\cdot\vec{A}\}$ is orthogonal and unimodular and thus represents a rotation.

In order to study the angle of rotation of the transformation (9.18), we introduce a pair of vectors $\vec{v}$ and $\vec{w}$ in the plane perpendicular to the rotation axis $\vec{n}$:

$$\vec{v} = \begin{pmatrix} n_2 - n_3 \\ n_3 - n_1 \\ n_1 - n_2 \end{pmatrix} \quad \text{and} \quad \vec{w} = \hat{n}\times\vec{v} = (n_1+n_2+n_3)\hat{n} - n\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \tag{9.22}$$

where n is defined by $n = \sqrt{n_1^2+n_2^2+n_3^2}$.

The vectors $\vec{v}$, $\vec{w}$ and $\vec{n}$ form an orthogonal set in three dimensions. Moreover, are the moduli of $\vec{v}$ and $\vec{w}$ equal.

Using formula (9.21), one finds that under the matrix $\hat{n}\cdot\vec{A}$ the vectors $\vec{v}$ and $\vec{w}$ transform according to:

$$(\hat{n}\cdot\vec{A})\vec{v} = \vec{w} \quad \text{and} \quad (\hat{n}\cdot\vec{A})\vec{w} = -\vec{v} \quad .$$

So, for the rotation $\exp(\vec{n}\cdot\vec{A})$ of formula (9.18) one obtains for the vectors $\vec{v}$ and $\vec{w}$ the following transformations:

$$\begin{aligned}\vec{v}\,' = \exp(\vec{n}\cdot\vec{A})\vec{v} &= \vec{v}+\sin(n)\vec{w}+(1-\cos(n))(-\vec{v}) = \vec{v}\cos(n)+\vec{w}\sin(n), \quad \text{and} \\ \vec{w}\,' = \exp(\vec{n}\cdot\vec{A})\vec{w} &= \vec{w}+\sin(n)(-\vec{v})+(1-\cos(n))(-\vec{w}) = -\vec{v}\sin(n)+\vec{w}\cos(n).\end{aligned}$$

The vectors $\vec{v}$ and $\vec{w}$ are rotated over an angle n in to the resulting vectors $\vec{v}\,'$ and $\vec{w}\,'$. This rotation is moreover in the positive sense with respect to the direction $\vec{n}$ of the rotation axis, because of the choice (9.22) for $\vec{w}$.

Notice that the case $n_1 = n_2 = n_3$, which is not covered by the choice (9.22), has to be studied separately. This is left as an exercise for the reader.

Concludingly, we may state that we found a second parametrization of a rotation around the origin in three dimensions, *i.e.*:

$$R(n_1, n_2, n_3) = \exp\{\vec{n}\cdot\vec{A}\}, \tag{9.23}$$

where the rotation angle is determined by:

$$n = \sqrt{n_1^2+n_2^2+n_3^2},$$

and where the rotation axis is indicated by the direction of $\vec{n}$.

The vector $\vec{n}$ can take any direction and its modulus can take any value. Consequently, $\exp(\vec{n}\cdot\vec{A})$ may represent any rotation and so all possible unimodular, orthogonal 3×3 matrices can be obtained by formula (9.18) once the appropriate vectors $\vec{n}$ are selected.

9.5 The Lie-algebra of $SO(3)$.

Instead of the anti-symmetric generators A_n $(n = 1, 2, 3)$ of formula (9.10), we prefer to continue with the Hermitean generators $L_n = iA_n$ $(n = 1, 2, 3)$, which in explicit form, using the expressions (9.10), can be given by:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \text{and} \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9.24}$$

An arbitrary rotation $R(\hat{n}, \alpha)$ with rotation angle α around the axis spanned by $\hat{n}$, can, using formula (9.18), be expressed in terms of those generators, according to:

$$R(\hat{n}, \alpha) = \exp\{-i\alpha\hat{n} \cdot \vec{L}\} = \mathbf{1} + \sin(\alpha)(-i\hat{n} \cdot \vec{L}) + (1 - \cos(\alpha))(-i\hat{n} \cdot \vec{L})^2. \tag{9.25}$$

$SO(3)$ is a Lie-group, because the derivatives (9.10) and all higher derivatives exist. The Lie-algebra is spanned either by A_1, A_2 and A_3 or by $-iL_1$, $-iL_2$ and $-iL_3$. The product of the Lie-algebra is given by the *Lie-product*, defined by the commutator of two elements of the algebra. For the generators (9.24), using formula (9.13), the Lie-products yield:

$$[L_i, L_j] = i\epsilon_{ijm}L_m. \tag{9.26}$$

This establishes, moreover, their relation with the so-called angular momentum operators in Quantum Mechanics.

The generator space of $SO(3)$ is an algebra. This implies that any real linear combination of $-iL_1$, $-iL_2$ and $-iL_3$ can serve as a generator of a rotation. However, the algebra may be extended as to include also complex linear combinations of the basic generators. Those operators do in general not represent rotations, but might be very helpfull for the construction of representations of $SO(3)$. Two such linear combinations L_+ and L_- are given by:

$$L_\pm = L_1 \pm iL_2 \quad . \tag{9.27}$$

They satisfy the following commutation relations amongst themselves and with the third generator L_3:

$$[L_3, L_\pm] = \pm L_\pm \quad \text{and} \quad [L_+, L_-] = 2L_3 \quad . \tag{9.28}$$

Notice moreover that, since L_1 and L_2 are Hermitean, the Hermitean conjugates of L_+ and L_- satisfy:

$$L_+^\dagger = L_- \quad \text{and} \quad L_-^\dagger = L_+ \quad . \tag{9.29}$$

9.6 The Casimir operator of $SO(3)$.

According to the first lemma of Schur (4.11), a matrix which commutes with all matrices $D(g)$ of an irreducible representation D for all elements of the group, must be proportional to the unit matrix. One irrep of $SO(3)$ is formed by the rotation matrices themselves. This will be shown in one of the next sections of this chapter. So, let us find a matrix which commutes with all rotation matrices. We represent such matrix by its generator X, *i.e.* by $\exp(X)$. Then, since the matrix is supposed to commute with a rotation, we have:

$$\exp(X)\exp(-i\alpha\hat{n}\cdot\vec{L}) = \exp(-i\alpha\hat{n}\cdot\vec{L})\exp(X) \quad .$$

Now, when X commutes with $\hat{n}\cdot\vec{L}$, then using the Baker-Campbell-Hausdorff formula (6.17), we find:

$$\exp(X)\exp(-i\alpha\hat{n}\cdot\vec{L}) = \exp(X - i\alpha\hat{n}\cdot\vec{L}) = \exp(-i\alpha\hat{n}\cdot\vec{L})\exp(X) \quad .$$

Consequently, X must be an operator which commutes with all three generators L_1, L_2 and L_3. A possible solution for such operator is given by:

$$X = \vec{L}^{\,2} = (L_1)^2 + (L_2)^2 + (L_3)^2 \quad , \tag{9.30}$$

for which it is not difficult to prove that the following commutation relations hold:

$$[\vec{L}^{\,2}, L_i] = 0 \quad \text{for} \quad i = 1, 2, 3 \ . \tag{9.31}$$

The operator X, which is usually refered to as L^2, is called the *Casimir* operator of $SO(3)$. Notice that L^2 is not an element of the Lie-algebra, because it can not be written as a linear combination of generators. Nevertheless, Casimir operators are important for the classification of irreps of Lie-groups, as we will see in the following.

For later use, we will give below some alternative expressions for L^2 in terms of the generator L_3 and the generators $L_\pm$ of formula (9.27):

$$\begin{aligned} L^2 &= (L_3)^2 + L_3 + L_-L_+ \\ &= (L_3)^2 - L_3 + L_+L_- \\ &= (L_3)^2 + \frac{1}{2}\left(L_+L_- + L_-L_+\right) \quad . \end{aligned} \tag{9.32}$$

9.7 The $(2\ell + 1)$-dimensional irrep $d^{(\ell)}$.

A representation of the Lie-algebra of $SO(3)$ and hence of the Lie-group itself, is fully characterized once the transformations $d^{(\ell)}(L_i)$ (for $i = 1, 2, 3$) of a certain complex vector space V_ℓ onto itself are known. Let us select an orthonormal basis in V_ℓ such that $d^{(\ell)}(L_3)$ is diagonal at that basis. From the considerations for $SO(2)$ which lead to the formulas (7.39) and (7.52), we know that the eigenvalues for the representations of the generator L_3 of rotations around the z-axis are integers. So, we might label the basis vectors of V_ℓ according to their eigenvalues for $d^{(\ell)}(L_3)$, *i.e.*

$$d^{(\ell)}(L_3)\ |\ell, m >= m|\ell, m > \quad \text{for} \quad m = 0, \pm 1, \pm 2, \ldots \tag{9.33}$$

The Casimir operator L^2 for the representation, given by

$$L^2 = \left[d^{(\ell)}(L_1)\right]^2 + \left[d^{(\ell)}(L_2)\right]^2 + \left[d^{(\ell)}(L_3)\right]^2 \quad , \tag{9.34}$$

commutes with $d^{(\ell)}(L_1)$, $d^{(\ell)}(L_2)$ and $d^{(\ell)}(L_3)$. Consequently, for an irrep one has, due the first lemma of Schur (4.11), that L^2 equals a matrix proportional to the unit matrix. Let us take for the constant of proportionality the symbol λ, *i.e.*

$$L^2\ |\ell, m >= \lambda|\ell, m > \quad \text{for all basis vectors.} \tag{9.35}$$

In the following we will show that $\lambda = \ell(\ell + 1)$ and that m runs from $-\ell$ to $+\ell$. First, we proof that

$$m^2 \leq \lambda \quad , \tag{9.36}$$

from which we conclude that there must exist a maximum and a minimum value for the integers m. Then we demonstrate that the representation of the operator L_+ in V_ℓ transforms a basis vector $|\ell, m >$ in to a vector proportional to the basis vector $|\ell, m + 1 >$ and similarly, the representation of the operator L_- in V_ℓ transforms a basis vector $|\ell, m >$ in to a vector proportional to the basis vector $|\ell, m - 1 >$, *i.e.*

$$d^{(\ell)}(L_\pm)\ |\ell, m >= C_\pm(\ell, m)|\ell, m \pm 1 > \quad , \tag{9.37}$$

where the coefficients $C_\pm(\ell, m)$ represent the constants of proportionality. Because of the property (9.37) the operator L_+ is called the *raising operator* of $SO(3)$ and similarly L_- the *lowering operator*. Property (9.37) leads us to conclude that operating on the basis vector which has the maximum eigenvalue m_{max} for $d^{(\ell)}(L_3)$, the raising operator must yield zero, and similarly on the basis vector with the minimum eigenvalue m_{min}, the lowering operator must yield zero.

$$d^{(\ell)}(L_+)|\ell, m_{max} >= d^{(\ell)}(L_-)|\ell, m_{min} >= 0 \quad . \tag{9.38}$$

Finally, we show that $m_{min} = -m_{max}$ and introduce the symbol ℓ for the maximum value of m.

The basis $|\ell, m >$ is orthonormal, which can be expressed by:

$$< \ell, m|\ell, m' >= \delta_{mm'} \quad . \tag{9.39}$$

Relations which hold for the generators, also hold for the representations of the generators. So, using (9.29) and (9.32) we find:

$$L^2 - \left[d^{(\ell)}(L_3)\right]^2 = \frac{1}{2}\left\{[d^{(\ell)}(L_+)]^\dagger d^{(\ell)}(L_+) + [d^{(\ell)}(L_-)]^\dagger d^{(\ell)}(L_-)\right\} \quad .$$

This leads, at the basis defined in formula (9.39), to the following property of the matrix elements for those operators:

$$\begin{aligned}
\lambda - m^2 &= <\ell, m|\left\{L^2 - \left[d^{(\ell)}(L_3)\right]^2\right\}|\ell, m> \\
&= <\ell, m|\frac{1}{2}\left\{[d^{(\ell)}(L_+)]^\dagger d^{(\ell)}(L_+) + [d^{(\ell)}(L_-)]^\dagger d^{(\ell)}(L_-)\right\}|\ell, m> \\
&= \frac{1}{2}\left\{\left|d^{(\ell)}(L_+)|\ell, m>\right|^2 + \left|d^{(\ell)}(L_-)|\ell, m>\right|^2\right\} \geq 0 \quad .
\end{aligned}$$

This proofs relation (9.36) and leads to the introduction of a maximum eigenvalue m_{max} of $d^{(\ell)}(L_3)$ as well as a minimum eigenvalue m_{min}.

Using the commutation relations (9.28), one obtains:

$$d^{(\ell)}(L_3)d^{(\ell)}(L_\pm) = d^{(\ell)}(L_\pm L_3 + [L_3, L_\pm]) = d^{(\ell)}(L_\pm)d^{(\ell)}(L_3) \pm d^{(\ell)}(L_\pm) \quad .$$

Acting with this operator on the basis (9.39), also using (9.33), one finds:

$$\begin{aligned}
d^{(\ell)}(L_3)\left\{d^{(\ell)}(L_\pm)|\ell, m>\right\} &= d^{(\ell)}(L_\pm)\left\{d^{(\ell)}(L_3) \pm 1\right\}|\ell, m> \\
&= (m \pm 1)\left\{d^{(\ell)}(L_\pm)|\ell, m>\right\} \quad .
\end{aligned} \tag{9.40}$$

Consequently, one may conclude that $d^{(\ell)}(L_+)|\ell, m>$ is proportional to the eigenvector of $d^{(\ell)}(L_3)$ with eigenvalue $m+1$, *i.e.* $|\ell, m+1>$, and similarly that the vector $d^{(\ell)}(L_-)|\ell, m>$ is proportional to $|\ell, m-1>$. Yet another consequence of (9.37) is that $d^{(\ell)}(L_+)|\ell, m_{max}>$ vanishes, since $|\ell, m_{max}+1>$ is not a basis vector of V_ℓ. Similarly, $d^{(\ell)}(L_-)|\ell, m_{min}>$ vanishes. So, using the relations (9.32), one finds:

$$\begin{aligned}
0 &= d^{(\ell)}(L_-)d^{(\ell)}(L_+)|\ell, m_{max}> = \left(L^2 - \left[d^{(\ell)}(L_3)\right]^2 - d^{(\ell)}(L_3)\right)|\ell, m_{max}> \\
&= (\lambda - (m_{max})^2 - m_{max})|\ell, m_{max}> \quad ,
\end{aligned}$$

and

$$\begin{aligned}
0 &= d^{(\ell)}(L_+)d^{(\ell)}(L_-)|\ell, m_{min}> = \left(L^2 - \left[d^{(\ell)}(L_3)\right]^2 + d^{(\ell)}(L_3)\right)|\ell, m_{min}> \\
&= (\lambda - (m_{min})^2 + m_{min})|\ell, m_{min}> \quad ,
\end{aligned}$$

which leads to the identification $m_{min} = -m_{max}$ and after defining $\ell = m_{max}$, to the equation:

$$\lambda = \ell^2 + \ell = \ell(\ell + 1) \quad . \tag{9.41}$$

This way we obtain for V_ℓ the following $(2\ell + 1)$-dimensional basis:

$$|\ell, -\ell>, \;\; |\ell, -\ell + 1>, \;\; \ldots, \;\; |\ell, \ell - 1>, \;\; |\ell, +\ell> \quad . \tag{9.42}$$

At this basis L_3 is represented by a diagonal $(2\ell + 1) \times (2\ell + 1)$ matrix, which has the following form:

$$d^{(\ell)}(L_3) = \begin{pmatrix} -\ell & & & & \\ & -\ell + 1 & & 0 & \\ & & \ddots & & \\ & 0 & & \ell - 1 & \\ & & & & +\ell \end{pmatrix} \quad . \tag{9.43}$$

Notice that $d^{(\ell)}(L_3)$ is traceless and Hermitean.

The non-negative integer ℓ characterizes the irrep $d^{(\ell)}$ of $SO(3)$. To each possible value for ℓ corresponds one non-equivalent unitary irrep. Its dimension is determined by the structure of the basis (9.42), *i.e.*

$$\dim(d^{(\ell)}) = 2\ell + 1 \quad . \tag{9.44}$$

The operator which characterizes the irrep, is the Casimir operator L^2, defined in formula (9.30). Its eigenvalues for any vector of V_ℓ have the same value, equal to $\ell(\ell + 1)$. The basis (9.42) is chosen such that the generator L_3 is represented by the diagonal matrix given in (9.43).

Because of the commutation relations (9.29), the operators $L_\pm$ defined in (9.27) serve as raising and lowering operators, usefull to construct the whole basis of V_ℓ out of one single basis vector.

Using equation (9.41) and once more the relations (9.29) and (9.32), we obtain for the constants of proportionality of formula (9.37), moreover:

$$\begin{aligned} \ell(\ell + 1) - m(m \pm 1) &= <\ell, m| \left\{ L^2 - \left(\left[d^{(\ell)}(L_3) \right]^2 \pm d^{(\ell)}(L_3) \right) \right\} |\ell, m> \\ &= <\ell, m| \left[d^{(\ell)}(L_\pm) \right]^\dagger d^{(\ell)}(L_\pm) |\ell, m> \\ &= \left| d^{(\ell)}(L_\pm) |\ell, m> \right|^2 = \left| C_\pm(\ell, m) \right|^2 \quad . \end{aligned}$$

It is convention not to consider a phase factor for the coefficients $C_\pm(\ell, m)$ and to take the following real solutions:

$$C_\pm(\ell, m) = \sqrt{\ell(\ell + 1) - m(m \pm 1)} \quad . \tag{9.45}$$

9.8 The three dimensional irrep for $\ell = 1$.

As an example, let us study the three dimensional irrep $d^{(1)}$ of the Lie-algebra of $SO(3)$, at the basis of eigenstates of $d^{(1)}(L_3)$ given by:

$$|1,-1>= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;,\quad |1,0>= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1,-1>= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad . \tag{9.46}$$

The matrix representation $d^{(1)}(L_3)$ of L_3 is, because of the above choice of basis, given by:

$$d^{(1)}(L_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad . \tag{9.47}$$

The matrices $d^{(1)}(L_\pm)$ at the basis (9.46) for the raising and lowering operators (9.27) of $SO(3)$ follow, using (9.45), from:

$$\begin{array}{lcllcl} d^{(1)}(L_+)|1,-1> & = & \sqrt{2}|1,0> & , \; d^{(1)}(L_-)|1,-1> & = & 0 \\ d^{(1)}(L_+)|1,0> & = & \sqrt{2}|1,+1> & , \; d^{(1)}(L_-)|1,0> & = & \sqrt{2}|1,-1> \\ d^{(1)}(L_+)|1,+1> & = & 0 & , \; d^{(1)}(L_-)|1,+1> & = & \sqrt{2}|1,0> \end{array}$$

So, the matrices for the choice of basis (9.46) are given by:

$$d^{(1)}(L_+) = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad d^{(1)}(L_-) = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{9.48}$$

The matrices $d^{(1)}(L_1)$ and $d^{(1)}(L_2)$ can be obtained using the definition (9.27) for the raising and lowering operators and the above result (9.48). This leads to:

$$d^{(1)}(L_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad d^{(1)}(L_2) = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad . \tag{9.49}$$

Now, we might also like to study the related representation $D^{(1)}$ of the group elements of $SO(3)$, in agreement with (9.25) given by:

$$D^{(1)}(\hat{n},\alpha) = \exp\{-i\alpha\hat{n}\cdot d^{(1)}(\vec{L})\} \quad . \tag{9.50}$$

Let us select the group elements shown in formula (9.1) and determine their representations for $\ell = 1$. For a rotation around the x-axis, using formulas (9.25) and (9.49), we find:

$$
\begin{aligned}
D^{(1)}\left(R(\hat{x},\alpha)\right) &= \exp\{-i\alpha d^{(1)}(L_1)\} \\
&= \mathbf{1} + \sin(\alpha)(-id^{(1)}(L_1)) + (1-\cos(\alpha))(-id^{(1)}(L_1))^2 \\
&= \begin{pmatrix} \frac{1}{2}(\cos(\alpha)+1) & -\frac{i}{\sqrt{2}}\sin(\alpha) & \frac{1}{2}(\cos(\alpha)-1) \\ -\frac{i}{\sqrt{2}}\sin(\alpha) & \cos(\alpha) & -\frac{i}{\sqrt{2}}\sin(\alpha) \\ \frac{1}{2}(\cos(\alpha)-1) & -\frac{i}{\sqrt{2}}\sin(\alpha) & \frac{1}{2}(\cos(\alpha)+1) \end{pmatrix} . \qquad (9.51)
\end{aligned}
$$

Similarly, for a rotation around the y-axis, one obtains:

$$
\begin{aligned}
D^{(1)}\left(R(\hat{y},\vartheta)\right) &= \exp\{-i\alpha d^{(1)}(L_2)\} \\
&= \mathbf{1} + \sin(\alpha)(-id^{(1)}(L_2)) + (1-\cos(\alpha))(-id^{(1)}(L_2))^2 \\
&= \begin{pmatrix} \frac{1}{2}(1+\cos(\vartheta)) & \frac{\sin(\vartheta)}{\sqrt{2}} & \frac{1}{2}(1-\cos(\vartheta)) \\ -\frac{\sin(\vartheta)}{\sqrt{2}} & \cos(\vartheta) & \frac{\sin(\vartheta)}{\sqrt{2}} \\ \frac{1}{2}(1-\cos(\vartheta)) & -\frac{\sin(\vartheta)}{\sqrt{2}} & \frac{1}{2}(1+\cos(\vartheta)) \end{pmatrix} . \qquad (9.52)
\end{aligned}
$$

And a rotation around the z-axis is here represented by:

$$
\begin{aligned}
D^{(1)}\left(R(\hat{z},\varphi)\right) &= \exp\{-i\alpha d^{(1)}(L_3)\} \\
&= \mathbf{1} + \sin(\alpha)(-id^{(1)}(L_3)) + (1-\cos(\alpha))(-id^{(1)}(L_3))^2 \\
&= \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix} . \qquad (9.53)
\end{aligned}
$$

The matrices (9.51), (9.52) and (9.53) of the three dimensional irreducible representation $D^{(1)}$ are not the same as the rotation matrices in three dimensions (9.1). However, one might notice that the traces are the same, respectively $2\cos(\alpha)+1$, $2\cos(\vartheta)+1$ and $2\cos(\varphi)+1$. So, one might expect a similarity transformation S which brings the rotation matrices (9.1) at the above form. This is most conveniently studied, using the expressions (9.24) for the generators and the expressions (9.47) and (9.49). But this is left as an exercise for the reader.

9.9 The standard irreps of $SO(3)$.

$SO(3)$ is the symmetry group of the sphere in three dimensions. Points at the sphere are characterized by the polar angle ϑ and the azimuthal angle φ. As a basis for the "well-behaved" functions $f(\vartheta,\varphi)$ for the variables ϑ and φ, serve the *spherical harmonics*:

$$Y_{\ell m}(\vartheta,\varphi), \quad \text{for} \quad \ell = 0,1,2,\ldots \quad \text{and} \quad m = -\ell,\ldots,+\ell \ . \tag{9.54}$$

The innerproduct for two functions χ and ψ in the space of functions on the sphere is defined as follows:

$$(\psi,\chi) = \int_{\text{sphere}} d\Omega\ \psi^*(\vartheta,\varphi)\chi(\vartheta,\varphi) \ . \tag{9.55}$$

The spherical harmonics (9.54) form an orthonormal basis for this innerproduct, *i.e.*

$$\int_{\text{sphere}} d\Omega\ Y^*_{\lambda\mu}(\vartheta,\varphi)Y_{\ell m}(\vartheta,\varphi) = \delta_{\lambda\ell}\delta_{\mu m} \ . \tag{9.56}$$

Each function $f(\vartheta,\varphi)$ on the sphere can be expanded in terms of the spherical harmonics, according to:

$$f(\vartheta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} B_{\ell m} Y_{\ell m}(\vartheta,\varphi) \ . \tag{9.57}$$

The coefficients $B_{\ell m}$ of the above expansion are, using the orthonormality relation (9.56), given by:

$$B_{\ell m} = \int_{\text{sphere}} d\Omega\ Y^*_{\ell m}(\vartheta,\varphi)f(\vartheta,\varphi) \ . \tag{9.58}$$

An active rotation $R(\hat{n},\alpha)$ in three dimensions induces a transformation in function space (compare equation 7.18), given by:

$$D(\hat{n},\alpha)f(\vec{r}) = f\left([R(\hat{n},\alpha)]^{-1}\vec{r}\right) \ , \tag{9.59}$$

where $\vec{r}$ indicates a point at the unit sphere, given by:

$$\vec{r}(\vartheta,\varphi) = \begin{pmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{pmatrix} \ . \tag{9.60}$$

Let us study here the differential operators which follow for the three generators L_1, L_2 and L_3, defined in formula (9.24), from the above equation. For a rotation $R(\hat{z},\alpha)$ around the z-axis we find, using formula (9.1), for $\Delta\vartheta$ and $\Delta\varphi$ to first order in α, the following:

$$\vec{r}(\vartheta+\Delta\vartheta,\varphi+\Delta\varphi) = \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{r}(\vartheta,\varphi) = \begin{pmatrix} \sin(\vartheta)\{\cos(\varphi)+\alpha\sin(\varphi)\} \\ \sin(\vartheta)\{-\alpha\cos(\varphi)+\sin(\varphi)\} \\ \cos(\vartheta) \end{pmatrix} \ ,$$

which is solved by:

$$\Delta\vartheta = 0 \quad \text{and} \quad \Delta\varphi = -\alpha \ .$$

We obtain then for (9.59), expanding the lefthand side in terms of the representation of L_3 and the righthand side in a Taylor series, both to first order in α, the following:

$$(1 - i\alpha d(L_3))\, f(\vec{r}) = \left(1 - \alpha\frac{\partial}{\partial\varphi}\right) f(\vec{r}) \ ,$$

from which we read the identification (compare also 7.51):

$$d(L_3) = -i\frac{\partial}{\partial\varphi} \ . \qquad (9.61)$$

For a rotation $R(\hat{x}, \alpha)$ around the x-axis we find, once more using formula (9.1), for $\Delta\vartheta$ and $\Delta\varphi$ to first order in α, the equations:

$$\vec{r}(\vartheta + \Delta\vartheta, \varphi + \Delta\varphi) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{array}\right) \vec{r}(\vartheta, \varphi) = \left(\begin{array}{c} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) + \alpha\cos(\vartheta) \\ -\alpha\sin(\vartheta)\sin(\varphi) + \cos(\vartheta) \end{array}\right) \ ,$$

which are solved by:

$$\Delta\vartheta = \alpha\sin(\varphi) \quad \text{and} \quad \Delta\varphi = \alpha\mathrm{cotg}(\vartheta)\cos(\varphi) \ .$$

Insertion of those results in the righthand side of (9.59), leads to the identification:

$$d(L_1) = i\left\{\sin(\varphi)\frac{\partial}{\partial\vartheta} + \mathrm{cotg}(\vartheta)\cos(\varphi)\frac{\partial}{\partial\varphi}\right\} \ . \qquad (9.62)$$

For a rotation $R(\hat{y}, \alpha)$ around the y-axis we find, again using formula (9.1), for $\Delta\vartheta$ and $\Delta\varphi$ to first order in α, the equations:

$$\vec{r}(\vartheta + \Delta\vartheta, \varphi + \Delta\varphi) = \left(\begin{array}{ccc} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{array}\right) \vec{r}(\vartheta, \varphi) = \left(\begin{array}{c} \sin(\vartheta)\cos(\varphi) - \alpha\cos(\vartheta) \\ \sin(\vartheta)\sin(\varphi) \\ \alpha\sin(\vartheta)\cos(\varphi) + \cos(\vartheta) \end{array}\right) \ ,$$

which are solved by:

$$\Delta\vartheta = -\alpha\cos(\varphi) \quad \text{and} \quad \Delta\varphi = \alpha\mathrm{cotg}(\vartheta)\sin(\varphi) \ ,$$

and lead to the identification:

$$d(L_2) = i\left\{-\cos(\varphi)\frac{\partial}{\partial\vartheta} + \mathrm{cotg}(\vartheta)\sin(\varphi)\frac{\partial}{\partial\varphi}\right\} \ . \qquad (9.63)$$

The raising and lowering operators $L_\pm$ are, according to their definition (9.27), given by:

$$d(L_\pm) = e^{\pm i\varphi}\left\{i\mathrm{cotg}(\vartheta)\frac{\partial}{\partial\varphi} \pm \frac{\partial}{\partial\vartheta}\right\} \ . \qquad (9.64)$$

For the Casimir operator L^2, using relation (9.30), we find the following differential operator:

$$
\begin{aligned}
L^2 &= [d(L_1)]^2 + [d(L_2)]^2 + [d(L_3)]^2 \\
&= -\left\{ \frac{\partial^2}{\partial\vartheta^2} + \mathrm{cotg}(\vartheta)\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\} \\
&= -\left\{ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\}
\end{aligned} \tag{9.65}
$$

The spherical harmonics $Y_{\ell m}(\vartheta, \varphi)$ for $m = -\ell, \ldots, +\ell$ form the standard basis of a $(2\ell + 1)$-dimensional irrep in function space. For L_3 their eigenvalues are given by:

$$
-i\frac{\partial}{\partial\varphi}Y_{\ell m}(\vartheta, \varphi) = d(L_3)Y_{\ell m}(\vartheta, \varphi) = mY_{\ell m}(\vartheta, \varphi) \ \ . \tag{9.66}
$$

And for the Casimir operator L^2, by:

$$
\begin{aligned}
-\left\{ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\} Y_{\ell m}(\vartheta, \varphi) &= L^2 Y_{\ell m}(\vartheta, \varphi) \\
&= \ell(\ell + 1)Y_{\ell m}(\vartheta, \varphi) \ \ .
\end{aligned} \tag{9.67}
$$

9.10 The spherical harmonics.

Equation (9.66) is solved by:

$$Y_{\ell m}(\vartheta,\varphi) = X_{\ell m}(\vartheta)e^{im\varphi} \quad , \tag{9.68}$$

where $X_{\ell m}(\vartheta)$ is some yet unknown function of the polar angle ϑ. For $m=0$ equation (9.67) takes the following form:

$$-\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}X_{\ell 0}(\vartheta) = \ell(\ell+1)X_{\ell 0}(\vartheta) \quad .$$

By a change of variables:

$$\xi = \cos(\vartheta) \quad \text{and} \quad P_\ell(\xi) = X_{\ell 0}(\vartheta) \quad , \tag{9.69}$$

this differential equation transforms into:

$$-\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}\, P_\ell(\xi) = \ell(\ell+1)P_\ell(\xi) \quad . \tag{9.70}$$

This equation is known as *Legendre's differential equation*, familiar in many problems of mathematical physics. Its solutions are well-known. The first few solutions are given by the following *Legendre polynomials*:

$$\begin{array}{rclcrcl} P_0(\xi) & = & 1 & , & P_1(\xi) & = & \xi \\ P_2(\xi) & = & \frac{1}{2}(3\xi^2-1) & , \text{ and } & P_3(\xi) & = & \frac{1}{2}(5\xi^3-3\xi) \end{array} \tag{9.71}$$

For the construction of the spherical harmonics for $m \neq 0$, we may use the raising and lowering operators, given in formula (9.64). Let us as an example, construct the complete basis of spherical harmonics for the standard irrep corresponding to $\ell = 1$. For $m=0$, using the expressions (9.68), (9.69) and (9.71), we find:

$$Y_{1,0}(\vartheta,\varphi) = N\cos(\vartheta) \quad ,$$

where the normalization constant N is determined by relation (9.56), *i.e.*

$$1 = |N|^2\int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos(\vartheta)\ \cos^2(\vartheta) \ = \ 4\pi|N|^2/3 \quad ,$$

leading to the conventional choice $N=\sqrt{3/4\pi}$. For $m=+1$, using the formulas (9.48) and (9.64), we obtain:

$$\begin{aligned} \sqrt{2}\,Y_{1,+1}(\vartheta,\varphi) &= d(L_+)Y_{1,0}(\vartheta,\varphi) \\ &= e^{+i\varphi}\left\{i\text{cotg}(\vartheta)\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\vartheta}\right\}\sqrt{\frac{3}{4\pi}}\ \cos(\vartheta) \\ &= -\sqrt{\frac{3}{4\pi}}\,e^{+i\varphi}\sin(\vartheta) \quad , \end{aligned}$$

leading to the solution:

$$Y_{1,+1}(\vartheta,\varphi) = -\sqrt{\frac{3}{8\pi}}\, e^{+i\varphi}\sin(\vartheta) \quad .$$

For $m=-1$, also using the formulas (9.48) and (9.64), we obtain:

$$\begin{aligned}
\sqrt{2}\, Y_{1,-1}(\vartheta,\varphi) &= d(L_{-})Y_{1,0}(\vartheta,\varphi) \\
&= e^{-i\varphi}\left\{ i\text{cotg}(\vartheta)\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\vartheta}\right\}\sqrt{\frac{3}{4\pi}}\;\cos(\vartheta) \\
&= +\sqrt{\frac{3}{4\pi}}\, e^{-i\varphi}\sin(\vartheta) \quad ,
\end{aligned}$$

leading to the solution:

$$Y_{1,-1}(\vartheta,\varphi) = +\sqrt{\frac{3}{8\pi}}\, e^{-i\varphi}\sin(\vartheta) \quad .$$

Applying once more the raising operator to the solution for $m=+1$, using formula (9.64), we obtain moreover:

$$\begin{aligned}
d(L_{+})Y_{1,+1}(\vartheta,\varphi) &= e^{+i\varphi}\left\{ i\text{cotg}(\vartheta)\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\vartheta}\right\}\left[-\sqrt{\frac{3}{8\pi}}\, e^{+i\varphi}\sin(\vartheta)\right] \\
&= -\sqrt{\frac{3}{8\pi}}\, e^{+2i\varphi}\left\{-\text{cotg}(\vartheta)\sin(\vartheta) + \cos(\vartheta)\right\} = 0 \quad ,
\end{aligned}$$

which demonstrates that the series of basis elements terminates at $m=+1$ for the raising operator. Similarly for the lowering operator, one finds that the series of basis elements terminates at $m=-1$. So, the complete basis of the standard irrep of $SO(3)$ for $\ell=1$, reads:

$$Y_{1,0}(\vartheta,\varphi) = \sqrt{\frac{3}{4\pi}}\;\cos(\vartheta) \quad\text{and}\quad Y_{1,\pm1}(\vartheta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\varphi}\sin(\vartheta) \quad .$$

9.11 Weight diagrams.

A *weight diagram* is a pictorial way to show the structure of the basis of an irrep. It shows in diagrammatic form the eigenvalues of those generators of the group which are represented by diagonal matrices. For an irrep of $SO(3)$ the weight diagram consists of one axis, along which the eigenvalues of the basis vectors of the irrep for the generator L_3 are indicated (in the case that the representation of L_3 is diagonal at the basis of the irrep). In the figure below, the weight diagram for the irreducible representation of $SO(3)$ corresponding to $\ell = 3$ is depicted.

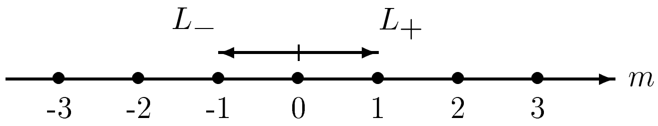


Figure 9.1: The weight diagram for $d^{(3)}$ of $SO(3)$. The dots ($\bullet$) at the horizontal line represent the basis vectors of the vector space V_3. The values below the dots represent their eigenvalues m for $d^{(3)}(L_3)$.

The raising operator $d(L_+)$ takes steps of one unit in the positive direction of the horizontal weight axis (see for example the weight diagram of figure 9.1), when operating on the basis vectors; the lowering operator $d(L_-)$, steps of one unit in the negative direction.

9.12 Equivalence classes and characters.

In this section we discuss the relation which exists between the structure of representations for finite groups and the structure of the irreps of $SO(3)$. For finite groups we discovered that the number of inequivalent irreps equals the number of equivalence classes of the group (see formula 4.23), which is intimately connected to the structure of the character space for representations of finite groups (see formula 4.17). So, let us begin by studying the equivalence classes of $SO(3)$.

In order to do so, we first perform the following exercise: We determine, using formula (9.21), the transformed matrix for $(\vec{n} \cdot \vec{A})$ under a rotation around the z-axis, for the following definition of the transformation:

$$\vec{n}\,' \cdot \vec{A} = R(\hat{z}, \alpha)(\vec{n} \cdot \vec{A})R^{-1}(\hat{z}, \alpha)$$

$$= \begin{pmatrix} 0 & -n_3 & n_1 \sin(\alpha) + n_2 \cos(\alpha) \\ n_3 & 0 & -n_1 \cos(\alpha) + n_2 \sin(\alpha) \\ -n_1 \sin(\alpha) - n_2 \cos(\alpha) & n_1 \cos(\alpha) - n_2 \sin(\alpha) & 0 \end{pmatrix} .$$

From this result we read for the transformed vector $\vec{n}\,'$ the relation:

$$\vec{n}\,' = \begin{pmatrix} n_1\cos(\alpha) - n_2\sin(\alpha) \\ \\ n_1\sin(\alpha) + n_2\cos(\alpha) \\ \\ n_3 \end{pmatrix} = R(\hat{z},\alpha)\vec{n} \ \ ,$$

which leads to the conclusion:

$$R(\hat{z},\alpha)(\vec{n}\cdot\vec{A})R^{-1}(\hat{z},\alpha) = \{R(\hat{z},\alpha)\vec{n}\}\cdot\vec{A} \ \ . \tag{9.72}$$

This result can be generalized to any rotation, not just rotations around the z-axis. In order to proof that, we need to mention a property of unimodular, orthogonal matrices, *i.e.*

$$\epsilon_{\ell mn}R_{im}R_{jn} = \epsilon_{ijk}R_{k\ell} \ \ .$$

Using this relation and formulas (6.20) and (9.11), we find:

$$\begin{aligned} \left[R(\vec{n}\cdot\vec{A})R^{-1}\right]_{ij} \ &= \\ &= \ n_k\left[RA_kR^T\right]_{ij} = n_kR_{ia}(A_k)_{ab}R_{jb} = n_kR_{ia}(-\epsilon_{kab})R_{jb} \\ &= \ = -n_k\epsilon_{ij\ell})R_{\ell k} = (R\vec{n})_\ell(A_\ell)_{ij} = \left[(R\vec{n})\cdot\vec{A}\right]_{ij} \ \ , \end{aligned}$$

which proofs the validity of relation (9.72) for an arbitrary rotation R. Using the above results, it is also not very difficult to proof that:

$$R\exp\{\vec{n}\cdot\vec{A}\}R^{-1} = \exp\left\{(R\vec{n})\cdot\vec{A}\right\} \ \ .$$

Consequently, the group element $\exp\{\vec{n}\cdot\vec{A}\}$ in the above equation is equivalent to the group element $\exp\left\{(R\vec{n})\cdot\vec{A}\right\}$ (see the definition of equivalence in formula 1.23). The rotation angles are equal for both group elements, because the modulus of vector $\vec{n}$ does not change under a rotation, and therefore:

$$\text{rotation angle of } \ e^{(R\vec{n})\cdot\vec{A}} = |R\vec{n}| = |\vec{n}| = \text{rotation angle of } \ e^{\vec{n}\cdot\vec{A}} \ \ .$$

Now, since R is arbitrary, we may conclude that all rotations which have the same angle of rotation form an equivalence class. The trace of the rotation matrices out of one class can be determined by taking the trace of the rotation matrix of one representant of the class. For example a rotation around the z-axis, which leads to:

$$\mathrm{Tr}\left\{e^{(R\vec{n})\cdot\vec{A}}\right\} = \mathrm{Tr}\left\{e^{\vec{n}\cdot\vec{A}}\right\} = \mathrm{Tr}\{R(\hat{z},n)\} = 1 + 2\cos(n) \ \ .$$

All rotation matrices for group elements of the same equivalence class, have the same character. Consequently, might one characterize the various classes of $SO(3)$

by the corresponding rotation angle. There are infinitely many angles and so, there exist infinitely many equivalence classes in $SO(3)$. Which settles moreover the point that there exist infinitely many irreps of $SO(3)$.

Also for a representation of $SO(3)$ one might select the matrix representation of rotations around the z-axis, in order to determine the characters for the various equivalence classes. The representation of L_3 is shown in formula (9.43). The resulting representation for a rotation around the z-axis is therefore given by:

$$
\begin{aligned}
D^{(\ell)}(R(\hat{z},\alpha)) &= e^{-i\alpha d^{(\ell)}(L_3)} \\
&= \begin{pmatrix} e^{+i\ell\alpha} & & & & \\ & e^{+i(\ell-1)\alpha} & & 0 & \\ & & \ddots & & \\ & 0 & & e^{-i(\ell-1)\alpha} & \\ & & & & e^{-i\ell\alpha} \end{pmatrix} .
\end{aligned}
$$

So, its character is readily determined to be equal to:

$$
\chi^{(\ell)}(R(\hat{z},\alpha)) = \sum_{m=-\ell}^{+\ell} e^{-im\alpha} = \frac{\sin\left((2\ell+1)\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} .
$$

For the particular case of the representation of the unit matrix $R(\hat{z},0)$, we have moreover that its character yields the dimension of the representation, *i.e.*

$$
\dim\left(d^{(\ell)}\right) = \chi^{(\ell)}(R(\hat{z},0)) = \sum_{m=-\ell}^{+\ell} 1 = 2\ell+1 ,
$$

in agreement with (9.44).

Chapter 10

The special unitary group in two dimensions.

In this chapter we study the representations of the special unitary group in two dimensions, $SU(2)$.

10.1 Unitary transformations in two dimensions.

Unitary 2×2 matrices form a group under normal matrix multiplication, because the product of two unitary matrices A and B is also unitary, *i.e.*

$$(AB)^{-1} = B^{-1}A^{-1} = B^{\dagger}A^{\dagger} = (AB)^{\dagger} \ . \tag{10.1}$$

The most general form of a unitary matrix in two dimensions is given by:

$$U(a,b) = \begin{pmatrix} a^* & -b^* \\ b & a \end{pmatrix} \ , \tag{10.2}$$

where a and b represent arbitrary complex numbers. When, moreover this matrix has unit determinant, we have the condition:

$$|a|^2 + |b|^2 = 1 \ . \tag{10.3}$$

A convenient paramatrization of unitary 2×2 matrices is by means of the *Cayley-Klein* parameters ξ_0, ξ_1, ξ_2 and ξ_3, which is a set of four real parameters related to the complex numbers a and b of formula (10.2) by:

$$a = \xi_0 + i\xi_3 \quad \text{and} \quad b = \xi_2 - i\xi_1 \ . \tag{10.4}$$

When one substitutes a and b in formula (10.2) by the Cayley-Klein parameters, then one finds for a unitary 2×2 matrix the expression:

$$U(\xi_0, \vec{\xi}) = \xi_0 \mathbf{1} - i\vec{\xi} \cdot \vec{\sigma} \ , \tag{10.5}$$

where σ_1, σ_2 and σ_3 represent the Hermitean Pauli matrices, defined by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{10.6}$$

When, moreover the matrix (10.5) is unimodular, we have the condition:

$$(\xi_0)^2 + (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = 1 \quad . \tag{10.7}$$

This way we obtain three free parameters for the characterization of a 2×2 unimodular and unitary matrix.

Let us combine three parameters n_1, n_2 and n_3 into a vector $\vec{n}$ given by:

$$\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad . \tag{10.8}$$

By means of those parameters, one might select for the Cayley-Klein parameters (10.4) which satisfy the condition (10.7), the following parametrization:

$$\xi_0 = \cos\left(\frac{n}{2}\right) \quad \text{and} \quad \vec{\xi} = \hat{n}\sin\left(\frac{n}{2}\right) \quad , \tag{10.9}$$

where $\hat{n}$ and n are defined by:

$$\hat{n} = \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{pmatrix} = \frac{1}{n}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad \text{and} \quad n = \sqrt{(n_1)^2 + (n_2)^2 + (n_3)^2} \quad .$$

This way we find for an arbitrary unimodular unitary 2×2 matrix the following general expression:

$$U(\vec{n}) = \mathbf{1}\cos\left(\frac{n}{2}\right) - i(\hat{n}\cdot\sigma)\sin\left(\frac{n}{2}\right) \quad . \tag{10.10}$$

Noticing moreover that $(\hat{n}\cdot\vec{\sigma})^2 = \mathbf{1}$, it is easy to show that $U(\vec{n})$ may also be written in the form:

$$U(\vec{n}) = \exp\left\{-\frac{i}{2}\vec{n}\cdot\vec{\sigma}\right\} \quad . \tag{10.11}$$

The Pauli matrices (10.6) satisfy the following relations:
For the product of two Pauli matrices one has:

$$\sigma_i\sigma_j = \mathbf{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k \quad , \tag{10.12}$$

for their commutator:

$$\left[\sigma_i, \sigma_j\right] = 2i\epsilon_{ijk}\sigma_k \quad , \tag{10.13}$$

and for their anti-commutator:

$$\left\{\sigma_i, \sigma_j\right\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbf{1} \tag{10.14}$$

10.2 The generators of $SU(2)$.

The group of special unitary transformations in two dimensions $SU(2)$ is a Lie-group. This can easily be understood if one considers the representation (10.10) for an arbitrary unimodular unitary transformation. Clearly, for that expression exist the partial derivatives with respect to its parameters to any order.

According to formula (10.11) we may select for the generators of $SU(2)$ the following set of 2×2 matrices:

$$J_1 = \frac{\sigma_1}{2} \quad , \quad J_2 = \frac{\sigma_2}{2} \quad \text{and} \quad J_3 = \frac{\sigma_3}{2} \quad . \tag{10.15}$$

The arbitrary special unitary transformation $U(\vec{n} = \alpha\hat{n})$ in two dimensions may then, according to formula (10.11), be expressed by:

$$U(\hat{n}, \alpha) = e^{-i\alpha\hat{n} \cdot \vec{J}} \quad . \tag{10.16}$$

The commutation relations for the generators $\vec{J}$ of $SU(2)$ defined in formula (10.15), can be read from the commutation relations (10.13), *i.e.*

$$\left[J_i \, , \, J_j\right] = i\epsilon_{ijk}J_k \quad . \tag{10.17}$$

The commutation relations (10.17) for the generators $\vec{J}$ of $SU(2)$ are identical to the commutation relations (9.26) for the generators $\vec{L}$ of $SO(3)$.

Notice that, as in the case of $SO(3)$, for $SU(2)$ the generators (10.15) are traceless and Hermitean.

10.3 The relation between $SU(2)$ and $SO(3)$.

In order to establish a relation between $SU(2)$ and $SO(3)$, we first study once more the effect of a rotation $R(\hat{n}, \alpha)$ in three dimensions on a vector $\vec{x}$. A vector $\vec{x}$ may be decomposed into a part parallel to the rotation axis $\hat{n}$ and a part perpendicular to this axis, as follows:

$$\vec{x} = (\hat{n} \cdot \vec{x})\hat{n} + \{\vec{x} - (\hat{n} \cdot \vec{x})\hat{n}\} \quad . \tag{10.18}$$

The vector $(\hat{n} \cdot \vec{x})\hat{n}$ does not suffer any transformation under a rotation $R(\hat{n}, \alpha)$ (see equation 9.20). But, the second part of (10.18), which represents a vector in the plane perpendicular to the rotation axis, transforms into another vector in that plane.

For an orthonormal basis in this plane one might select the unit vectors $\hat{v}$ and $\hat{w}$, given by:

$$\hat{v} = \frac{\vec{x} - (\hat{n} \cdot \vec{x})\hat{n}}{\sqrt{\vec{x}^{\,2} - (\hat{n} \cdot \vec{x})^2}} \quad \text{and} \quad \hat{w} = \frac{\hat{n} \times \vec{x}}{\sqrt{\vec{x}^{\,2} - (\hat{n} \cdot \vec{x})^2}} \quad . \tag{10.19}$$

The second part of (10.18) is before the transformation given by:

$$\vec{x} - (\hat{n}\cdot\vec{x})\hat{n} = \hat{v}\sqrt{\vec{x}^{\,2} - (\hat{n}\cdot\vec{x})^{\,2}} \ \ .$$

And after the rotation over an angle α in the plane perpendicular to the rotation axis $\hat{n}$ in the positive sense, by:

$$\{\hat{v}\cos(\alpha) + \hat{w}\sin(\alpha)\}\sqrt{\vec{x}^{\,2} - (\hat{n}\cdot\vec{x})^2} \ \ . \tag{10.20}$$

So, using formulas (10.18), (10.19) and (10.20), we find for the transformed vector $\vec{x}\,'$ resulting from $\vec{x}$ under a rotation $R(\hat{n},\alpha)$, the following:

$$\vec{x}\,' = R(\hat{n},\alpha)\vec{x} = \vec{x}\cos(\alpha) + (\hat{n}\times\vec{x})\sin(\alpha) + (\hat{n}\cdot\vec{x})\hat{n}(1-\cos(\alpha)) \ \ . \tag{10.21}$$

A more elegant way to obtain the same result, is by means of the expression (9.18) for a rotation of an angle α around the axis indicated by $\hat{n}$, *i.e.*

$$R(\hat{n},\alpha) = e^{\alpha\hat{n}\cdot\vec{A}} = \mathbf{1} + \sin(\alpha)(\hat{n}\cdot\vec{A}) + (1-\cos(\alpha))(\hat{n}\cdot\vec{A})^2. \tag{10.22}$$

Using formula (9.11) for the matrix elements of $\vec{A}$, we find for $(\vec{n}\cdot\vec{A})\vec{x}$ the result:

$$\left[(\vec{n}\cdot\vec{A})\vec{x}\right]_i = n_k(A_k)_{ij}x_j = -\epsilon_{kij}n_kx_j = [\vec{n}\times\vec{x}]_i \ \ . \tag{10.23}$$

Moreover, using formula (9.12) for the contraction of two Levi-Civita tensors, we obtain for $(\vec{n}\cdot\vec{A})^2\vec{x}$ the expression:

$$\left[(\vec{n}\cdot\vec{A})^2\vec{x}\right]_i = n_an_b(A_a)_{ij}(A_b)_{jk}x_k = \epsilon_{aij}\epsilon_{bjk}n_an_bx_k = (\vec{n}\cdot\vec{x})n_i - n^2x_i \ \ . \tag{10.24}$$

Insertion of the result (10.23) and (10.24) in (10.22), yields once more the expression (10.21) for $R(\hat{n},\alpha)\vec{x}$.

Having derived formula (10.21) for a rotation in three dimensions by two alternative methods, we return to the unitary transformations in two dimensions. Let us introduce, by means of the Pauli matrices given in formula (10.6), a Hermitean 2×2 matrix X defined by:

$$X = \vec{x}\cdot\vec{\sigma} \ \ . \tag{10.25}$$

The components of the vector $\vec{x}$ can, with the use of formula (10.14) for the anti-commutator of the Pauli matrices, uniquely be recovered from any Hermitean matrix X by taking the anti-commutator of X and $\vec{\sigma}$, *i.e.*

$$\vec{x} = \tfrac{1}{2}\{X,\vec{\sigma}\} = \tfrac{1}{2}(X\sigma + \sigma X) \ \ . \tag{10.26}$$

For an arbitrary unimodular unitary 2×2 matrix $U(\hat{n},\alpha)$, we define moreover the following transformation for the matrix X:

$$X' = U(\hat{n},\alpha)XU^\dagger(\hat{n},\alpha) \ \ . \tag{10.27}$$

Since $U(\hat{n},\alpha)$ is unitary and X Hermitean by definition (10.25), the matrix X' is also Hermitean and thus by the procedure (10.26) uniquely related to a vector $\vec{x}\,'$.

Inserting the expressions (10.10) for $U(\hat{n},\alpha)$ and (10.25) for X, we obtain for the matrix X' the result:

$$\begin{aligned} X' &= \left\{\mathbf{1}\cos\left(\frac{\alpha}{2}\right) - i(\hat{n}\cdot\sigma)\sin\left(\frac{\alpha}{2}\right)\right\}(\vec{x}\cdot\vec{\sigma})\left\{\mathbf{1}\cos\left(\frac{\alpha}{2}\right) + i(\hat{n}\cdot\sigma)\sin\left(\frac{\alpha}{2}\right)\right\} \\ &= (\vec{x}\cdot\vec{\sigma})\cos^2\left(\frac{\alpha}{2}\right) + i\left[(\vec{x}\cdot\vec{\sigma}),(\hat{n}\cdot\vec{\sigma})\right]\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \\ &\qquad + (\hat{n}\cdot\vec{\sigma})(\vec{x}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma})\sin^2\left(\frac{\alpha}{2}\right) \quad . \end{aligned} \tag{10.28}$$

Using the relations (10.13) for the commutator of two Pauli-matrices and (10.12) for their product, it is easy to verify that:

$$i\left[(\vec{x}\cdot\vec{\sigma}),(\hat{n}\cdot\vec{\sigma})\right] = 2(\hat{n}\times\vec{x})\cdot\vec{\sigma} \quad \text{and}$$

$$(\hat{n}\cdot\vec{\sigma})(\vec{x}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma}) = 2(\hat{n}\cdot\vec{x})(\hat{n}\cdot\vec{\sigma}) - (\vec{x}\cdot\vec{\sigma}) \quad .$$

Inserting those equations in (10.28), we find:

$$X' = (\vec{x}\cdot\vec{\sigma})\cos(\alpha) + (\hat{n}\times\vec{x})\cdot\vec{\sigma}\sin(\alpha) + (\hat{n}\cdot\vec{x})(\hat{n}\cdot\vec{\sigma})(1-\cos(\alpha)) \quad . \tag{10.29}$$

The vector $\vec{x}\,'$ can be extracted from this matrix using the relation (10.26), for which one obtains:

$$\vec{x}\,' = \vec{x}\cos(\alpha) + (\hat{n}\times\vec{x})\sin(\alpha) + (\hat{n}\cdot\vec{x})\hat{n}(1-\cos(\alpha)) \quad .$$

Comparison of this result to the result of formula (10.21) for the expression of the vector $\vec{x}$ rotated around a rotation axis $\hat{n}$ by an angle α, leads us to the conclusion that:

$$R(\hat{n},\alpha)\vec{x} = \tfrac{1}{2}\left\{U(\hat{n},\alpha)(\vec{x}\cdot\vec{\sigma})U^\dagger(\hat{n},\alpha)\ ,\ \vec{\sigma}\right\} \quad , \tag{10.30}$$

which formula establishes the relation between a rotation in three dimensions and a unitary transformation in two dimensions.

Notice however, that although for rotations one has:

$$R(\hat{n},\alpha+2\pi) = R(\hat{n},\alpha) \quad ,$$

for unitary transformations in two dimensions, we have (see for example the expression 10.10 for $U(\hat{n},\alpha)$):

$$U(\hat{n},\alpha+2\pi) \neq U(\hat{n},\alpha) \quad ! \tag{10.31}$$

This implies that each rotation $R(\hat{n},\alpha)$ corresponds according to expression (10.30), to two different unitary transformations $U(\hat{n},\alpha)$ and $U(\hat{n},\alpha+2\pi)$.

10.4 The subgroup $U(1)$.

The matrices $U(\hat{z},\alpha)$ form a subgroup $U(1)$ of $SU(2)$, because of the following property:

$$U(\hat{z},\alpha_2)U(\hat{z},\alpha_1) = U(\hat{z},\alpha_1+\alpha_2) \ . \tag{10.32}$$

The explicit form of $U(\hat{z},\alpha)$, using relation (10.10), is given by:

$$U(\hat{z},\alpha) = \begin{pmatrix} e^{-\frac{i}{2}\alpha} & 0 \\ & \\ 0 & e^{+\frac{i}{2}\alpha} \end{pmatrix} \ . \tag{10.33}$$

At the subspace spanned by the vector $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $U(\hat{z},\alpha)$, is represented by $\exp\{-i\alpha/2\}$, according to:

$$U(\hat{z},\alpha)a\hat{e}_1 = e^{-\frac{i}{2}\alpha}a\hat{e}_1 \ \text{, so} \ \ d(U(\hat{z},\alpha)) = e^{-\frac{i}{2}\alpha} \ . \tag{10.34}$$

The group $U(1)$ is Abelian, as can be read from the property (10.32) for the matrices $U(\hat{z},\alpha)$. Consequently its irreps are one-dimensional (see also formula 4.24).

Other one-dimensional irreps of $U(\hat{z},\alpha)$ may be given by:

$$d(U(\hat{z},\alpha)) = e^{-im\alpha} \ . \tag{10.35}$$

Now, since:

$$U(\hat{z},2\pi) = -\mathbf{1} \ \text{ and } \ U(\hat{z},4\pi) = \mathbf{1} \ , \tag{10.36}$$

we have for the parameters m of formula (10.35), the condition that $4\pi m = 2\pi$, which is solved by:

$$m = 0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \pm 2, \ldots \tag{10.37}$$

The representation (10.34) corresponds to the case $m = \frac{1}{2}$.

10.5 The $(2j+1)$-dimensional irrep $\{2j+1\}$.

In section (10.2) we have seen that the generators $\vec{J}$ of $SU(2)$ satisfy the same commutation relations as the generators $\vec{L}$ of $SO(3)$. Consequently, the structure of the Lie-algebras for the two groups is the same. Therefore, we can repeat the construction of irreps followed for $SO(3)$. However, there is one important difference: In the case of $SO(3)$, because of condition (9.33) for m, only *odd* values $(2\ell+1)$ for the dimensions of its irreps are possible. For $SU(2)$, because of the condition (10.37), also *even* values $(2j+1)$ are possible for the dimensions of its irreps.

The irreps of $SU(2)$ are characterized by the parameter j, which, because of the result (10.37), may take the values:

$$j = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ 2, \ldots \tag{10.38}$$

The $(2j+1)$-dimensional irrep of the Lie-algebra of $SU(2)$, $d^{(j)}$, is characterized by the $(2j+1)$ orthonormal basis vectors, given by:

$$|j,+j> \ \ , \ \ |j,j-1> \ \ , \ \ldots \ , |j,-j+1> \ \ , \ \ |j,-j> \ \ . \tag{10.39}$$

The basis is chosen such that J_3 is represented by a diagonal matrix, *i.e.*

$$d^{(j)}(J_3)|j,m>= m|j,m> \ \ . \tag{10.40}$$

The raising and lowering operators $J_\pm$ are defined by the linear combinations of J_1 and J_2 corresponding to those for $L_\pm$ in $SO(3)$, *i.e.*

$$J_\pm = J_1 \pm iJ_2 \ \ . \tag{10.41}$$

At the basis (10.39) for the $(2j+1)$-dimensional irrep $d^{(j)}$ they are represented by:

$$d^{(j)}(J_\pm)|j,m>= \sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1> \ \ . \tag{10.42}$$

The Casimir operator, J^2, defined by:

$$J^2 = \left(d^{(j)}(J_1)\right)^2 + \left(d^{(j)}(J_2)\right)^2 + \left(d^{(j)}(J_3)\right)^2 \ \ , \tag{10.43}$$

has eigenvalues:

$$J^2\,|j,m>= j(j+1)|j,m> \ \ . \tag{10.44}$$

Besides allowing for more possible irreps, for $SU(2)$ the representations have the same structure as the representations of $SO(3)$. In this chapter we use the symbol $\{2j+1\}$, which indicates its dimension, for the irrep $D^{(2j+1)}$ of $SU(2)$.

10.6 The two-dimensional irrep $\{2\}$.

For $j=\frac{1}{2}$ one has the two-dimensional irrep $d^{(1/2)}$ of the Lie-algebra of $SU(2)$. We denote the basis vectors by:

$$|\tfrac{1}{2},+\tfrac{1}{2}>=\left(\begin{array}{c}1\\0\end{array}\right) \quad \text{and} \quad |\tfrac{1}{2},-\tfrac{1}{2}>=\left(\begin{array}{c}0\\1\end{array}\right) \tag{10.45}$$

The generator J_3 is at this basis represented by a diagonal matrix. Its eigenvalues, as indicated by the notation for the basis vectors in formula (10.45), are respectively $+\frac{1}{2}$ and $-\frac{1}{2}$. So, we find for $d^{(1/2)}(J_3)$ the following matrix:

$$d^{(1/2)}(J_3)=\tfrac{1}{2}\left(\begin{array}{cc}+1 & 0\\0 & -1\end{array}\right)=\frac{\sigma_3}{2} \quad . \tag{10.46}$$

For the raising and lowering operators we have, using formula (10.42), the transformations:

$$\begin{array}{rcccl}
d^{(1/2)}(J_+)|\frac{1}{2},+\frac{1}{2}> & = & \sqrt{\frac{1}{2}(\frac{1}{2}+1)-\frac{1}{2}(\frac{1}{2}+1)}|\quad > & = & 0 \ , \\
d^{(1/2)}(J_+)|\frac{1}{2},-\frac{1}{2}> & = & \sqrt{\frac{1}{2}(\frac{1}{2}+1)+\frac{1}{2}(-\frac{1}{2}+1)}|\frac{1}{2},+\frac{1}{2}> & = & |\frac{1}{2},+\frac{1}{2}> \ , \\
d^{(1/2)}(J_-)|\frac{1}{2},+\frac{1}{2}> & = & \sqrt{\frac{1}{2}(\frac{1}{2}+1)-\frac{1}{2}(\frac{1}{2}-1)}|\frac{1}{2},-\frac{1}{2}> & = & |\frac{1}{2},-\frac{1}{2}> \ \text{ and} \\
d^{(1/2)}(J_-)|\frac{1}{2},-\frac{1}{2}> & = & \sqrt{\frac{1}{2}(\frac{1}{2}+1)+\frac{1}{2}(-\frac{1}{2}-1)}|\quad > & = & 0 \ .
\end{array} \tag{10.47}$$

Consequently, the raising and lowering operators $J_\pm$ are here represented by the following matrices:

$$d^{(1/2)}(J_+)=\left(\begin{array}{cc}0 & 1\\0 & 0\end{array}\right)=\tfrac{1}{2}(\sigma_1+i\sigma_2) \quad \text{and} \quad d^{(1/2)}(J_-)=\left(\begin{array}{cc}0 & 0\\1 & 0\end{array}\right)=\tfrac{1}{2}(\sigma_1-i\sigma_2) \quad . \tag{10.48}$$

This leads for the matrix representations of J_1 and J_2 furthermore to the result:

$$d^{(1/2)}(J_1)=\frac{\sigma_1}{2} \quad \text{and} \quad d^{(1/2)}(J_2)=\frac{\sigma_2}{2} \quad . \tag{10.49}$$

The Casimir operator J^2, which is defined in formula (10.43), is for $j=\frac{1}{2}$ represented by $\frac{3}{4}\mathbf{1}$, in agreement with:

$$J^2|\tfrac{1}{2},m>=\tfrac{1}{2}(\tfrac{1}{2}+1)|\tfrac{1}{2},m>=\tfrac{3}{4}|\tfrac{1}{2},m> \quad . \tag{10.50}$$

The matrices of the representation $d^{(1/2)}$ are exactly the same as the matrices of the definition of the Lie-algebra of $SU(2)$.

10.7 The three-dimensional irrep $\{3\}$.

For $j = 1$ one has the three-dimensional irrep $d^{(1)}$ of $SU(2)$. We denote the basis vectors by:

$$|1,+1>=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad,\quad |1,0>=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad\text{and}\quad |1,-1>=\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{10.51}$$

The generator J_3 is at this basis represented by a diagonal matrix. Its eigenvalues, as indicated by the notation for the basis vectors in formula (10.51), are respectively $+1$, 0 and -1. So, we find for $d^{(1)}(J_3)$ the following matrix:

$$d^{(1)}(J_3)=\begin{pmatrix}+1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\quad. \tag{10.52}$$

For the raising and lowering operators we have at the basis (10.51), using formula (10.42), the matrices:

$$d^{(1)}(J_+)=\sqrt{2}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\quad\text{and}\quad d^{(1)}(J_-)=\sqrt{2}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}\quad. \tag{10.53}$$

Notice that those matrices are not exactly equal to the corresponding matrix representations (9.48) of $SO(3)$, because the choice of the basis (9.46) for $SO(3)$ is not the same as the above choice (10.51) for $SU(2)$.

The representation of the group elements $U(\hat{n},\alpha)$ of $SU(2)$ is, according to the relation (10.16), given by:

$$D^{(1)}(U(\hat{n},\alpha))=e^{-i\alpha\hat{n}\cdot d^{(1)}(\vec{J})}\quad. \tag{10.54}$$

For example for $U(\hat{z},\alpha)$, also using formula (10.52), one has:

$$D^{(1)}(U(\hat{z},\alpha))=\begin{pmatrix}e^{-i\alpha}&0&0\\0&1&0\\0&0&e^{+i\alpha}\end{pmatrix}\quad. \tag{10.55}$$

Now, the group elements $U(\hat{z},\alpha)$ and $U(\hat{z},\alpha+2\pi)$ are not equal, but their matrix representations as given in the above formula (10.55), are equal. The representation $D^{(1)}$ is therefore not faithfull. This is true for all odd dimensions. The reason is that there are allways two different group elements $U(\hat{n},\alpha)$ and $U(\hat{n},\alpha+2\pi)$ which correspond to one rotation $R(\hat{n},\alpha)$ (see formula 10.30) and that the odd dimensional irreps are equivalent for $SO(3)$ and $SU(2)$.

In order to determine the matrix representation of J^2 we use the following relation (see also formula 9.32):

$$J^2 = (J_3)^2 + J_3 + J_- J_+ \quad . \tag{10.56}$$

Using also the formulas (10.52) and (10.53), we find for the Casimir operator in this case:

$$J^2 = 2\mathbf{1} = 1(1+1)\mathbf{1} \quad . \tag{10.57}$$

10.8 The four-dimensional irrep $\{4\}$.

For $j = \frac{3}{2}$ one has the four-dimensional irrep $d^{(3/2)}$ of $SU(2)$. We denote the basis vectors by:

$$|\tfrac{3}{2},+\tfrac{3}{2}>= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \; |\tfrac{3}{2},+\tfrac{1}{2}>= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \; |\tfrac{3}{2},-\tfrac{1}{2}>= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } |\tfrac{3}{2},-\tfrac{3}{2}>= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{10.58}$$

The generator J_3 is at this basis represented by a diagonal matrix. Its eigenvalues, as indicated by the notation for the basis vectors in formula (10.58), are respectively $+\frac{3}{2}$, $+\frac{1}{2}$, $-\frac{1}{2}$ and $-\frac{3}{2}$. So, we find for $d^{(3/2)}(J_3)$ the following matrix:

$$d^{(3/2)}(J_3) = \tfrac{1}{2} \begin{pmatrix} +3 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \quad . \tag{10.59}$$

For the raising and lowering operators we have at the basis (10.58), using formula (10.42), the matrices:

$$d^{(3/2)}(J_+) = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d^{(3/2)}(J_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \quad . \tag{10.60}$$

Using the formulas (10.56), (10.52) and (10.53), we find for the Casimir operator in this case:

$$J^2 = \tfrac{15}{4}\mathbf{1} = \tfrac{3}{2}(\tfrac{3}{2}+1)\mathbf{1} \quad . \tag{10.61}$$

10.9 The product space $\{2\} \otimes \{3\}$.

As an example of the representation of $SU(2)$ in a product space, we select here the product space $D^{(1/2)} \otimes D^{(1)}$. The basis vectors of this space are, by means of the basis vectors of the two bases (10.45) and (10.51), defined by:

$$|\tfrac{1}{2}, +\tfrac{1}{2} > \otimes |1, m >= \begin{pmatrix} |1, m > \\ 0 \end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2}, -\tfrac{1}{2} > \otimes |1, m >= \begin{pmatrix} 0 \\ |1, m > \end{pmatrix} \quad . \tag{10.62}$$

Before we continue, for typographical reasons, we first simplify the notation of the basis vectors to:

$$|m_1 > |m_2 >= |\tfrac{1}{2}, m_1 > \otimes |1, m_2 > \quad . \tag{10.63}$$

In a more explicit form one obtains for the basis defined in (10.62), column vectors of length six, $\hat{e}_1$, . . . , $\hat{e}_6$, where $\hat{e}_1$ represents a column vector which has only zeroes except for a 1 in the upper position, $\hat{e}_2$ represents a column vector which has only zeroes except for a 1 in the second position, etc., *i.e.*

$$\begin{array}{lll} | + \tfrac{1}{2} > | + 1 >= \hat{e}_1 \ , & | + \tfrac{1}{2} > | \, 0 >= \hat{e}_2 \ , & | + \tfrac{1}{2} > | - 1 >= \hat{e}_3 \ , \\ | - \tfrac{1}{2} > | + 1 >= \hat{e}_4 \ , & | - \tfrac{1}{2} > | \, 0 >= \hat{e}_5 \ \text{and} & | - \tfrac{1}{2} > | - 1 >= \hat{e}_6 \ . \end{array} \tag{10.64}$$

Using the matrix representations $D^{(1/2)}(U(\vec{n}))$ and $D^{(1)}(U(\vec{n}))$ of a group element $U(\vec{n})$ of $SU(2)$ in the spaces respectively defined by the bases (10.45) and (10.51), we let $U(\vec{n})$ in the product space $D^{(1/2)} \otimes D^{(1)}$ be represented by the following matrix:

$$D(U(\vec{n})) = \begin{pmatrix} \left[D^{(1/2)}(U(\vec{n}))\right]_{11} D^{(1)}(U(\vec{n})) & \left[D^{(1/2)}(U(\vec{n}))\right]_{12} D^{(1)}(U(\vec{n})) \\ \left[D^{(1/2)}(U(\vec{n}))\right]_{21} D^{(1)}(U(\vec{n})) & \left[D^{(1/2)}(U(\vec{n}))\right]_{22} D^{(1)}(U(\vec{n})) \end{pmatrix} \quad . \tag{10.65}$$

The above matrix represents a 6×6 matrix, because $D^{(1)}(U(\vec{n}))$ stands for a 3×3 matrix. The matrix (10.65) acts as follows at the basis vectors (10.62):

$$\begin{aligned} & D(U(\vec{n}))| + \tfrac{1}{2} > |m >= \\ &= D(U(\vec{n})) \begin{pmatrix} |1, m > \\ 0 \end{pmatrix} = \begin{pmatrix} \left[D^{(1/2)}(U(\vec{n}))\right]_{11} D^{(1)}(U(\vec{n}))|1, m > \\ \left[D^{(1/2)}(U(\vec{n}))\right]_{12} D^{(1)}(U(\vec{n}))|1, m > \end{pmatrix} \\ &= D^{(1/2)}(U(\vec{n})) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes D^{(1)}(U(\vec{n}))|1, m > \\ &= \left\{D^{(1/2)}(U(\vec{n}))|\tfrac{1}{2}, +\tfrac{1}{2} >\right\} \otimes \left\{D^{(1)}(U(\vec{n}))|1, m >\right\} \quad , \end{aligned}$$

and similar:

$$D(U(\vec{n}))|-\tfrac{1}{2}>|m>=\left\{D^{(1/2)}(U(\vec{n}))|\tfrac{1}{2},-\tfrac{1}{2}>\right\}\otimes\left\{D^{(1)}(U(\vec{n}))|1,m>\right\} \quad . \tag{10.66}$$

In terms of the generators (see formula 10.16), we have

$$\begin{aligned} e^{-i\vec{n}\cdot d(\vec{J})} &= D(U(\vec{n})) \qquad (10.67) \\ &= D^{(1/2)}(U(\vec{n}))\otimes D^{(1)}(U(\vec{n})) = e^{-i\vec{n}\cdot d^{(1/2)}(\vec{J})}\otimes e^{-i\vec{n}\cdot d^{(1)}(\vec{J})} \quad , \end{aligned}$$

from which formula one deduces for the representation of the generators in the product space, the following:

$$\begin{aligned} d(J_i) &= i\left\{\left.\frac{\partial D(U(\vec{n}))}{\partial n_i}\right|_{\vec{n}=0}\right\} \\ &= i\left\{\left.\frac{\partial D^{(1/2)}(U(\vec{n}))}{\partial n_i}\right|_{\vec{n}=0}\right\}\otimes D^{(1)}(U(\vec{n}=0)) + \\ &\qquad + iD^{(1/2)}(U(\vec{n}=0))\otimes\left\{\left.\frac{\partial D^{(1)}(U(\vec{n}))}{\partial n_i}\right|_{\vec{n}=0}\right\} \\ &= d^{(1/2)}(J_i)\otimes\mathbf{1}_{3\times 3}+\mathbf{1}_{2\times 2}\otimes d^{(1)}(J_i) \qquad (10.68) \end{aligned}$$

So, the representation $d(O)$ in the product space $D^{(1/2)}\otimes D^{(1)}$ of an arbitrary element O of the Lie-algebra of $SU(2)$ is, using the irreps $d^{(1/2)}$ and $d^{(1)}$, given by the following transformation rule:

$$d(O)|m_1>|m_2>=\left\{d^{(1/2)}(O)|m_1>\right\}\ |m_2>+|m_1>\left\{d^{(1)}(O)|m_2>\right\} \quad . \tag{10.69}$$

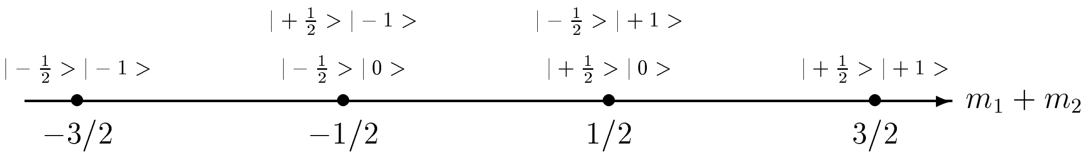


Figure 10.1: The weight diagram for the product representation $D^{(1/2)}\otimes D^{(1)}$. The dots at the horizontal line represent as indicated, the basis vectors (10.63) of the product vector space. The values below the dots represent their eigenvalues for $d(J_3)$.

At the basis (10.64) one obtains for J_3 obviously a diagonal matrix, which is given by:

$$d(J_3)|m_1>|m_2>=(m_1+m_2)|m_1>|m_2> \quad . \tag{10.70}$$

The weight diagram which results from formula (10.70) is depicted in figure (10.1). It does not have the form of a possible weight diagram for an irrep of $SU(2)$, since there exist different eigenvectors of $d(J_3)$ with the same eigenvalue. Consequently, the representation of $SU(2)$ in the product space $D^{(1/2)} \otimes D^{(1)}$ must be reducible. The principal issue of this section is therefore the reduction of the representation d into irreps of $SU(2)$.

To that aim we first determine the matrix representation of J^2 in the product space, which for arbitrary representations neither is proportional to the unit matrix, nor diagonal. Diagonalization of the resulting matrix J^2 leads to another basis of the product space, at which basis the matrix representations of the elements of the Lie-algebra obtain the form shown in formula (2.34).

We use the relation (10.56) for J^2. The representation of J_3 is, according to the result (10.70), diagonal at the basis (10.62). So, $d((J_3)^2) = [d(J_3)]^2$ is also diagonal and can be determined using formula (10.70). Consequently, what is left to be calculated for the representation of J^2 as given by the expression (10.56), is the product of $d(J_-)$ and $d(J_+)$.

At the basis (10.64), using the results (10.48) for $d^{(1/2)}(J_-)$ and (10.53) for $d^{(1)}(J_-)$, one finds for $d(J_-)$ the following matrix:

$$
\begin{aligned}
d(J_-) &= d^{(1/2)}(J_-) \otimes \mathbf{1}_{3\times 3} + \mathbf{1}_{2\times 2} \otimes d^{(1)}(J_-) \\
&= \begin{pmatrix} 0 & 0 \\ & \\ \mathbf{1}_{3\times 3} & 0 \end{pmatrix} + \begin{pmatrix} d^{(1)}(J_-) & 0 \\ & \\ 0 & d^{(1)}(J_-) \end{pmatrix} \\
&= \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \sqrt{2} & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \sqrt{2} & \cdot
\end{pmatrix} ,
\end{aligned}
\tag{10.71}
$$

where the dots in the latter matrix represent zeroes.

Let us verify one of the columns of the matrix obtained in formula (10.71). We determine by means of the definition (10.69) for the representation of an arbitrary element of the Lie-algebra of $SU(2)$, the transformation of one of the basis vectors (10.64):

$$
\begin{aligned}
d(J_-)\hat{e}_2 &= d(J_-)|+\tfrac{1}{2}>|0> \\
&= |-\tfrac{1}{2}>|0> + \sqrt{2}|+\tfrac{1}{2}>|-1> = \hat{e}_5 + \sqrt{2}\hat{e}_3 \ .
\end{aligned}
$$

The result corresponds to the second column of the matrix (10.71).

For the representation of the raising operator J_+ in the product space $D^{(1/2)} \otimes D^{(1)}$, one finds similarly the transposed matrix:

$$d(J_+) = \begin{pmatrix} \cdot & \sqrt{2} & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \sqrt{2} & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad . \tag{10.72}$$

Next, one may determine the product of the matrices (10.71) and (10.72) and add the result to the sum of the square of the matrix for $d(J_3)$ and the matrix for $d(J_3)$ itself, to find according to formula (10.56)for J^2, the following matrix representation in the product space:

$$J^2 = \begin{pmatrix} \frac{15}{4} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{11}{4} & \cdot & \sqrt{2} & \cdot & \cdot \\ \cdot & \cdot & \frac{7}{4} & \cdot & \sqrt{2} & \cdot \\ \cdot & \sqrt{2} & \cdot & \frac{7}{4} & \cdot & \cdot \\ \cdot & \cdot & \sqrt{2} & \cdot & \frac{11}{4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{15}{4} \end{pmatrix} \quad . \tag{10.73}$$

The resulting matrix is not even diagonal, let aside being proportional to the unit matrix. In studying this matrix, one finds that out of the six basis vectors (10.64), two are eigenvectors of J^2, *i.e.*

$$J^2|\pm\tfrac{1}{2}>|\pm 1> = \tfrac{15}{4}|\pm\tfrac{1}{2}>|\pm 1> = \tfrac{3}{2}(\tfrac{3}{2}+1)|\pm\tfrac{1}{2}>|\pm 1> \quad . \tag{10.74}$$

The eigenvalues for $d(J_3)$ for those two vectors can be read from equation (10.70) to be equal to $\pm\frac{3}{2}$, which leads us to conclude that the two basis vectors (10.74) of the product space form also basis vectors of the irrep $d^{(3/2)}$ of $SU(2)$ for $j = \frac{3}{2}$. Therefore we identify:

$$|\tfrac{3}{2},\pm\tfrac{3}{2}> = |\pm\tfrac{1}{2}>|\pm 1> \quad . \tag{10.75}$$

Now, we might search for the other eigenvectors of J^2 in order to obtain the remaining basis vectors of the four dimensional irrep $d^{(3/2)}$ of $SU(2)$ and the basis vectors of the other, yet unknown, irreps. But, there exists a more elegant method: Starting from one basis vector, one can construct the complete basis for an irrep by repeatedly applying the raising or lowering operator.

Let us start with the basis vector $|\frac{3}{2},+\frac{3}{2}>$ and apply the representation of the lowering operator to it. For the lefthand side of equation (10.75) we find:

$$d^{(3/2)}(J_-)|\tfrac{3}{2},+\tfrac{3}{2}> = \sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\,|\tfrac{3}{2},+\tfrac{1}{2}> = \sqrt{3}|\tfrac{3}{2},+\tfrac{1}{2}> \quad . \tag{10.76}$$

And applying $d(J_-)$ to the righthand side of equation (10.75), one obtains:

$$d(J_-)|+\tfrac{1}{2}>|+1> = |-\tfrac{1}{2}>|+1> + \sqrt{2}|+\tfrac{1}{2}>|0> \quad . \tag{10.77}$$

The results (10.76) and (10.77) lead to the identification:

$$|\tfrac{3}{2},+\tfrac{1}{2}>=\sqrt{\tfrac{1}{3}}|-\tfrac{1}{2}>|+1>+\sqrt{\tfrac{2}{3}}|+\tfrac{1}{2}>|0> \quad . \tag{10.78}$$

The eigenvalue of $d(J_3)$ for the vector at the righthand side of this formula equals $+\frac{1}{2}$, since both terms in the linear combination have that eigenvalue for $d(J_3)$. Using the matrix representation (10.73) for J^2 in the product space and the identifications (10.64), then we find that the righthand side of (10.78) is also an eigenvector of J^2 with eigenvalue $\frac{15}{4}$, as expected.

Applying next $d^{(3/2)}(J_-)$ to the lefthand side of (10.78) and $d(J_-)$ to the righthand side, we come to the identification:

$$|\tfrac{3}{2},-\tfrac{1}{2}>=\sqrt{\tfrac{2}{3}}|-\tfrac{1}{2}>|0>+\sqrt{\tfrac{1}{3}}|+\tfrac{1}{2}>|-1> \quad . \tag{10.79}$$

So, with the above procedure we found the four basis vectors (10.75), (10.78) and (10.79) of the four dimensional irrep $\{4\}$ of $SU(2)$ for $j=\frac{3}{2}$.

At the subspace of the product space, which is spanned by the basis vectors which have eigenvalue $+\frac{1}{2}$ for $d(J_3)$, *i.e.*

$$|\tfrac{1}{2},+\tfrac{1}{2}>\otimes|1,0> \quad \text{and} \quad |\tfrac{1}{2},-\tfrac{1}{2}>\otimes|1,+1> \quad .$$

we found one linear combination, (10.78), which is an eigenvector of J^2. A possible vector orthogonal to that linear combination is given by:

$$\sqrt{\tfrac{1}{3}}|\tfrac{1}{2},+\tfrac{1}{2}>\otimes|1,0>-\sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}>\otimes|1,+1> \quad .$$

This vector is also an eigenvector of J^2 with eigenvalue $\frac{3}{4}$. Its eigenvalue for $d(J_3)$ is evidently equal to $+\frac{1}{2}$. Consequently, we may identify:

$$|\tfrac{1}{2},+\tfrac{1}{2}>=\sqrt{\tfrac{1}{3}}|+\tfrac{1}{2}>|0>-\sqrt{\tfrac{2}{3}}|-\tfrac{1}{2}>|+1> \quad . \tag{10.80}$$

Applying $d^{(1/2)}(J_-)$ to the lefthand side and $d(J_-)$ to the righthand side, gives us the other basis vector of the two-dimensional irrep $d^{(1/2)}$ of $SU(2)$, *i.e.*

$$|\tfrac{1}{2},-\tfrac{1}{2}>=\sqrt{\tfrac{2}{3}}|+\tfrac{1}{2}>|-1>-\sqrt{\tfrac{1}{3}}|-\tfrac{1}{2}>|0> \quad . \tag{10.81}$$

When we perform a basis transformation in the product space from the old basis (10.64) to a new basis given by the expressions (10.75), (10.78), (10.79), (10.80) and (10.81), then one obtains for the matrix representations of all elements of the algebra and hence of the group, matrices of the form of a direct sum of a 4×4 matrix equivalent to the irrep $d^{(3/2)}$ and a 2×2 matrix equivalent to $d^{(1/2)}$. Consequently, the product space $D^{(1/2)}\otimes D^{(1)}$ yields a representation which is equivalent to the direct sum of the four-dimensional and the two-dimensional irreps of $SU(2)$.

For example, when we select in the direct product space $\{2\}\otimes\{3\}$ the basis given by:

$$\begin{aligned}
\hat{e}'_1 &= |+\tfrac{1}{2}>|+1> \ , \\
\hat{e}'_2 &= \sqrt{\tfrac{1}{3}}|-\tfrac{1}{2}>|+1>+\sqrt{\tfrac{2}{3}}|+\tfrac{1}{2}>|0> \ ,
\end{aligned}$$

$$
\begin{aligned}
\hat{e}'_3 &= \sqrt{\tfrac{2}{3}}|-\tfrac{1}{2}>|\,0> +\sqrt{\tfrac{1}{3}}|+\tfrac{1}{2}>|-1> \ , \\
\hat{e}'_4 &= |-\tfrac{1}{2}>|-1> \ , \\
\hat{e}'_5 &= \sqrt{\tfrac{1}{3}}|+\tfrac{1}{2}>|\,0> -\sqrt{\tfrac{2}{3}}|-\tfrac{1}{2}>|+1> \quad \text{and} \\
\hat{e}'_6 &= \sqrt{\tfrac{2}{3}}|+\tfrac{1}{2}>|-1> -\sqrt{\tfrac{1}{3}}|-\tfrac{1}{2}>|\,0> \ ,
\end{aligned}
\tag{10.82}
$$

then we find for all matrices $d'(O)$ which represent elements O of the Lie-algebra of $SU(2)$ at the new basis (10.82), the form:

$$
d'(O) = \begin{pmatrix} d^{(3/2)}(O) & 0 \\ \\ 0 & d^{(1/2)}(O) \end{pmatrix} \ . \tag{10.83}
$$

For J^2 we obtain at the new basis (10.82), the matrix representation given by:

$$
J^2 = \begin{pmatrix} \frac{15}{4}\mathbf{1}_{4\times 4} & 0 \\ \\ 0 & \frac{3}{4}\mathbf{1}_{2\times 2} \end{pmatrix} = \begin{pmatrix} J^2(j=\frac{3}{2}) & 0 \\ \\ 0 & J^2(j=\frac{1}{2}) \end{pmatrix} \ . \tag{10.84}
$$

For the matrix representations $D'(U)$ at the basis (10.82) of all group elements U of $SU(2)$ we find, using the fact that the product of two matrices which have the form (10.83) also has that form and hence the exponent $\exp\{-iO\}$, the general expression:

$$
D'(U) = \begin{pmatrix} D^{(3/2)}(U) & 0 \\ \\ 0 & D^{(1/2)}(U) \end{pmatrix} \ . \tag{10.85}
$$

This result may symbolically be represented by:

$$
\{2\} \otimes \{3\} = \{4\} \oplus \{2\} \ . \tag{10.86}
$$

10.10 Clebsch-Gordan or Wigner coefficients.

The identifications between the basis vectors (10.62) of the product space $\{2\}\otimes\{3\}$ and the basis vectors of the irreps $\{4\}$ and $\{2\}$, involve the linear combinations given in the formulas (10.75), (10.78), (10.79), (10.80) and (10.81). In general, let us for the two irreps $d^{(j_1)}$ and $d^{(j_2)}$ consider the product space given by:

$$\{2j_1+1\}\otimes\{2j_2+1\} \quad . \tag{10.87}$$

Let moreover, in the reduction of the representation d in this product space, appear the irrep $d^{(j)}$ of $SU(2)$. The identification of a basis vector $|j,m>$ with a linear combination of basis vectors of the product space (10.87) can then be written as:

$$|j,m>=\sum_{m_1,m_2} C\begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} |j_1,m_1>\otimes|j_2,m_2> \quad . \tag{10.88}$$

The coefficients $C\{j_1,j_2,j;m_1,m_2,m\}$ in this linear combination are called *Clebsch-Gordan* or *Wigner* coefficients.

In formula (10.75) we find the following Clebsch-Gordan coefficients:

$$C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{3}{2} \\ +\frac{1}{2} & +1 & +\frac{3}{2} \end{array} = 1 \quad \text{and} \quad C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & -1 & -\frac{3}{2} \end{array} = 1 \quad ,$$

in formula (10.78)

$$C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & +1 & +\frac{1}{2} \end{array} = \sqrt{\tfrac{1}{3}} \quad \text{and} \quad C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{3}{2} \\ +\frac{1}{2} & 0 & +\frac{1}{2} \end{array} = \sqrt{\tfrac{2}{3}} \quad ,$$

in formula (10.79)

$$C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{3}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{array} = \sqrt{\tfrac{2}{3}} \quad \text{and} \quad C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{3}{2} \\ +\frac{1}{2} & -1 & -\frac{1}{2} \end{array} = \sqrt{\tfrac{1}{3}} \quad ,$$

in formula (10.80)

$$C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{1}{2} \\ +\frac{1}{2} & 0 & +\frac{1}{2} \end{array} = \sqrt{\tfrac{1}{3}} \quad \text{and} \quad C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & +1 & +\frac{1}{2} \end{array} = -\sqrt{\tfrac{2}{3}} \quad ,$$

and in formula (10.81)

$$C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{1}{2} \\ +\frac{1}{2} & -1 & -\frac{1}{2} \end{array} = \sqrt{\tfrac{2}{3}} \quad \text{and} \quad C\begin{array}{ccc} \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{array} = -\sqrt{\tfrac{1}{3}} \quad ,$$

From the construction it might be clear that the Clebsch-Gordan coefficient given by $C(\frac{1}{2},1,\frac{1}{2};+\frac{1}{2},0,+\frac{1}{2})$, which relates the various basis vectors given in formula (10.80), could as well have been chosen negative and even complex. It is however convention to choose one specific Clebsch-Gordan coefficient in each irrep $d^{(j)}$ which is contained in the direct product $\{2j_1+1\}\otimes\{2j_2+1\}$, according to the following condition:

$$C^{j_1 \quad j_2 \quad j}_{j_1 \quad j-j_1 \quad j} \quad \text{real and positive.} \tag{10.89}$$

Notice that also the identification (10.75) implies the rather arbitrary, although the most obvious, choice $C(\frac{1}{2}, 1, \frac{3}{2}; +\frac{1}{2}, +1, +\frac{3}{2}) = +1$, only at this stage explained by the above condition.

As a consequence of condition (10.89) are all Clebsch-Gordan coefficients real. In that case is the matrix which describes the transformation from the orthonormal basis of the product space to the orthonormal bases of the various irreps, orthogonal. The matrix elements of this matrix are the Clebsch-Gordan coefficients (10.88). Consequently, using the orthonormality conditions of rows and columns of an orthogonal matrix, one obtains:

$$\sum_{m_1, m_2} C^{j_1 \quad j_2 \quad j}_{m_1 \quad m_2 \quad m} C^{j_1 \quad j_2 \quad j'}_{m_1 \quad m_2 \quad m'} = \delta_{jj'}\delta_{mm'}$$

and

$$\sum_{j, m} C^{j_1 \quad j_2 \quad j}_{m_1 \quad m_2 \quad m} C^{j_1 \quad j_2 \quad j}_{m_1' \quad m_2' \quad m} = \delta_{m_1 m_1'}\delta_{m_2 m_2'} \ . \tag{10.90}$$

Notice furthermore the following property of Clebsch-Gordan coefficients:

$$C^{j_1 \quad j_2 \quad j}_{m_1 \quad m_2 \quad m} = 0 \quad \text{for} \quad m_1 + m_2 \neq m \ . \tag{10.91}$$

However, it is not generally true that Clebsch-Gordan coefficients are nonzero if $m_1 + m_2 = m$; for example:

$$C^{1 \quad 1 \quad 1}_{0 \quad 0 \quad 0} = 0 \ .$$

Some of the most useful symmetry relations for the Clebsch-Gordan coefficients are summarized by the equation:

$$C^{j_2 \quad j_1 \quad j}_{m_2 \quad m_1 \quad m} = (-1)^{j-j_1-j_2} C^{j_1 \quad j_2 \quad j}_{m_1 \quad m_2 \quad m} = C^{j_1 \quad j_2 \quad j}_{-m_1 \quad -m_2 \quad -m} \ . \tag{10.92}$$

10.11 The Lie-algebra of $SU(2)$.

The generator subspace of the Lie-algebra of $SU(2)$ consists of purely imaginary linear combinations of the basic generators $\vec{J} = \vec{\sigma}/2$. Linear combinations which are not purely imaginary, like the raising and the lowering operators $J_{\pm}$, do not represent generators of the group, but can be very useful for the construction and classification of irreps. For $SU(2)$, we define the following set of traceless *standard* matrices A_{ij} $(i,j = 1,2)$, by:

$$\left[A_{ij}\right]_{k\ell} = \delta_{ik}\delta_{j\ell} - \tfrac{1}{2}\delta_{ij}\delta_{k\ell} \quad \text{for} \quad i,j,k,\ell = 1,2. \tag{10.93}$$

Their relation with the generators $\vec{J}$ is given by:

$$\begin{array}{rclcl} A_{11} &=& \frac{1}{2}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} &=& J_3 \quad , \quad A_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = J_+ \ , \\ A_{21} &=& \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &=& J_- \quad \text{and} \quad A_{22} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} = -J_3 \ . \end{array} \tag{10.94}$$

Using their definition (10.93) or alternatively the above explicit expressions, it is easy to verify the following commutation relations for the standard matrices:

$$\left[A_{ij}, A_{k\ell}\right] = \delta_{jk}A_{i\ell} - \delta_{i\ell}A_{kj} \ . \tag{10.95}$$

The Casimir operator J^2 can, using the relations (9.32) and (10.94), be expressed in terms of the standard matrices, *i.e.*

$$\begin{aligned} J^2 &= (J_3)^2 + \tfrac{1}{2}(J_+J_- + J_-J_+) = \tfrac{1}{2}\left\{(J_3)^2 + J_+J_- + J_-J_+ + (-J_3)^2\right\} \\ &= \tfrac{1}{2}A_{ij}A_{ji} \ . \end{aligned} \tag{10.96}$$

It is easy to show that this operator commutes with all standard matrices (10.93) and hence with all generators of $SU(2)$. Using the commutation relations (10.95), we find:

$$\begin{aligned} 2\left[J^2, A_{k\ell}\right] &= \left[A_{ij}A_{ji}, A_{k\ell}\right] = A_{ij}\left[A_{ji}, A_{k\ell}\right] + \left[A_{ij}, A_{k\ell}\right]A_{ji} \\ &= A_{ij}(\delta_{ik}A_{j\ell} - \delta_{j\ell}A_{ki}) + (\delta_{jk}A_{i\ell} - \delta_{i\ell}A_{kj})A_{ji} = 0 \end{aligned} \tag{10.97}$$

In general, one might, using the standard matrices (10.93), construct similar operators which commute with all generators of $SU(2)$. For example, it is straightforward to demonstrate that the operator $A_{ij}A_{jk}A_{ki}$ commutes with all standard matrices and so, could perfectly serve as a Casimir operator. However, one has, using the commutation relations (9.28), the identity:

$$
\begin{aligned}
A_{ij}A_{jk}A_{ki} &= A_{11}A_{11}A_{11} + A_{11}A_{12}A_{21} + A_{12}A_{21}A_{11} + A_{12}A_{22}A_{21} + \\
&\quad + A_{21}A_{11}A_{12} + A_{21}A_{12}A_{22} + A_{22}A_{21}A_{12} + A_{22}A_{22}A_{22} \\
&= (J_3)^3 + J_3J_+J_- + J_+J_-J_3 + J_+(-J_3)J_- + \\
&\quad + J_-J_3J_+ + J_-J_+(-J_3) + (-J_3)J_-J_+ + (-J_3)^3 \\
&= J_3\left[J_+, J_-\right] + \left[J_+, J_-\right]J_3 + J_+\left[J_-, J_3\right] + \left[J_-J_3, J_+\right] \\
&= 2\left\{(J_3)^2 + \tfrac{1}{2}(J_+J_- + J_-J_+)\right\} = A_{ij}A_{ji} \quad . \qquad (10.98)
\end{aligned}
$$

Consequently, no new Casimir operator results from the contraction of three standard matrices.

10.12 The product space $\{2\} \otimes \{2\} \otimes \{2\}$.

At the basis (10.45) for the irrep $D^{(1/2)}$ of $SU(2)$, the matrix representations of the generators and hence of all elements of the Lie-algebra, are identical to the defining matrices (10.15). So, for the standard matrices (10.93), we must also have:

$$
\left[d^{(1/2)}(A_{ij})\right]_{k\ell} = \delta_{ik}\delta_{j\ell} - \tfrac{1}{2}\delta_{ij}\delta_{k\ell} \quad . \qquad (10.99)
$$

The transformation of the basis vectors (10.45), which we shall denote here by respectively $\hat{e}_1$ and $\hat{e}_2$, is then given by:

$$
\hat{e}'_\ell = \left[d^{(1/2)}(A_{ij})\right]_{k\ell}\hat{e}_k \quad . \qquad (10.100)
$$

Let us study the reduction into irreps of the product space $\{2\} \otimes \{2\} \otimes \{2\}$. We denote the basis of this space by:

$$
E_{abc} = \hat{e}_a \otimes \hat{e}_b \otimes \hat{e}_c \quad \text{for} \quad a, b, c = 1, 2. \qquad (10.101)
$$

The representation $d(A_{ij})$ of standard matrix A_{ij} in this product space, is defined by (compare formula 10.68):

$$
d(A_{ij})E_{abc} = \left[d^{(1/2)}(A_{ij})\right]_{\lambda a}E_{\lambda bc} + \left[d^{(1/2)}(A_{ij})\right]_{\lambda b}E_{a\lambda c} + \left[d^{(1/2)}(A_{ij})\right]_{\lambda c}E_{ab\lambda} \quad . \qquad (10.102)
$$

Using relation (10.99), one obtains the following explicit expression:

$$
\begin{aligned}
d(A_{ij})E_{abc} &= (\delta_{i\lambda}\delta_{ja} - \tfrac{1}{2}\delta_{ij}\delta_{a\lambda})E_{\lambda bc} + (\delta_{i\lambda}\delta_{jb} - \tfrac{1}{2}\delta_{ij}\delta_{b\lambda})E_{a\lambda c} + \\
&\quad + (\delta_{i\lambda}\delta_{jc} - \tfrac{1}{2}\delta_{ij}\delta_{c\lambda})E_{ab\lambda} \\
&= \delta_{ja}E_{ibc} + \delta_{jb}E_{aic} + \delta_{jc}E_{abi} - \tfrac{3}{2}\delta_{ij}E_{abc} \qquad (10.103)
\end{aligned}
$$

In order to determine the representation of the Casimir operator (10.96) in the product space, we first, using the expression (10.103), determine:

$$
\begin{aligned}
2J^2E_{abc} &= d(A_{ji})d(A_{ij})E_{abc} \\
&= \delta_{ja}d(A_{ji})E_{ibc} + \delta_{jb}d(A_{ji})E_{aic} + \delta_{jc}d(A_{ji})E_{abi} + \\
&\qquad - \tfrac{3}{2}\delta_{ij}d(A_{ji})E_{abc} \\
&= \delta_{ja}\left\{\delta_{ii}E_{jbc} + \delta_{ib}E_{ijc} + \delta_{ic}E_{ibj} - \tfrac{3}{2}\delta_{ji}E_{ibc}\right\} + \\
&\quad + \delta_{jb}\left\{\delta_{ia}E_{jic} + \delta_{ii}E_{ajc} + \delta_{ic}E_{aij} - \tfrac{3}{2}\delta_{ji}E_{aic}\right\} + \\
&\quad + \delta_{jc}\left\{\delta_{ia}E_{jbi} + \delta_{ib}E_{aji} + \delta_{ii}E_{abj} - \tfrac{3}{2}\delta_{ji}E_{abi}\right\} + \\
&\quad - \tfrac{3}{2}\delta_{ij}\left\{\delta_{ia}E_{jbc} + \delta_{ib}E_{ajc} + \delta_{ic}E_{abj} - \tfrac{3}{2}\delta_{ji}E_{abc}\right\} \\
&= 2E_{abc} + E_{bac} + E_{cba} - \tfrac{3}{2}E_{abc} + \\
&\quad + E_{bac} + 2E_{abc} + E_{acb} - \tfrac{3}{2}E_{abc} + \\
&\quad + E_{cba} + E_{acb} + 2E_{abc} - \tfrac{3}{2}E_{abc} + \\
&\quad - \tfrac{3}{2}E_{abc} - \tfrac{3}{2}E_{abc} - \tfrac{3}{2}E_{abc} + \tfrac{9}{2}E_{abc} \\
&= \left[3(2-\tfrac{3}{2}) - \tfrac{3}{2}(3 - 2\cdot\tfrac{3}{2})\right]E_{abc} + 2\left[E_{acb} + E_{cba} + E_{bac}\right] \\
&= 2\left\{\tfrac{3}{4}(4-3)E_{abc} + \sum_{\sigma} E_{\sigma(abc)}\right\} \ ,
\end{aligned} \tag{10.104}
$$

where $\sigma(abc)$ stands for the interchange of two indices and where the summation is over all possible combinations in which two indices are interchanged.

Using formula (10.104), we find in the product space $\{2\}\otimes\{2\}\otimes\{2\}$ at the basis (10.101), for the Casimir operator (10.96) the following representation:

$$
J^2E_{abc} = \tfrac{1}{2}d(A_{ji})d(A_{ij})E_{abc}\tfrac{3}{4}E_{abc} + E_{acb} + E_{cba} + E_{bac} \ . \tag{10.105}
$$

In order to find the reduction of the representation d of $SU(2)$ in the product space $\{2\}\otimes\{2\}\otimes\{2\}$ into irreps of $SU(2)$, we must first determine the matrices of the representation of the Casimir operator and of the representation of $J_3 = A_{11}$ at the basis (10.101).

For $J_3 = A_{11}$, we find in the product space, using formula (10.103), the transformation:

$$
d(A_{11})E_{abc} = \delta_{1a}E_{1bc} + \delta_{1b}E_{a1c} + \delta_{1c}E_{ab1} - \tfrac{3}{2}E_{abc} = \left\{\delta_{1a} + \delta_{1b} + \delta_{1c} - \tfrac{3}{2}\right\}E_{abc} \ . \tag{10.106}
$$

We find that the basis vectors (10.101) are eigenvectors of A_{11}. Consequently, at the basis vectors (10.101) the matrix representation of $A_{11} = J_3$ is diagonal. The eigenvalues of $d(A_{11})$ as given in formula (10.106) are collected in table (10.1) below. From this table we read that the eigenvalues for the representation of J_3 take values $\pm\frac{3}{2}$ and $\pm\frac{1}{2}$. So, we may expect that in the reduction of the representation d will appear doublets (*i.e.* $\{2\}$) and quadruplets (*i.e.* $\{4\}$).

vector T	111	112	121	122	211	212	221	222
eigenvalue	$+\frac{3}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$

Table 10.1: The eigenvalues of $d(A_{11})$ as defined in formula (10.106), for the various basis vectors (10.101).

Notice from the results of table (10.1) that, as also can be read from formula (10.106), there is a direct relation between the number of times that appears a 1 or a 2 in its indices, and the eigenvalue of a basis vector.

The action of the transformation $J^2 = \frac{1}{2}d(A_{ji})d(A_{ij})$ at the basis vectors (10.101) is presented in table (10.2) below.

initial basis vector E	transformed vector $J^2E = \frac{1}{2}d(A_{ji})d(A_{ij})E$
111	$\frac{15}{4}E_{111}$
112	$\frac{7}{4}E_{112} + E_{121} + E_{211}$
121	$\frac{7}{4}E_{121} + E_{112} + E_{211}$
122	$\frac{7}{4}E_{122} + E_{212} + E_{221}$
211	$\frac{7}{4}E_{211} + E_{121} + E_{112}$
212	$\frac{7}{4}E_{212} + E_{122} + E_{221}$
221	$\frac{7}{4}E_{221} + E_{212} + E_{122}$
222	$\frac{15}{4}E_{222}$

Table 10.2: The transformations of the Casimir operator as defined in formula (10.105), for the various basis vectors (10.101).

Notice from the results of table (10.2), that the number of times it appears a 1 or a 2 in the indices of the basis vectors involved in one linear combination which represents a transformed basis vector, is a constant.

The matrix for the Casimir operator is not diagonal at the basis (10.101). But, we notice from table (10.2) that the Casimir operator has two eigenvectors, namely E_{111} and E_{222}, both with eigenvalue $\frac{15}{4} = \frac{3}{2}(\frac{3}{2}+1)$. Their eigenvalues for $d(J_3 = A_{11})$ are respectively equal to $+3/2$ and $-3/2$ (see table 10.1). Consequently, we might identify:

$$|\tfrac{3}{2},+\tfrac{3}{2}>= E_{111} \quad \text{and} |\tfrac{3}{2},-\tfrac{3}{2}>= E_{222} \ . \tag{10.107}$$

Repeated action of $d(J_- = A_{21})$ on E_{111} or $d(J_+ = A_{12})$ on E_{222}, using formula (10.103), leads us to the other two basis vectors for the four-dimensional irrep $\{4\}$ contained in the product space given by (10.101). The final result is collected in table (10.3).

$\{4\}$	$\{2\} \otimes \{2\} \otimes \{2\}$
$\|\tfrac{3}{2},+\tfrac{3}{2}>$	E_{111}
$\|\tfrac{3}{2},+\tfrac{1}{2}>$	$\sqrt{\tfrac{1}{3}}\{E_{211}+E_{121}+E_{112}\}$
$\|\tfrac{3}{2},-\tfrac{1}{2}>$	$\sqrt{\tfrac{1}{3}}\{E_{221}+E_{212}+E_{122}\}$
$\|\tfrac{3}{2},-\tfrac{3}{2}>$	E_{222}

Table 10.3: The irrep $\{4\}$ contained in $\{2\} \otimes \{2\} \otimes \{2\}$.

In the space spanned by the basis vectors E_{211}, E_{121} and E_{112}, which have eigenvalue $+\frac{1}{2}$ for $d(J_3 = A_{11})$, we construct two more orthonormal basis vectors perpendicular to $\sqrt{\frac{1}{3}}\{E_{211}+E_{121}+E_{112}\}$, *i.e.*

$$\sqrt{\tfrac{1}{2}}\{E_{211}-E_{121}\} \quad \text{and} \quad \sqrt{\tfrac{1}{6}}\{E_{211}+E_{121}-2E_{112}\} \ . \tag{10.108}$$

So, we find two subspaces of the product space $\{2\} \otimes \{2\} \otimes \{2\}$ in which the representation d is equivalent to the two-dimensional irrep $\{2\}$. Using formula (10.103) for $d(J_- = A_{21})$, we find the other basis vector in each of the two subspaces. The resulting bases are collected in table (10.4).

$\{2\}$	$\{2\} \otimes \{2\} \otimes \{2\}$ (I)	$\{2\} \otimes \{2\} \otimes \{2\}$ (II)
$\|\tfrac{1}{2},+\tfrac{1}{2}>$	$\sqrt{\tfrac{1}{2}}\{E_{211}-E_{121}\}$	$\sqrt{\tfrac{1}{6}}\{E_{211}+E_{121}-2E_{112}\}$
$\|\tfrac{1}{2},-\tfrac{1}{2}>$	$\sqrt{\tfrac{1}{2}}\{E_{212}-E_{122}\}$	$\sqrt{\tfrac{1}{6}}\{2E_{221}-E_{212}-E_{122}\}$

Table 10.4: The two irreps $\{2\}$ contained in $\{2\} \otimes \{2\} \otimes \{2\}$.

When we select the basis:

$$\begin{aligned}
\hat{e}'_1 &= E_{111} \ , \\
\hat{e}'_2 &= \sqrt{\tfrac{1}{3}}\{E_{211}+E_{121}+E_{112}\} \ , \\
\hat{e}'_3 &= \sqrt{\tfrac{1}{3}}\{E_{221}+E_{212}+E_{122}\} \ , \\
\hat{e}'_4 &= E_{222} \ ,
\end{aligned}$$

$$\begin{aligned}
\hat{e}'_5 &= \sqrt{\tfrac{1}{2}}\{E_{211} - E_{121}\} \ , \\
\hat{e}'_6 &= \sqrt{\tfrac{1}{2}}\{E_{212} - E_{122}\} \ , \\
\hat{e}'_7 &= \sqrt{\tfrac{1}{6}}\{E_{211} + E_{121} - 2E_{112}\} \ \text{and} \\
\hat{e}'_8 &= \sqrt{\tfrac{1}{6}}\{2E_{221} - E_{212} - E_{122}\} \ ,
\end{aligned}$$

then we find for the representation $D'(U)$ of all group elements U of $SU(2)$ in the product space $\{2\} \otimes \{2\} \otimes \{2\}$, the form:

$$D'(U) = \begin{pmatrix} D^{(3/2)}(U) & 0 & 0 \\ 0 & D^{(1/2)}(U) & 0 \\ 0 & 0 & D^{(1/2)}(U) \end{pmatrix} \ . \tag{10.109}$$

Symbolically, we may write this result as:

$$\{2\} \otimes \{2\} \otimes \{2\} = \{4\} \oplus \{2\} \oplus \{2\} \ . \tag{10.110}$$

10.13 Tensors for $SU(2)$.

The basis vectors of the irrep $\{4\}$, which are collected in table (10.3), all have the property that interchanging two indices does not alter the expression. Let us study this phenomenon in a bit more detail.

An arbitrary vector $\boldsymbol{T}$ in the product space $\{2\} \otimes \{2\} \otimes \{2\}$ can be written as a linear combination of the basis vectors (10.101), as follows:

$$\boldsymbol{T} = T_{ijk} E_{ijk} \ . \tag{10.111}$$

When the coefficients T in (10.111) are symmetric in their indices, *i.e.* when:

$$T_{112} = T_{121} = T_{211} \quad \text{and} \quad T_{122} = T_{212} = T_{221} \quad ,$$

then we find for the vector $\boldsymbol{T}$ (10.111) the expression:

$$\boldsymbol{T} = T_{111}E_{111} + T_{112}\{E_{211} + E_{121} + E_{112}\} + T_{122}\{E_{221} + E_{212} + E_{122}\} + T_{222}E_{222} \ . \tag{10.112}$$

Consequently, the vector $\boldsymbol{T}$ is entirely in the subspace spanned by the basis vectors of the irrep $\{4\}$ given in table (10.3). Under any transformation $D(U)$, for U in $SU(2)$, such a vector transforms into a vector $\boldsymbol{T}$' which is also entirely in that subspace. So, the subspace of the irrep $\{4\}$ can be defined as the space of all vectors $\boldsymbol{T}$ which have symmetric coefficients T.

The objects $\boldsymbol{T}$ of formula (10.111) are usually not called vectors but *tensors* for $SU(2)$.

Notice that $\{4\}$ has the highest possible dimension of an irrep contained in the product space $\{2\} \otimes \{2\} \otimes \{2\}$. Its eigenvalue j for the Casimir operator can be determined from:

$$j = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{3}{2} \ . \tag{10.113}$$

10.14 Irreducible tensors for $SU(2)$.

In general, one might study tensor product spaces of length p, given by the direct product of p times the defining space $\{2\}$ of $SU(2)$, *i.e.*

$$\{2\} \otimes \{2\} \otimes \cdots \otimes \{2\} \quad p \ \text{ times.} \tag{10.114}$$

The basis vectors of this space can, in analogy with expression (10.101) be represented by:

$$E_{i_1 i_2 \ldots i_p} = \hat{e}_{i_1} \otimes \hat{e}_{i_2} \otimes \cdots \otimes \hat{e}_{i_p} \quad \text{for } \ i_1, i_2, \ldots, i_p = 1, 2. \tag{10.115}$$

Tensors $\boldsymbol{T}$ are linear combinations of those basis vectors and take the general form:

$$\boldsymbol{T} = T_{i_1 i_2 \ldots i_p} E_{i_1 i_2 \ldots i_p} \ . \tag{10.116}$$

Such tensor is a vector in the 2^p-dimensional product space given in (10.114) and is said to have *rank* p.

For the representations $d(A_{ij})$ of the standard matrices at the basis given in formula (10.115), we find (compare formulas 10.102 and 10.103):

$$\begin{aligned} d(A_{ij}) E_{i_1 i_2 \ldots i_p} &= \\ &= \left[d^{(1/2)}(A_{ij}) \right]_{k i_1} E_{k i_2 \ldots i_p} + \left[d^{(1/2)}(A_{ij}) \right]_{k i_2} E_{i_1 k i_3 \ldots i_p} + \cdots \\ &\qquad \cdots + \left[d^{(1/2)}(A_{ij}) \right]_{k i_p} E_{i_1 i_2 \ldots i_{p-1} k} \\ &= \delta_{j i_1} E_{i i_2 \ldots i_p} + \cdots + \delta_{j i_p} E_{i_1 i_2 \ldots i_{p-1} i} - \frac{p}{2} \delta_{ij} E_{i_1 i_2 \ldots i_p} \ . \end{aligned} \tag{10.117}$$

And for the Casimir operator (compare formula 10.104), we find:

$$\tfrac{1}{2} d(A_{ij}) d(A_{ji}) E_{i_1 i_2 \ldots i_p} = \tfrac{1}{4} p(4-p) E_{i_1 i_2 \ldots i_p} + \tfrac{1}{2} \sum_{\sigma} E_{\sigma(i_1 i_2 \ldots i_p)} \ , \tag{10.118}$$

where $\sigma(i_1 i_2 \ldots i_p)$ stands for the interchange of two of the indices. The summation is over all possible combinations where two indices are interchanged. Notice that there are $p(p-1)$ such combinations.

Let us consider the subspace of the product space (10.114) which is spanned by all tensors $\boldsymbol{T}$ (10.116) for which the coefficients T are symmetric under the interchange of any pair of indices. It is easy to understand that such tensors are eigenvectors for the Casimir operator (10.118). Their eigenvalues are given by:

$$\tfrac{1}{2}d(A_{ij})d(A_{ji})\boldsymbol{T} = \left\{\tfrac{1}{4}p(4-p) + \tfrac{1}{2}p(p-1)\right\}\boldsymbol{T} = \frac{p}{2}(\frac{p}{2}+1)\boldsymbol{T} \quad . \tag{10.119}$$

Consequently, symmetric tensors span the subspace of the product space in which the vectors transform according to the irrep for $j = p/2$ of $SU(2)$. Notice, that $j = p/2$ is the maximum value for j possible in the product space given by formula (10.114).

Using formula (10.117) one can also easily find the two eigenvectors of $d(J_3 = A_{11})$ which have the maximum and minimum eigenvalues, *i.e.*

$$d(A_{11})E_{11\cdots 1} = +\frac{p}{2}E_{11\cdots 1} \quad \text{and} \quad d(A_{11})E_{22\cdots 2} = -\frac{p}{2}E_{22\cdots 2} \quad . \tag{10.120}$$

Conclusion: "*The vector spaces for the irreps $\{2j+1\}$ of $SU(2)$ are formed by the symmetric tensors of rank $p = 2j$*".

Chapter 11

The special unitary group in three dimensions.

In this chapter we study the representations of the special unitary group in three dimensions, $SU(3)$.

11.1 The generators.

The generators H of $SU(3)$ are traceless and Hermitean 3×3 matrices (see formulas 6.26 and 7.49), and thus have the following general form:

$$H(a,b;\alpha,\beta,\gamma)=\begin{pmatrix} a & \alpha^* & \beta^* \\ \alpha & b & \gamma^* \\ \beta & \gamma & -a-b \end{pmatrix} \quad \begin{array}{l}\text{with } a \text{ and } b \text{ real, and} \\ \alpha,\ \beta \text{ and } \gamma \text{ complex.}\end{array} \tag{11.1}$$

Now, since a and b are real and α, β and γ complex, such a matrix has $2+6=8$ free parameters. Consequently, $SU(3)$ is an eight parameter group and hence has eight independent generators. A possible set is the Gell-Mann set of generators:

$$I_1\ ,\ \ I_2\ ,\ \ I_3\ ,\ \ K_1\ ,\ \ K_2\ ,\ \ L_1\ ,\ \ L_2\ ,\ \ \text{and}\ \ M\ , \tag{11.2}$$

given by the following matrices:

$$I_1=\tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,\ I_2=\tfrac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,\ I_3=\tfrac{1}{2}\begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,$$

$$K_1=\tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\ ,\ K_2=\tfrac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}\ ,\ L_1=\tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\ ,$$

$$L_2=\tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\ ,\ \text{ and }\ M=\tfrac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\ . \tag{11.3}$$

The arbitrary traceless and Hermitean 3×3 matrix H given in formula (11.1), can be written as a linear combination of the Gell-Mann set of generators, as follows:

$$\begin{aligned} H(a,b;\alpha,\beta,\gamma) &= 2\Re e(\alpha)I_1 + 2\Im m(\alpha)I_2 + (a-b)I_3 + 2\Re e(\beta)K_1 + \\ &\quad + 2\Im m(\beta)K_2 + 2\Re e(\gamma)L_1 + 2\Im m(\gamma)L_2 + (a+b)\sqrt{3}M \ , \end{aligned} \tag{11.4}$$

where the symbols $\Re e(\ )$ and $\Im m(\ )$ stand respectively for the real and imaginary part of a complex number. When this matrix is exponentiated, we obtain an arbitrary unitary 3×3 matrix U with unit determinant, *i.e.*

$$U(a,b;\alpha,\beta,\gamma) = e^{iH(a,b;\alpha,\beta,\gamma)} \ . \tag{11.5}$$

The set of eight Gell-Mann generators forms a basis of the Lie-algebra of $SU(3)$. We define in this algebra, the following linear combinations of the generators:

$$I_\pm = I_1 \pm iI_2 \ , \ K_\pm = K_1 \pm iK_2 \ \text{ and } \ L_\pm = L_1 \pm iL_2 \ . \tag{11.6}$$

They satisfy the following commutation relations with I_3 and M (compare to the commutation relations given in formula 9.28):

$$\begin{array}{lcllcll} \left[I_3\ ,\ I_\pm\right] &=& \pm I_\pm & , \ \left[M\ ,\ I_\pm\right] &=& 0 & , \\ \left[I_3\ ,\ K_\pm\right] &=& \pm\frac{1}{2}K_\pm & , \ \left[M\ ,\ K_\pm\right] &=& \pm\frac{\sqrt{3}}{2}K_\pm & , \\ \left[I_3\ ,\ L_\pm\right] &=& \mp\frac{1}{2}L_\pm & , \ \left[M\ ,\ L_\pm\right] &=& \pm\frac{\sqrt{3}}{2}L_\pm & . \end{array} \tag{11.7}$$

The operators defined in formula (11.6) will show very useful in the construction of irreducible representations.

11.2 The representations of $SU(3)$.

The subset of generators $\{\ I_1,\ I_2,\ I_3\ \}$ satisfies the same type of commutation relations as the generators of $SU(2)$ (see formula 10.17), *i.e.*

$$\left[I_i\ ,\ I_j\right] = i\epsilon_{ijk}I_k \quad . \tag{11.8}$$

They generate a subgroup of $SU(3)$ which is equivalent to $SU(2)$ and which is called the subgroup for *isospin*. In the following we will study representations of $SU(3)$ for which the bases are chosen such that I_3 is represented by a diagonal matrix. We have then moreover the operator I^2, given by:

$$I^2 = (I_1)^2 + (I_2)^2 + (I_3)^2 \quad , \tag{11.9}$$

which commutes with the generators I_1, I_2 and I_3, *i.e.*

$$\left[I^2\ ,\ I_i\right] = 0 \ \text{ for } \ i = 1,2,3. \tag{11.10}$$

This operator does however not commute with all other generators and hence is not a Casimir operator for $SU(3)$. But, it distinguishes within the irreps of $SU(3)$ the so-called *isomultiplets*, as we will see in the following.

Using the representation (11.3) of the generators (11.2), we find that M commutes with I_1, I_2 and I_3, *i.e.*

$$\left[M\ ,\ I_i\right] = 0 \ \text{ for } \ i = 1,2,3. \tag{11.11}$$

This implies that M commutes with I^2, *i.e.*

$$\left[I^2\ ,\ M\right] = 0 \quad . \tag{11.12}$$

Relation (11.11) implies also that it is possible to construct such a basis for a representation of $SU(3)$ that both $d(I_3)$ and $d(M)$ are diagonal. So, we might indicate the basis vectors for an irrep of $SU(3)$ by their eigenvalues i_3 and m of respectively $d(I_3)$ and $d(M)$ and moreover by their eigenvalues $i(i+1)$ of I^2, *i.e.*:

$$\begin{aligned} d(I_3)|i,i_3,m> &= i_3|i,i_3,m> \ , \\ d(M)|i,i_3,m> &= m|i,i_3,m> \quad \text{and} \\ I^2|i,i_3,m> &= i(i+1)|i,i_3,m> \quad . \end{aligned} \tag{11.13}$$

The unitary matrices generated by I_3, form an $U(1)$ subgroup of $SU(3)$. Consequently, the eigenvalues, i_3, can take any integer or half-integer value (see the discussion preceding formula 10.37), *i.e.*

$$i_3 = \ 0,\ \pm\tfrac{1}{2},\ \pm1,\ \pm\tfrac{3}{2},\ \pm2, \ldots \tag{11.14}$$

The unitary matrices generated by M also form an $U(1)$ subgroup of $SU(3)$ and which, because of relation (11.11) commute with the unitary matrices generated by

I_3. Due to the normalization of (11.3) of M, the eigenvalues which m can take are given by:

$$m = 0,\ \pm\frac{\sqrt{3}}{6},\ \pm\frac{\sqrt{3}}{3}\ \pm\frac{\sqrt{3}}{2},\ \pm\frac{2\sqrt{3}}{3},\ldots \tag{11.15}$$

Now, because of the commutation relations (11.11) and (11.12), the basis vectors within an irrep which have the same eigenvalue m for M, can be subdivided into *isomultiplets*, which are sets of basis vectors which have the same eigenvalue $i(i+1)$ for I^2. Within each isomultiplet the values for i_3 run from $-i$ to $+i$ with integer steps, because of the commutation relations (11.7) for I_3 and $I\pm$. The operators $I_\pm$, $K_\pm$ and $L_\pm$ defined in (11.6), function as step operators at the basis of an irrep of $SU(3)$, because of the commutation relations (11.7). At the above basis vectors (11.13) one obtains:

$$\begin{aligned}
d(I_\pm)|i,i_3,m> &= \sqrt{i(i+1)-i_3(i_3\pm 1)}\,|i,i_3\pm 1,m> \ , \\
d(K_\pm)|i,i_3,m> &= A_\pm(i,i_3,m)\,|i+\tfrac{1}{2},i_3\pm\tfrac{1}{2},m\pm\tfrac{\sqrt{3}}{2}> + \\
& B_\pm(i,i_3,m)\,|i-\tfrac{1}{2},i_3\pm\tfrac{1}{2},m\pm\tfrac{\sqrt{3}}{2}> \quad \text{and} \\
d(L_\pm)|i,i_3,m> &= C_\pm(i,i_3,m)\,|i+\tfrac{1}{2},i_3\mp\tfrac{1}{2},m\pm\tfrac{\sqrt{3}}{2}> + \\
& D_\pm(i,i_3,m)\,|i-\tfrac{1}{2},i_3\mp\tfrac{1}{2},m\pm\tfrac{\sqrt{3}}{2}> \quad .
\end{aligned} \tag{11.16}$$

The matrix elements A, B, C and D are rather complicated functions of i, i_3 and m and moreover of some other "quantum" numbers characterizing the irrep under consideration. We will discuss them in some more detail at a later stage (see section 11.11 formula 11.68). Notice that since I^2 does not commute with $K_\pm$ and $L_\pm$, the eigenvalues for I^2 change under the action of $d(K_\pm)$ and $d(L_\pm)$ at the basis vectors. In general one even might obtain mixtures of eigenvectors for I^2.

To define uniquely the matrix elements A, B, C and D of formula (11.16), certain conventions about the relative phases between the basis vectors within an irrep have to be established. For the basis vectors within the same isomultiplet, we take the standard Condon and Shortley phase convention. This says that the nonzero matrix elements of $d(I_\pm)$ are all real and positive. This convention defines uniquely the phases between the different basis vectors within the same isomultiplet. But leaves undetermined an overall phase between the various isomultiplets of an irrep.

The relative phases between the different isomultiplets are defined by the requirement that the nonzero matrix elements of $d(K_\pm)$ are all positive.

So, we find from formula (11.16) the following transformation rules:

$$\begin{aligned}
d(I_\pm) &: \ \Delta i = 0 \ , \ \Delta i_3 = \pm 1 \ , \ \Delta m = 0 \ , \\
d(K_\pm) &: \ |\Delta i| = \tfrac{1}{2} \ , \ \Delta i_3 = \pm\tfrac{1}{2} \ , \ \Delta m = \pm\tfrac{\sqrt{3}}{2} \ \text{ and} \\
d(L_\pm) &: \ |\Delta i| = \tfrac{1}{2} \ , \ \Delta i_3 = \mp\tfrac{1}{2} \ , \ \Delta m = \pm\tfrac{\sqrt{3}}{2} \ .
\end{aligned} \tag{11.17}$$

The above formula (11.17) summarizes the center piece of the irreducible representations of $SU(3)$. It shows the sizes and the directions of the steps which lead from one basis vector to another.

11.3 Weight diagrams for $SU(3)$.

A weight diagram for an irrep of $SU(3)$ has two dimensions. Along the horizontal axis of the plane we indicate i_3, the eigenvalues of $d(I_3)$ and along the vertical axis we indicate m, the eigenvalues of $d(M)$. For an example, we show in figure (11.1) the weight diagram for the ten-dimensional irreducible representation $\{10\}$ of $SU(3)$.

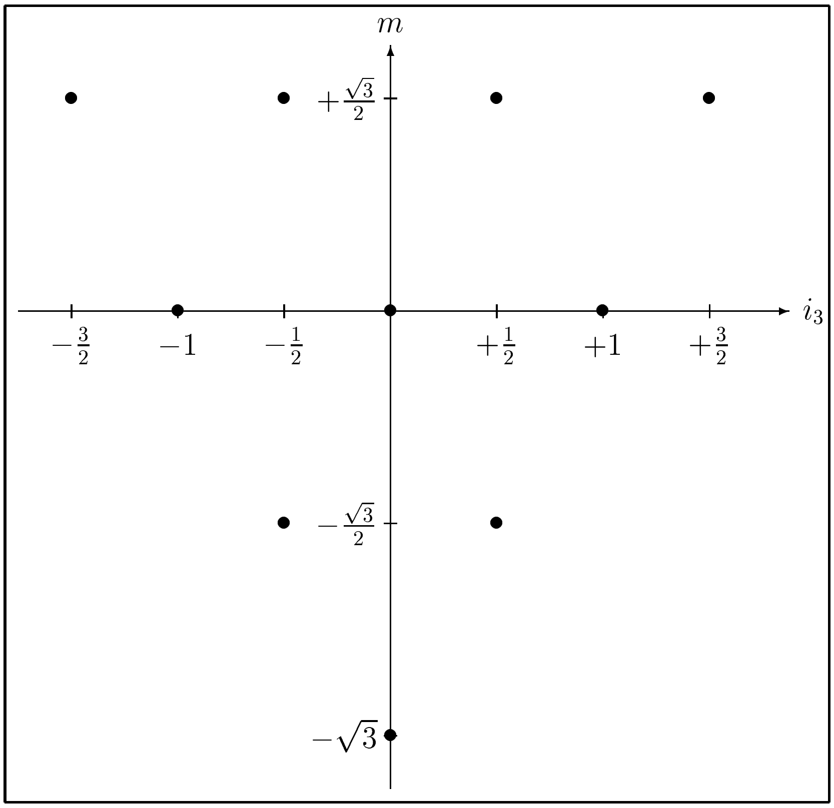


Figure 11.1: The weight diagram for the ten-dimensional irrep $\{10\}$ of $SU(3)$. The ten dots in the figure represent the ten basis vectors of this irrep.

For the eigenvalue $m = +\frac{\sqrt{3}}{2}$ of $d^{\{10\}}(M)$ we find an isoquartet with $i = \frac{3}{2}$ for the eigenvalues $i_3 = -\frac{3}{2}$, $i_3 = -\frac{1}{2}$, $i_3 = +\frac{1}{2}$ and $i_3 = +\frac{3}{2}$ for $d^{\{10\}}(I_3)$. Restricted to the set of generators $\{ I_1, I_2, I_3 \}$ the four basis vectors of this isoquartet transform like the irrep $\{4\}$ of $SU(2)$ under $d^{\{10\}}(I_1)$, $d^{\{10\}}(I_2)$ and $d^{\{10\}}(I_3)$. The step operator I_+ generates in this isoquartet the following sequence of transformations:

$$|\tfrac{3}{2},-\tfrac{3}{2},+\tfrac{\sqrt{3}}{2}> \stackrel{I_+}{\longrightarrow} |\tfrac{3}{2},-\tfrac{1}{2},+\tfrac{\sqrt{3}}{2}> \stackrel{I_+}{\longrightarrow} |\tfrac{3}{2},+\tfrac{1}{2},+\tfrac{\sqrt{3}}{2}> \stackrel{I_+}{\longrightarrow} |\tfrac{3}{2},+\tfrac{3}{2},+\tfrac{\sqrt{3}}{2}> \stackrel{I_+}{\longrightarrow} 0. \tag{11.18}$$

For the eigenvalue $m = 0$ of $d^{\{10\}}(M)$ we find an isotriplet with $i = 1$ for the eigenvalues $i_3 = -1$, $i_3 = 0$ and $i_3 = +1$ for $d^{\{10\}}(I_3)$. Restricted to the set of generators $\{ I_1, I_2, I_3 \}$ the three basis vectors of this isotriplet transform like the irrep $\{3\}$ of $SU(2)$ under $d^{\{10\}}(I_1)$, $d^{\{10\}}(I_2)$ and $d^{\{10\}}(I_3)$. The step operator I_+ generates in this isotriplet the following sequence of transformations:

$$|1,-1,0> \stackrel{I_+}{\longrightarrow} |1,0,0> \stackrel{I_+}{\longrightarrow} |1,+1,0> \stackrel{I_+}{\longrightarrow} 0. \qquad (11.19)$$

For the eigenvalue $m = -\frac{\sqrt{3}}{2}$ of $d^{\{10\}}(M)$ we find an isodoublet with $i = \frac{1}{2}$ for the eigenvalues $i_3 = -\frac{1}{2}$ and $i_3 = +\frac{1}{2}$ for $d^{\{10\}}(I_3)$. Restricted to the set of generators $\{\ I_1,\ I_2,\ I_3\ \}$ the two basis vectors of this isodoublet transform as the irrep $\{2\}$ of $SU(2)$ under $d^{\{10\}}(I_1)$, $d^{\{10\}}(I_2)$ and $d^{\{10\}}(I_3)$. The step operator I_+ generates in this isodoublet the following sequence of transformations:

$$|\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}> \stackrel{I_+}{\longrightarrow} |\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}> \stackrel{I_+}{\longrightarrow} 0. \qquad (11.20)$$

For the eigenvalue $m = -\sqrt{3}$ of $d^{\{10\}}(M)$ we find an isosinglet with $i = 0$ for the eigenvalue $i_3 = 0$ for $d^{\{10\}}(I_3)$. Restricted to the set of generators $\{\ I_1,\ I_2,\ I_3\ \}$ the only basis vectors of this isosinglet transforms like the trivial irrep $\{1\}$ of $SU(2)$ under $d^{\{10\}}(I_1)$, $d^{\{10\}}(I_2)$ and $d^{\{10\}}(I_3)$. The step operator I_+ generates in this isosinglet the following transformation:

$$|0,0,-\sqrt{3}> \stackrel{I_+}{\longrightarrow} 0. \qquad (11.21)$$

In figure (11.2) we show the directions of action of the various step operators defined in formula (11.6), in a weight diagram.

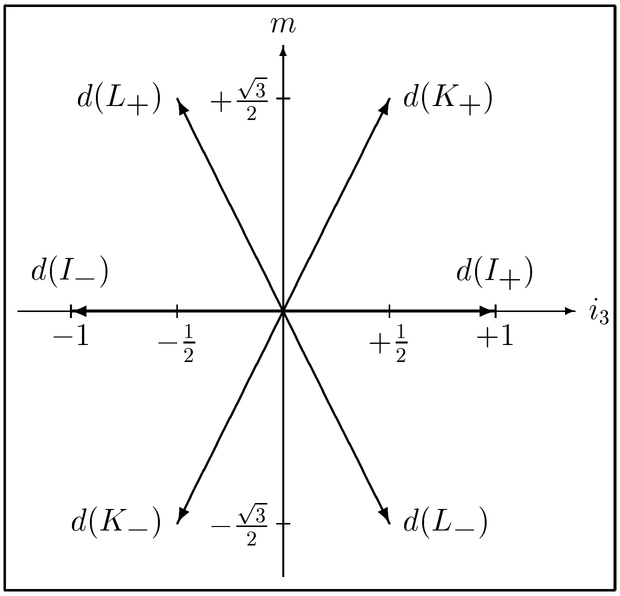


Figure 11.2: The action of the six step operators $d(I_\pm)$, $d(K_\pm)$ and $d(L_\pm)$ defined in formula (11.6), in the weight diagram for an irrep of $SU(3)$.

In the irrep $\{10\}$ of $SU(3)$ as show in figure (11.1), the step operator K_+ generates the following sequences:

$$|0,0,-\sqrt{3}> \stackrel{K_+}{\longrightarrow} |\tfrac{1}{2},+\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}> \stackrel{K_+}{\longrightarrow} |1,+1,0> \stackrel{K_+}{\longrightarrow} |\tfrac{3}{2},+\tfrac{3}{2},+\tfrac{\sqrt{3}}{2}> \stackrel{K_+}{\longrightarrow} 0 \quad ,$$

$$|\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{\sqrt{3}}{2}> \xrightarrow{K_+} |1,0,0> \xrightarrow{K_+} |\tfrac{3}{2},+\tfrac{1}{2},+\tfrac{\sqrt{3}}{2}> \xrightarrow{K_+} 0 \quad ,$$

$$|1,-1,0> \xrightarrow{K_+} |\tfrac{3}{2},-\tfrac{1}{2},+\tfrac{\sqrt{3}}{2}> \xrightarrow{K_+} 0 \quad \text{and}$$

$$|\tfrac{3}{2},-\tfrac{3}{2},+\tfrac{\sqrt{3}}{2}> \xrightarrow{K_+} 0 \qquad (11.22)$$

11.4 The Lie-algebra of $SU(3)$.

Like in the case of $SU(2)$ (see formula 10.93), we select for the basis of the Lie-algebra of $SU(3)$ a set of traceless standard matrices A_{ij} $(i,j=1,2,3)$, given by:

$$\left[A_{ij}\right]_{kl} = \delta_{ik}\delta_{jl} - \tfrac{1}{3}\delta_{ij}\delta_{kl} \ \text{ for } \ i,j,k,l = 1,2,3. \qquad (11.23)$$

Their relation to the generators (11.2) also using the definitions (11.6) for the step operators, is as follows:

$$\begin{array}{llllll} A_{11} = I_3 + \frac{1}{\sqrt{3}}M & , & A_{12} = I_+ & , & A_{13} = K_+ & , \\ A_{21} = I_- & , & A_{22} = -I_3 + \frac{1}{\sqrt{3}}M & , & A_{23} = L_+ & , \\ A_{31} = K_- & , & A_{32} = L_- & , & A_{33} = -\frac{2}{\sqrt{3}}M & . \end{array} \qquad (11.24)$$

The commutation relations for the standard matrices are given by (see also formula 10.95 for the standard matrices of $SU(2)$):

$$\left[A_{ij}\ ,\ A_{k\ell}\right] = \delta_{jk}A_{i\ell} - \delta_{i\ell}A_{kj} \quad . \qquad (11.25)$$

For $SU(3)$ one has two Casimir operators, given by:

$$F^2 = \tfrac{1}{2}A_{ij}A_{ji} \ \text{ and } \ G^3 = \tfrac{1}{2}\left\{A_{ij}A_{jk}A_{ki} + A_{ij}A_{ki}A_{jk}\right\} \quad . \qquad (11.26)$$

Unlike in the case of $SU(2)$ (see formula 10.98), the contraction of three standard matrices is in the case of $SU(3)$ not equal to the contraction of two standard matrices and hence give different operators.

When we use the representation (11.5) for an arbitrary unitary 3×3 matrix and formula (11.4), or alternatively formula (11.1), for its generator, then by defining:

$$\begin{array}{llllll} \alpha_{11} = 0 & , & \alpha_{12} = \alpha^* & , & \alpha_{13} = \beta^* & , \\ \alpha_{21} = \alpha & , & \alpha_{22} = b-a & , & \alpha_{23} = \gamma^* & , \\ \alpha_{31} = \beta & , & \alpha_{32} = \gamma & \text{and} & \alpha_{33} = -2a-b & , \end{array}$$

we may denote the arbitrary unitary 3×3 matrix U of formula (11.5) by:

$$U(\alpha_{ij}) = e^{i\alpha_{ij}A_{ij}} \quad . \qquad (11.27)$$

11.5 The three-dimensional irrep $\{3\}$ of $SU(3)$.

For the basis of the three-dimensional vector space V for the defining representation $\{3\}$ of $SU(3)$, we select the notation:

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad . \tag{11.28}$$

The unitary 3×3 matrices U of $SU(3)$ are represented by themselves, *i.e.*

$$D^{\{3\}}(U) = U \quad . \tag{11.29}$$

Consequently, the transformation of a vector $\vec{x} = x^i \hat{e}_i$ is given by:

$$D^{\{3\}}(U)\vec{x} = U_{ij} x^j \hat{e}_i \quad . \tag{11.30}$$

One might alternatively view this equation as the transformation of the basis vectors, given by:

$$\hat{e}'_j = D^{\{3\}}(U)\hat{e}_j = U_{ij}\hat{e}_i \quad , \tag{11.31}$$

or as a transformation of the components of the vector $\vec{x}$, given by:

$$x'^i = \left[D^{\{3\}}(U)\vec{x}\right]^i = U_{ij} x^j \quad . \tag{11.32}$$

The components of a vector transform *contra-variant*, *i.e.* the summation in (11.32) runs over the second index of U, whereas the summation in (11.31) for the basis vectors, runs over the first index of U.

At the level of the algebra, using formulas (11.23) and (11.31), we have for the standard matrices the following transformation rule at the basis vectors:

$$\hat{e}'_\ell = d^{\{3\}}(A_{ij})\hat{e}_\ell = \left[A_{ij}\right]_{k\ell}\hat{e}_k = \delta_{j\ell}\hat{e}_i - \tfrac{1}{3}\delta_{ij}\hat{e}_\ell \quad . \tag{11.33}$$

Notice that this transformation, except for the factor $\frac{1}{3}$, is the same as in the case of the defining representation of $SU(2)$ (see formula 10.100).

We are interested here in the eigenvalues for $d^{\{3\}}(I_3)$ and $d^{\{3\}}(M)$ for the various basis vectors (11.28), for which we assume that the corresponding matrices are diagonal. From the relations (11.24) we understand that it is sufficient to determine the matrices for $d^{\{3\}}(A_{11})$ and $d^{\{3\}}(A_{22})$. Hence, using formula (11.33), we find the following:

$$\begin{aligned} d^{\{3\}}(A_{11})\hat{e}_\ell &= \delta_{1\ell}\hat{e}_1 - \tfrac{1}{3}\hat{e}_\ell \quad , \\ d^{\{3\}}(A_{22})\hat{e}_\ell &= \delta_{2\ell}\hat{e}_2 - \tfrac{1}{3}\hat{e}_\ell \ \text{ and} \\ d^{\{3\}}(A_{33})\hat{e}_\ell &= \delta_{3\ell}\hat{e}_3 - \tfrac{1}{3}\hat{e}_\ell \quad . \end{aligned} \tag{11.34}$$

The resulting eigenvalues (using the relations 11.24) for the matrices $d^{\{3\}}(I_3) = \frac{1}{2}\left\{d^{\{3\}}(A_{11}) - d^{\{3\}}(A_{22})\right\}$ and $d^{\{3\}}(M) = \frac{\sqrt{3}}{2}\left\{d^{\{3\}}(A_{11}) + d^{\{3\}}(A_{22})\right\}$ are collected in table (11.1).

$\hat{e}$	$d^{\{3\}}(A_{11})$	$d^{\{3\}}(A_{22})$	$d^{\{3\}}(I_3)$	$d^{\{3\}}(M)$
$\hat{e}_1$	$\frac{2}{3}\hat{e}_1$	$-\frac{1}{3}\hat{e}_1$	$+\frac{1}{2}\hat{e}_1$	$\frac{\sqrt{3}}{6}\hat{e}_1$
$\hat{e}_2$	$-\frac{1}{3}\hat{e}_2$	$\frac{2}{3}\hat{e}_2$	$-\frac{1}{2}\hat{e}_2$	$\frac{\sqrt{3}}{6}\hat{e}_2$
$\hat{e}_3$	$-\frac{1}{3}\hat{e}_3$	$-\frac{1}{3}\hat{e}_3$	0	$-\frac{\sqrt{3}}{3}\hat{e}_3$

Table 11.1: The eigenvalues for $d^{\{3\}}(A_{11})$ and $d^{\{3\}}(A_{22})$ as defined in formula (11.34), and for $d^{\{3\}}(I_3)$ and $d^{\{3\}}(M)$, using the relations (11.24), for the basis vectors (11.28).

Notice from table (11.1) that the basis vectors $\hat{e}_1$ and $\hat{e}_2$ have the same eigenvalue with respect to $d^{\{3\}}(M)$, and respectively eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$ with respect to $d^{\{3\}}(I_3)$. The latter eigenvalues correspond to the eigenvalues for $\{2\}$ of $SU(2)$ with respect to $d^{\{2\}}(J_3)$ (compare formula 10.46).

According to the relations (11.24), we may associate the step operators to the operation of replacing indices: The Kronecker delta δ_{ij} in formula (11.33) vanishes for all step operators. The other Kronecker delta replaces the index ℓ by i in the case that j equals ℓ, or else vanishes. This process is expressed in the following equation:

$$d^{\{3\}}(A_{i\neq j})\hat{e}_\ell = \delta_{j\ell}\hat{e}_i = \begin{cases} \hat{e}_i & \text{if} \quad j = \ell \\ \\ 0 & \text{if} \quad j \neq \ell \end{cases} \quad . \tag{11.35}$$

The resulting transformations $d^{\{3\}}(I_\pm)$, $d^{\{3\}}(K_\pm)$ and $d^{\{3\}}(L_\pm)$ at the basis (11.28), are collected in table (11.2).

$\hat{e}$	$d^{\{3\}}(I_+)$	$d^{\{3\}}(I_-)$	$d^{\{3\}}(K_+)$	$d^{\{3\}}(K_-)$	$d^{\{3\}}(L_+)$	$d^{\{3\}}(L_-)$
$\hat{e}_1$	0	$\hat{e}_2$	0	$\hat{e}_3$	0	0
$\hat{e}_2$	$\hat{e}_1$	0	0	0	0	$\hat{e}_3$
$\hat{e}_3$	0	0	$\hat{e}_1$	0	$\hat{e}_2$	0

Table 11.2: The eigenvalues for the step operators defined in formula (11.6), using the relations (11.24) and (11.35), for the basis vectors (11.28).

Notice from table (11.2) that the isospin raising and lowering operators $d^{\{3\}}(I_+)$ and $d^{\{3\}}(I_-)$ connect the basis vectors $\hat{e}_1$ and $\hat{e}_2$ in the same way that the basis vectors of the irrep $\{2\}$ of $SU(2)$ are connected by the raising and lowering operators of $SU(2)$ (compare formula 10.47).

The information of tables (11.1) and (11.2) may be graphically represented in the weight diagram for $\{3\}$. This is depicted in figure (11.3).

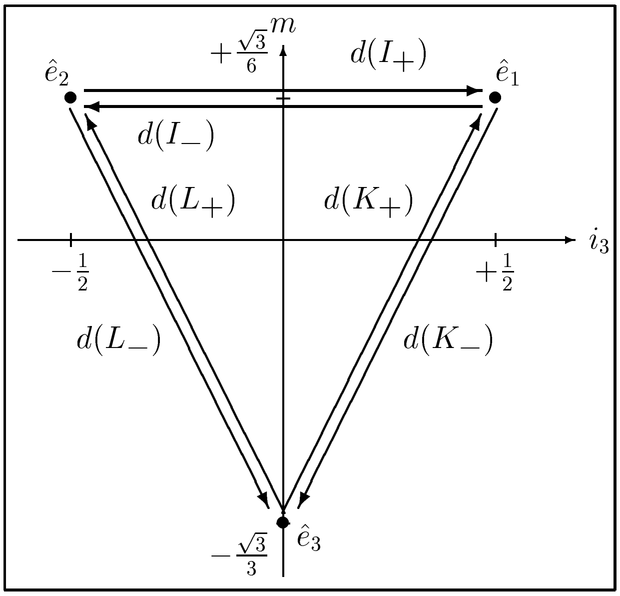


Figure 11.3: The action of the six step operators $d^{\{3\}}(I_\pm)$, $d^{\{3\}}(K_\pm)$ and $d^{\{3\}}(L_\pm)$ given in table (11.2), in the weight diagram for the irrep $\{3\}$ of $SU(3)$. The dots in the diagram represent the basis vectors (11.28). Their respective eigenvalues i_3 and m can be read from table (11.1).

From figure (11.3) one might guess that the irrep $\{3\}$ can be subdivided into an isodoublet ($\hat{e}_1$ and $\hat{e}_2$) and an isosinglet ($\hat{e}_3$). So, we might like to verify the values of i for the isomultiplet Casimir operator I^2. To this aim, we use the relation (compare the corresponding relations 9.32 and 10.56):

$$I^2 = (I_3)^2 + I_3 + I_-I_+ \quad . \tag{11.36}$$

Using the information of tables (11.2) for $d^{\{3\}}(I_3)$ and (11.1) for $d^{\{3\}}(I_-)$ and $d^{\{3\}}(I_+)$, we find:

$$I^2\hat{e}_1 = \tfrac{3}{4}\hat{e}_1 \ \ , \ \ I^2\hat{e}_2 = \tfrac{3}{4}\hat{e}_2 \ \text{ and } \ I^2\hat{e}_3 = 0 \quad . \tag{11.37}$$

Formula (11.37) indeed confirms that the set of basis vectors $\{\ \hat{e}_1,\ \hat{e}_2\ \}$ span the two-dimensional space of an isodoublet. Any linear combination of those two basis vectors transforms under the action of $d^{\{3\}}(I_1)$, $d^{\{3\}}(I_2)$ and $d^{\{3\}}(I_3)$ into another (or the same) linear combination of those two basis vectors. With respect to the three isospin operators $d^{\{3\}}(I_1)$, $d^{\{3\}}(I_2)$ and $d^{\{3\}}(I_3)$ the space spanned by $\hat{e}_1$ and $\hat{e}_2$ transforms like the irrep $\{2\}$ of $SU(2)$. Similarly, spans the basis vector $\hat{e}_3$ an one-dimensional isosinglet space (*i.e.* the trivial representation) for $SU(2)$ with respect to the three isospin operators.

Finally, we must determine the eigenvalues for the two Casimir operators of $SU(3)$ which are defined in formula (11.26). Also using the formulas (11.23) and (11.33), we determine:

$$
\begin{aligned}
F^2 \hat{e}_\ell &= \tfrac{1}{2} d^{\{3\}}(A_{ji}) d^{\{3\}}(A_{ij}) \hat{e}_\ell \\
&= \tfrac{1}{2} \left[d^{\{3\}}(A_{ji}) d^{\{3\}}(A_{ij}) \right]_{m\ell} \hat{e}m \\
&= \tfrac{1}{2} \left[A_{ji} \right]_{mk} \left[A_{ij} \right]_{k\ell} \hat{e}m \\
&= \tfrac{1}{2} \left(\delta_{jm}\delta_{ik} - \tfrac{1}{3}\delta_{ji}\delta_{mk} \right) \left(\delta_{ik}\delta_{j\ell} - \tfrac{1}{3}\delta_{ij}\delta_{k\ell} \right) \hat{e}m \\
&= \tfrac{4}{3} \hat{e}_\ell \ , \qquad (11.38)
\end{aligned}
$$

where we used moreover the identity $\delta_{ij}\delta_{ji} = \delta_{ii} = 3$.

So, we find the eigenvalue $\frac{4}{3}$ for F^2 at the irreducible representation $\{3\}$ of $SU(3)$. Similarly, one obtains, after some algebra, for the Casimir operator G^3 the result:

$$
G^3 \hat{e}_\ell = \tfrac{20}{9} \hat{e}_\ell \ . \qquad (11.39)
$$

We find the eigenvalue $\frac{20}{9}$ for the Casimir operator G^3 at the irrep $\{3\}$ of $SU(3)$.

11.6 The conjugate representation $\{3^*\}$.

In the case of $SU(2)$ the two-dimensional conjugate representation $D^{(*)}(U) = U^*$ of the 2×2 unitary matrices U is equivalent to the defining representation $D(U) = U$ of $SU(2)$. Using the representation of formula (10.2) of a 2×2 unitary matrix U, it is easy to demonstrate this, *i.e.*

$$\text{For } S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ one has } S^{-1}US = S^{-1}\begin{pmatrix} a^* & -b^* \\ b & a \end{pmatrix} S = \begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix} = U^* \ . \tag{11.40}$$

Consequently, the representation $D^{(*)}(U) = U^*$ is equivalent to the representation $D(U) = U$ for the 2×2 unitary matrices U of $SU(2)$.

However, in the case of 3×3 unitary matrices U this is not true, as can be seen from one example: The matrix S which solves the equation $S^{-1}US = U^*$ for the 3×3 unitary matrix U given by:

$$U = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i & 0 \end{pmatrix} \ ,$$

is equal to $S = 0$. Consequently, the conjugate representation for $SU(3)$ is not equivalent to the defining representation of $SU(3)$.

We denote the vector space in this case by V^* and its basis vectors by:

$$\hat{e}^1 \ , \ \hat{e}^2 \text{ and } \hat{e}^3 \ . \tag{11.41}$$

The unitary 3×3 matrices U of $SU(3)$ are represented by their complex conjugate matrices, *i.e.*

$$D^{\{3^*\}}(U) = U^* = (U^\dagger)^T = (U^{-1})^T \ . \tag{11.42}$$

Consequently, the transformation of a vector $\vec{y} = y_i\hat{e}^i$ is given by:

$$D^{\{3^*\}}(U)\vec{y} = (U^{-1})_{ij}y_i\hat{e}^j \ . \tag{11.43}$$

One might alternatively view this equation as the transformation of the basis vectors, given by:

$$\hat{e}'^{\,i} = D^{\{3^*\}}(U)\hat{e}^i = (U^{-1})_{ij}\hat{e}^j \ , \tag{11.44}$$

or as a transformation of the components of the vector $\vec{y}$, given by:

$$y'_j = \left[D^{\{3^*\}}(U)\vec{y}\right]_j = (U^{-1})_{ij}y_i \ . \tag{11.45}$$

The components of a vector transform *co-variant*, *i.e.* the summation in (11.45) runs over the first index of U, which is the same index of summation as for the transformation (11.31) of the basis vectors (11.28) of V.

In order to find the transformation rules for the standard matrices defined in (11.23), we must recall that using formula (11.27)

$$\left(\left[U(\alpha_{ij})\right]^{-1}\right)^T = e^{-i\alpha_{ij}(A_{ij})^T} \quad . \tag{11.46}$$

Therefore, at the level of the algebra, using formulas (11.23), (11.44) and the above relation (11.46), we have for the standard matrices the following transformation rule at the basis vectors:

$$\hat{e}'^{\,\ell} = d^{\{3^*\}}(A_{ij})\hat{e}^{\ell} = -\left[A_{ij}\right]_{\ell k}\hat{e}^{k} = -\delta_{i\ell}\hat{e}^{j} + \tfrac{1}{3}\delta_{ij}\hat{e}^{\ell} \quad . \tag{11.47}$$

The weight diagram for the irrep $\{3^*\}$ of $SU(3)$ is depicted in figure (11.4) below.

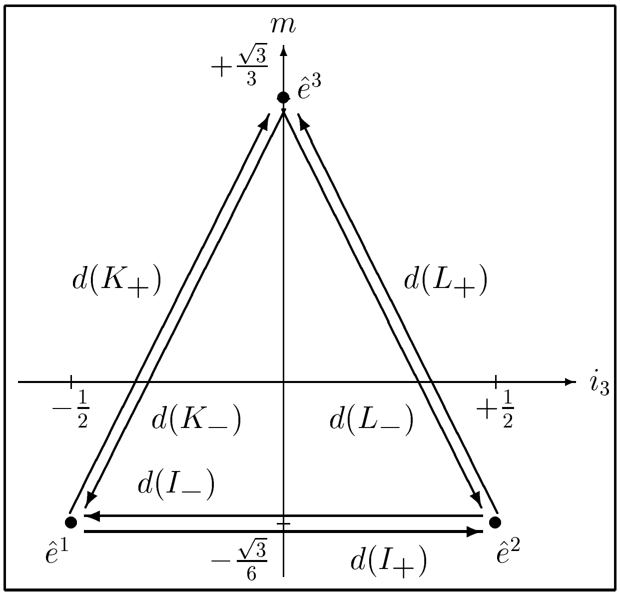


Figure 11.4: The action of the six step operators $d^{\{3^*\}}(I_\pm)$, $d^{\{3^*\}}(K_\pm)$ and $d^{\{3^*\}}(L_\pm)$ in the weight diagram for the irrep $\{3^*\}$ of $SU(3)$. The dots in the diagram represent the basis vectors (11.41).

The eigenvalues for the Casimir operators F^2 and G^3 are in this case respectively given by:

$$F^2\hat{e}_\ell = \tfrac{4}{3}\hat{e}_\ell \;\text{ and }\; G^3\hat{e}_\ell = -\tfrac{20}{9}\hat{e}_\ell \quad . \tag{11.48}$$

Notice that the eigenvalue for F^2 for the irrep $\{3^*\}$ is the same as for the irrep $\{3\}$ (see formula 11.38). This demonstrates that we really need one more Casimir operator (*i.e.* G^3) for $SU(3)$ in order to distinguish between irreps.

11.7 Tensors for $SU(3)$.

For $SU(3)$ one has two non-equivalent three-dimensional irreducible representations, namely the irrep $\{3\}$, which is discussed in section (11.5), and the conjugate irrep $\{3^*\}$, which is studied in section (11.6). The basis for the irrep $\{3\}$ in the three-dimensional vector space V is given in formula (11.28) and for the irrep $\{3^*\}$ in the conjugate vector space V^* in formula (11.41). In this section we study the direct product space $V^{(p+q)}$ given by the direct product of p times V and q times V^*, *i.e.*

$$V^{(p+q)} = \overbrace{V \otimes V \otimes \cdots \otimes V}^{p \text{ times}} \otimes \overbrace{V^* \otimes V^* \cdots \otimes V^*}^{q \text{ times}}$$

In the literature one finds the alternative notation:

$$V^{(p+q)} = \overbrace{\{3\} \otimes \{3\} \otimes \cdots \otimes \{3\}}^{p \text{ times}} \otimes \overbrace{\{3^*\} \otimes \{3^*\} \cdots \otimes \{3^*\}}^{q \text{ times}} \quad . \tag{11.49}$$

Using the bases given in (11.28) and (11.41), one can define the basis vectors of the direct product space $V^{(p+q)}$ by:

$$E^{j_1,\, j_2\, \ldots,\, j_q}_{i_1,\, i_2\, \ldots,\, i_p} = \hat{e}_{i_1} \otimes \hat{e}_{i_2} \otimes \cdots \otimes \hat{e}_{i_p} \otimes \hat{e}^{j_1} \otimes \hat{e}^{j_2} \otimes \cdots \otimes \hat{e}^{j_q} \quad . \tag{11.50}$$

As one can read from the construction, there are $3^{(p+q)}$ linearly independent basis vectors in the product space and therefore is its dimension equal to $3^{(p+q)}$.

Tensors $\boldsymbol{T}$ for $SU(3)$ are vectors in the direct product space $V^{(p+q)}$ and thus linear combinations of the basis elements of this vector space, *i.e.*

$$\boldsymbol{T} = T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \, E^{j_1 \ldots j_q}_{i_1 \ldots i_p} \quad . \tag{11.51}$$

An arbitrary element U of $SU(3)$ induces a transformation $D^{(p+q)}(U)$ in the vector space $V^{(p+q)}$, represented by $3^{(p+q)} \times 3^{(p+q)}$ unitary matrices. Using the formulas (11.30) and (11.43), we find for this transformation the expression:

$$\begin{aligned} \boldsymbol{T}' &= D^{(p+q)}(U)\boldsymbol{T} \\ &= U_{i_1k_1} \cdots U_{i_pk_p} \left[U^{-1}\right]_{\ell_1 j_1} \cdots \left[U^{-1}\right]_{\ell_q j_q} \, T^{k_1 \ldots k_p}_{\ell_1 \ldots \ell_q} \, E^{j_1 \ldots j_q}_{i_1 \ldots i_p} \quad . \end{aligned} \tag{11.52}$$

One may view this expression as either the transformation of the components of the tensor $\boldsymbol{T}$, *i.e.*

$$T'^{i_1 \ldots i_p}_{j_1 \ldots j_q} = U_{i_1k_1} \cdots U_{i_pk_p} \left[U^{-1}\right]_{\ell_1 j_1} \cdots \left[U^{-1}\right]_{\ell_q j_q} \, T^{k_1 \ldots k_p}_{\ell_1 \ldots \ell_q} \quad , \tag{11.53}$$

or as a transformation of the basis vectors:

$$E'^{\ell_1 \ldots \ell_p}_{k_1 \ldots k_q} = U_{i_1 k_1} \cdots U_{i_p k_p} \left[U^{-1}\right]_{\ell_1 j_1} \cdots \left[U^{-1}\right]_{\ell_q j_q} \; E^{j_1 \ldots j_q}_{i_1 \ldots i_p} \quad . \tag{11.54}$$

At the level of the algebra of $SU(3)$, using the formulas (11.33) and (11.47), we find for the basis vectors (11.50) under the transformation induced by a standard matrix A_{ij} the following:

$$\begin{aligned}
& d^{(p+q)}(A_{ij}) E^{j_1 \ldots j_q}_{i_1 \ldots i_p} = \\
& = \left\{ d^{\{3\}}(A_{ij}) \hat{e}_{i_1} \right\} \otimes \hat{e}_{i_2} \otimes \cdots \otimes \hat{e}_{i_p} \otimes \hat{e}^{j_1} \otimes \cdots \otimes \hat{e}^{j_q} + \cdots \\
& + \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_{p-1}} \otimes \left\{ d^{\{3\}}(A_{ij}) \hat{e}_{i_p} \right\} \otimes \hat{e}^{j_1} \otimes \cdots \otimes \hat{e}^{j_q} + \\
& + \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_p} \otimes \left\{ d^{\{3^*\}}(A_{ij}) \hat{e}^{j_1} \right\} \otimes \hat{e}^{j_2} \otimes \cdots \otimes \hat{e}^{j_q} + \cdots \\
& + \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_p} \otimes \hat{e}^{j_1} \otimes \cdots \otimes \hat{e}^{j_{q-1}} \otimes \left\{ d^{\{3^*\}}(A_{ij}) \hat{e}^{j_q} \right\} \\
& = \left\{ \delta_{j i_1} \hat{e}_i - \tfrac{1}{3} \delta_{ij} \hat{e}_{i_1} \right\} \otimes \hat{e}_{i_2} \otimes \cdots \otimes \hat{e}_{i_p} \otimes \hat{e}^{j_1} \otimes \cdots \otimes \hat{e}^{j_q} + \cdots \\
& + \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_{p-1}} \otimes \left\{ \delta_{j i_p} \hat{e}_i - \tfrac{1}{3} \delta_{ij} \hat{e}_{i_p} \right\} \otimes \hat{e}^{j_1} \otimes \cdots \otimes \hat{e}^{j_q} + \\
& + \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_p} \otimes \left\{ -\delta_{i j_1} \hat{e}^{j} + \tfrac{1}{3} \delta_{ji} \hat{e}^{j_1} \right\} \otimes \hat{e}^{j_2} \otimes \cdots \otimes \hat{e}^{j_q} + \cdots \\
& + \hat{e}_{i_1} \otimes \cdots \otimes \hat{e}_{i_p} \otimes \hat{e}^{j_1} \otimes \cdots \otimes \hat{e}^{j_{q-1}} \otimes \left\{ -\delta_{i j_q} \hat{e}^{j} + \tfrac{1}{3} \delta_{ji} \hat{e}^{j_q} \right\} \\
& = \delta_{j i_1} E^{j_1 \ldots j_q}_{i i_2 \ldots i_p} + \cdots + \delta_{j i_p} E^{j_1 \ldots j_q}_{i_1 \ldots i_{p-1} i} + \\
& - \left\{ \delta_{i j_1} E^{j j_2 \ldots j_q}_{i_1 \ldots i_p} + \cdots + \delta_{i j_q} E^{j_1 \ldots j_{q-1} j}_{i_1 \ldots i_p} \right\} + \\
& + \tfrac{1}{3}(q-p) \delta_{ij} E^{j_1 \ldots j_q}_{i_1 \ldots i_p} \quad .
\end{aligned} \tag{11.55}$$

The representation $D^{(p+q)}$ is reducible. In the following, we first study some special tensors and show how the representation $D^{(p+q)}$ can be reduced. Then we concentrate on irreducible tensors and the related irreducible representations of $SU(3)$.

11.8 Traceless tensors.

Let us in the direct product space $V^{(p+q)}$ defined in formula (11.49), study tensors $\boldsymbol{T}$ for which the components are given by:

$$T^{i_1\cdots i_a\cdots i_p}_{j_1\cdots j_b\cdots j_q} = \begin{cases} A^{i_1\cdots i_{a-1}i_{a+1}\cdots i_p}_{j_1\cdots j_{b-1}j_{b+1}\cdots j_q} & \text{if} \quad i_a = j_b \\ 0 & \text{if} \quad i_a \neq j_b \end{cases} \quad . \tag{11.56}$$

Such tensors can be written in the form:

$$\boldsymbol{T} = A^{i_1\cdots i_{a-1}i_{a+1}\cdots i_p}_{j_1\cdots j_{b-1}j_{b+1}\cdots j_q}\,\delta_{i_aj_b}E^{j_1\cdots j_q}_{i_1\cdots i_p} \quad . \tag{11.57}$$

For an arbitrary $SU(3)$-induced transformation $D^{(p+q)}(U)$, we obtain for a tensor of the form (11.57), the following:

$$\begin{aligned} \boldsymbol{T}' &= D^{(p+q)}(U)\boldsymbol{T} \\ &= U_{i_1k_1}\cdots U_{i_pk_p}\left[U^{-1}\right]_{\ell_1j_1}\cdots\left[U^{-1}\right]_{\ell_qj_q}\cdot \\ &\cdot A^{k_1\cdots k_{a-1}k_{a+1}\cdots k_p}_{\ell_1\cdots \ell_{b-1}\ell_{b+1}\cdots \ell_q}\,\delta_{k_a\ell_b}E^{j_1\cdots j_q}_{i_1\cdots i_p} \quad . \end{aligned} \tag{11.58}$$

When moreover, we use the relation:

$$U_{i_ak_a}\left[U^{-1}\right]_{\ell_bj_b}\delta_{k_a\ell_b} = \delta_{i_aj_b} \quad ,$$

then we find for the components of the transformed tensor $\boldsymbol{T}'$ the expression:

$$(T')^{i_1\cdots i_p}_{j_1\cdots j_q} = (A')^{i_1\cdots i_{a-1}i_{a+1}\cdots i_p}_{j_1\cdots j_{b-1}j_{b+1}\cdots j_q}\,\delta_{i_aj_b} \quad , \tag{11.59}$$

where:

$$\begin{aligned} &(A')^{i_1\cdots i_{a-1}i_{a+1}\cdots i_p}_{j_1\cdots j_{b-1}j_{b+1}\cdots j_q} = \\ &= U_{i_1k_1}\cdots U_{i_{a-1}k_{a-1}}U_{i_{a+1}k_{a+1}}\cdots U_{i_pk_p}\cdot \\ &\cdot\left[U^{-1}\right]_{\ell_1j_1}\cdots\left[U^{-1}\right]_{\ell_{b-1}j_{b-1}}\left[U^{-1}\right]_{\ell_{b+1}j_{b+1}}\cdots\left[U^{-1}\right]_{\ell_qj_q}\cdot \\ &\cdot A^{k_1\cdots k_{a-1}k_{a+1}\cdots k_p}_{\ell_1\cdots \ell_{b-1}\ell_{b+1}\cdots \ell_q} \quad . \end{aligned} \tag{11.60}$$

The components (11.59) of the transformed tensor (11.58) have exactly the same form as the components (11.56) of the original tensor (11.57). Consequently, those tensors form a subspace of $V^{(p+q)}$ which is invariant under the transformations

$D^{(p+q)}(U)$ for all elements U of $SU(3)$. Furthermore, counting the number of U's and $[U^{-1}]$'s in formula (11.60), we find that at the subspace of tensors of the form (11.57) the transformations $D^{(p+q)}(U)$ form the representation $D^{((p-1)+(q-1))}$ of $SU(3)$.

So, we must conclude that $D^{(p+q)}$ is reducible, since we found a subspace of $V^{(p+q)}$ at which $SU(3)$ is represented by a representation of a lower dimension.

In fact, we found a whole set of such subspaces, because i_a and j_b in formula (11.56) may be chosen freely amongst respectively the contra-variant and covariant indices of the tensor $\boldsymbol{T}$. When we remove all those subspaces from $V^{(p+q)}$, then we are left with *traceless* tensors.

In order to define what is understood by *traceless*, we first study contractions: A *contraction* of a pair of one contra-variant (for example i_a) and one covariant index (for example j_b) of the components of a tensor, is defined by:

$$T^{i_1 \ldots i_a = \alpha \ldots i_p}_{j_1 \ldots j_b = \alpha \ldots j_q} = \sum_{\alpha=1}^{3} T^{i_1 \ldots i_{a-1} \alpha\, i_{a+1} \ldots i_p}_{j_1 \ldots j_{b-1} \alpha\, j_{b+1} \ldots j_q} \quad . \tag{11.61}$$

Such a contraction of one upper and one lower index of the components of a tensor, is referred to as a *trace* of the tensor $\boldsymbol{T}$. A tensor $\boldsymbol{T}$ is said to be *traceless* if for the contraction of any pair of one upper and one lower index one finds vanishing result, *i.e.*

$$T^{i_1 \ldots i_{a-1} \alpha\, i_{a+1} \ldots i_p}_{j_1 \ldots j_{b-1} \alpha\, j_{b+1} \ldots j_q} = 0 \quad . \tag{11.62}$$

A traceless tensor $\boldsymbol{T}$ remains moreover traceless under $SU(3)$-induced transformations. This can easily be demonstrated: Using expression (11.53) and the relation

$$U_{\alpha k_a} \left[U^{-1}\right]_{\ell_b \alpha} = \delta_{k_a \ell_b} \quad ,$$

we find for the trace of the transformed tensor $\boldsymbol{T}'$ the following:

$$\begin{aligned} & T'^{i_1 \ldots i_{a-1} \alpha\, i_{a+1} \ldots i_p}_{j_1 \ldots j_{b-1} \alpha\, j_{b+1} \ldots j_q} = \\ & \quad U_{i_1 k_1} \cdots U_{i_{a-1} k_{a-1}} U_{i_{a+1} k_{a+1}} \cdots U_{i_p k_p} \cdot \\ & \quad \cdot \left[U^{-1}\right]_{\ell_1 j_1} \cdots \left[U^{-1}\right]_{\ell_{b-1} j_{b-1}} \left[U^{-1}\right]_{\ell_{b+1} j_{b+1}} \cdots \left[U^{-1}\right]_{\ell_q j_q} \cdot \\ & \quad \cdot T^{k_1 \ldots k_{a-1} \alpha\, k_{a+1} \ldots k_p}_{\ell_1 \ldots \ell_{b-1} \alpha\, \ell_{b+1} \ldots \ell_q} \quad . \end{aligned} \tag{11.63}$$

So, when $\boldsymbol{T}$ is traceless, then $\boldsymbol{T}'$ is also traceless. This implies that the subspace of $V^{(p+q)}$ which is spanned by traceless tensors, is invariant under $SU(3)$-induced transformations.

11.9 Symmetric tensors.

Using the Levi-Civita tensors which are defined in formula (9.4), let us consider in this section tensors $\boldsymbol{T}$ of the direct product space $V^{(p+q)}$ (11.49) of the form:

$$\boldsymbol{T} = B^{i_1 \ldots i_{a-1} i_{a+1} \ldots i_{b-1} i_{b+1} \ldots i_{c-1} i_{c+1} \ldots i_p}_{j_1 \ldots j_q} \epsilon_{i_a i_b i_c} E^{j_1 \ldots j_q}_{i_1 \ldots i_p} \quad . \tag{11.64}$$

Notice that the components of this tensor are *antisymmetric* under the interchange of any pair of the indices i_a, i_b and i_c.

In the expression for the transformation of this tensor under $D^{(p+q)}(U)$, we find the following terms:

$$\cdots U_{i_a k_a} \cdots U_{i_b k_b} \cdots U_{i_c k_c} \cdots \epsilon_{k_a k_b k_c} \cdots$$

Due to a property of unitary unimodular 3×3 matrices, the product of those four terms gives:

$$U_{i_a k_a} U_{i_b k_b} U_{i_c k_c} \epsilon_{k_a k_b k_c} = \epsilon_{i_a i_b i_c} \quad .$$

So, we end up with the expression:

$$\begin{aligned} \boldsymbol{T}' &= \\ &= = U_{i_1 k_1} \cdots U_{i_{a-1} k_{a-1}} U_{i_{a+1} k_{a+1}} \cdots U_{i_{b-1} k_{b-1}} U_{i_{b+1} k_{b+1}} \cdots \\ & \quad \cdots U_{i_{c-1} k_{c-1}} U_{i_{c+1} k_{c+1}} \cdots U_{i_p k_p} \left[U^{-1}\right]_{\ell_1 j_1} \cdots \left[U^{-1}\right]_{\ell_q j_q} \cdot \\ & \quad \cdot B^{k_1 \ldots k_{a-1} k_{a+1} \ldots k_{b-1} k_{b+1} \ldots k_{c-1} k_{c+1} \ldots k_p}_{l_1 \ldots l_q} \epsilon_{i_a i_b i_c} E^{j_1 \ldots j_q}_{i_1 \ldots i_p} \quad . \end{aligned} \tag{11.65}$$

This is again a tensor of the form (11.64). Consequently, the subspace of $V^{(p+q)}$ which is spanned by tensors of the form (11.64) is invariant under $SU(3)$-induced transformations. Moreover, is the representation $D^{(p+q)}$ at this subspace reduced to the representation $D^{((p-3)+q)}$ of $SU(3)$.

There are as many of those subspaces in $V^{(p+q)}$ as one can take sets of three upper indices. Similarly, one can find comparable subspaces for the lower indices.

When all such subspaces are removed from $V^{(p+q)}$, then the remaining tensors are all symmetric under the interchange of any pair of upper indices and of any pair of lower indices. Such tensors are called *symmetric*. A symmetric tensor remains symmetric under $SU(3)$-induced transformations. This can easily be understood from equation (11.53).

11.10 Irreducible tensors $D^{(p,q)}$.

Tensors which are symmetric (as defined in section 11.9) and traceless (as defined in section 11.8) form a subspace $V^{(p,q)}$ of $V^{(p+q)}$ which is invariant under $SU(3)$-induced transformations. The group $SU(3)$ is in $V^{(p,q)}$ represented by the irreducible representation $D^{(p,q)}$.

The dimension of the irrep $D^{(p,q)}$ is given by:

$$f(p,q) = \tfrac{1}{2}(p+1)(q+1)(p+q+2) \ . \tag{11.66}$$

Formula (11.66) can be derived by considering the number of possible different components of a tensor with p upper indices, q lower indices and which is symmetric and traceless: The number of ways in which one can arrange ones, twos and threes on p places such that one gets first all ones, then all twos and then all threes, is given by:

$$\begin{pmatrix} p+2 \\ 2 \end{pmatrix} = \tfrac{1}{2}(p+1)(p+2) \ .$$

So, a tensor with p upper and q lower indices, totally symmetric in the upper indices and totally symmetric in the lower indices, has

$$N_1 = \tfrac{1}{4}(p+1)(p+2)(q+1)(q+2)$$

linearly independent components. Moreover, because of the symmetry, there exists only one trace tensor. This tensor has $(p-1)$ symmetric upper indices and $(q-1)$ symmetric lower indices, and therefore

$$N_2 = \tfrac{1}{4}p(p+1)q(q+1)$$

linearly independent components. The elements of the trace tensor must be zero, since the original tensor is traceless. This gives N_2 linear relations between the N_1 components. So, the total number of linearly independent traceless and symmetric tensors in $V^{(p+q)}$ equals:

$$N_1 - N_2 = \tfrac{1}{2}(p+1)(q+1)(p+q+2) \ .$$

The trivial one-dimensional representation is denoted by $D^{(0,0)}$. The two three-dimensional irreducible representations are given by $\{3\} = D^{(1,0)}$ and $\{3^*\} = D^{(0,1)}$.

The eigenvalues of the Casimir operators F^2 and G^3 defined in (11.26), are at the irrep $D^{(p,q)}$ respectively given by:

$$f^2 = \tfrac{1}{3}(p^2+pq+q^2)+p+q \ \text{ and } \ g^3 = \tfrac{1}{9}(p-q)(2p+q+3)(p+2q+3) \ . \tag{11.67}$$

Those results can be obtained by applying the Casimir operators F^2 and G^3 to the basis tensor for which all indices are equal to 1, using repeatedly formula (11.53). One may compare the results (11.39) and (11.48) for the eigenvalues of G^3 with the above formula (11.67) in order to find how G^3 distinguishes between different irreps for which the number p of upper indices and q of lower indices are interchanged; cases for which F^2 gives the same eigenvalue.

For each value of p and q there exists an irrep of $SU(3)$ and there are also not more irreps for $SU(3)$. The irrep $D^{(p,q)}$ can moreover be represented by a Young diagram (see 1.3 for the definition), as follows:

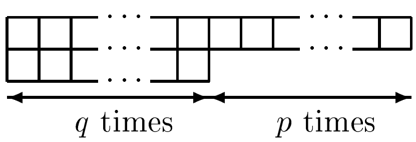


Figure 11.5: The Young diagram which represents the irrep $D^{(p,q)}$ of $SU(3)$.

There is one exception, which is the Young diagram for the trivial representation $D^{(0,0)}$. This representation is represented by a Young diagram of three rows of length one.

In table (11.3) we have collected some information on the irreps for $p+q = 0, 1, 2$ and 3.

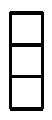
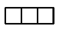
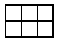
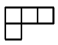
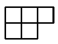
irrep	p	q	Young diagram	$f(p,q)$	f^2	g^3
$\{1\}$	0	0		1	0	0
$\{3\}$	1	0		3	$\frac{4}{3}$	$+\frac{20}{9}$
$\{3^*\}$	0	1		3	$\frac{4}{3}$	$-\frac{20}{9}$
$\{6\}$	2	0		6	$\frac{10}{3}$	$+\frac{70}{9}$
$\{6^*\}$	0	2		6	$\frac{10}{3}$	$-\frac{70}{9}$
$\{8\}$	1	1		8	3	0
$\{10\}$	3	0		10	6	$+18$
$\{10^*\}$	0	3		10	6	-18
$\{15\}$	2	1		15	$\frac{16}{3}$	$+\frac{56}{9}$
$\{15^*\}$	1	2		15	$\frac{16}{3}$	$-\frac{56}{9}$

Table 11.3: The Young diagrams, dimensions and eigenvalues for the Casimir operators of some irreps of $SU(3)$.

Notice that there may be more than just two non-equivalent irreps of $SU(3)$ which have the same dimension. For example $f(0,4) = f(1,2) = f(2,1) = f(4,0) = 15$ and hence four non-equivalent irreps of $SU(3)$ with dimension 15 exist. With dimension 9240 one finds ten non-equivalent irreps of $SU(3)$, according to: $f(2,76) = f(3,65)$

$= f(6,47) = f(13,29) = f(19,21) = f(21,19) = f(29,13) = f(47,6) = f(65,3) = f(76,2) = 9240.$

In the following we study the sextet and octet irreps of $SU(3)$.

11.11 The matrix elements of the step operators.

We postponed to give the relations for the matrix elements A, B, C and D for the step operators which are defined in formula (11.16), untill all quantum numbers of $SU(3)$ irreps had been discussed. Since, with the definitions of p and q we completed the set of quantum numbers which fully characterize the states of the various $SU(3)$ irreps, we are now prepared to discuss those matrix elements for the step operators.

The matrix elements $A_+(i, i_3, m)$ and $B_+(i, i_3, m)$ are at the irrep $D^{(p,q)}$ given by:

$$A_+(p, q; i, i_3, y = 2m/\sqrt{3}) =$$

$$\sqrt{\frac{(i + i_3 + 1)\left[\frac{1}{3}(p - q) + i + \frac{y}{2} + 1\right]\left[\frac{1}{3}(p + 2q) + i + \frac{y}{2} + 2\right]\left[\frac{1}{3}(2p + q) - i - \frac{y}{2}\right]}{2(i + 1)(2i + 1)}}$$

and

$$B_+(p, q; i, i_3, y = 2m/\sqrt{3}) =$$

$$\sqrt{\frac{(i - i_3)\left[\frac{1}{3}(q - p) + i - \frac{y}{2}\right]\left[\frac{1}{3}(p + 2q) - i + \frac{y}{2} + 1\right]\left[\frac{1}{3}(2p + q) + i - \frac{y}{2} + 1\right]}{2i(2i + 1)}} \tag{11.68}$$

The expressions for $A_-(p, q; i, i_3, m)$ and $B_-(p, q; i, i_3, m)$ can be obtained by using the relation:

$$K_- = (K_+)^\dagger \ .$$

And the matrix elements $C_\pm(p, q; i, i_3, m)$ and $D_\pm(p, q; i, i_3, m)$ follow when one explores the relations:

$$L_+ = \left[\, I_- \ , \ K_+ \,\right] \quad \text{and} \quad L_- = (L_+)^\dagger \ .$$

11.12 The six-dimensional irrep $D^{(2,0)}$.

In the subspace $V^{(2,0)}$ of the direct product space $V^{(2+0)} = \{3\} \otimes \{3\}$ we define as a basis for the symmetric tensors (there are no traces in this case) the following linear combinations of the basisvectors (11.50) for $\{3\} \otimes \{3\}$:

$$
\begin{array}{lllllllll}
\hat{u}_1 & = & E_{11} & , & \hat{u}_2 & = & \frac{1}{\sqrt{2}}(E_{12}+E_{21}) & , & \hat{u}_3 = E_{22} \ , \\
\hat{u}_4 & = & \frac{1}{\sqrt{2}}(E_{13}+E_{31}) & , & \hat{u}_5 & = & \frac{1}{\sqrt{2}}(E_{23}+E_{32}) & \text{and} & \hat{u}_6 = E_{33} \ .
\end{array}
\tag{11.69}
$$

Notice that the basis vectors (11.50) are orthonormal. Consequently, for the definition of the above basis vectors (11.69) for the subspace $V^{(2,0)}$, we have to introduce some normalization constants $\frac{1}{\sqrt{2}}$. Forgetting about those normalization constants, one might compactify the definitions of the above formula (11.69) into:

$$\phi_{k\ell} = E_{k\ell} + E_{\ell k} \ \text{ for } \ k \leq \ell = 1,2,3. \tag{11.70}$$

This way one obtains an elegant way to study the transformations of the irreducible representation $D^{(2,0)}$ at the six-dimensional subspace $V^{(2,0)}$. Exploring expression (11.54), one obtains for the $SU(3)$-induced transformations of the basis vectors (11.70) the relations:

$$d^{(2,0)}(A_{ij})\phi_{k\ell} = \delta_{jk}\phi_{i\ell} + \delta_{j\ell}\phi_{ik} - \tfrac{2}{3}\delta_{ij}\phi_{k\ell} \ . \tag{11.71}$$

For the step operators one may forget about the last term containing the Kronecker delta δ_{ij} in formula (11.71), since according to the relations (11.24) for those operators one has standard matrices for which $i \neq j$. The resulting transformations for those operators at the basis vectors (11.69) are collected in table (11.4).

$\hat{u}$	I_+	I_-	K_+	K_-	L_+	L_-
$\hat{u}_1$	0	$\sqrt{2}\hat{u}_2$	0	$\sqrt{2}\hat{u}_4$	0	0
$\hat{u}_2$	$\sqrt{2}\hat{u}_1$	$\sqrt{2}\hat{u}_3$	0	$\hat{u}_5$	0	$\hat{u}_4$
$\hat{u}_3$	$\sqrt{2}\hat{u}_2$	0	0	0	0	$\sqrt{2}\hat{u}_5$
$\hat{u}_4$	0	$\hat{u}_5$	$\sqrt{2}\hat{u}_1$	$\sqrt{2}\hat{u}_6$	$\hat{u}_2$	0
$\hat{u}_5$	$\hat{u}_4$	0	$\hat{u}_2$	0	$\sqrt{2}\hat{u}_3$	$\sqrt{2}\hat{u}_6$
$\hat{u}_6$	0	0	$\sqrt{2}\hat{u}_4$	0	$\sqrt{2}\hat{u}_5$	0

Table 11.4: The action of the step operators on the basis defined in formula (11.69) for the sextet representation of $SU(3)$.

Notice from table (11.4) that all matrix elements for $I_\pm$ and $K_\pm$ are positive, as required by the phase conventions mentioned in section (11.2) in the discussion following formula (11.16).

Next, let us determine the Casimir operators in this case. As may be verified by explicit calculations using repeatedly formula (11.71), we find for the Casimir operators F^2 and G^3, defined in formula (11.26), at the tensors defined in (11.70), the following eigenvalues:

$$F^2\phi_{k\ell} = \tfrac{10}{3}\phi_{k\ell} \quad \text{and} \quad G^3\phi_{k\ell} = \tfrac{70}{9}\phi_{k\ell} \quad . \tag{11.72}$$

Those values are in perfect agreement with formula (11.67) for $p = 2$ and $q = 0$, as also shown in table (11.3).

In order to obtain the full structure of the irrep $D^{(2,0)}$, let us furthermore determine the eigenvalues i_3 and m for respectively the operators $I_3 = \frac{1}{2}(A_{11} - A_{22})$ and $M = \frac{\sqrt{3}}{2}(A_{11} + A_{22})$, and also the eigenvalues $i(i+1)$ for $I^2 = (I_3)^2 + I_3 + I_-I_+$. The results, with the use of the formulas (11.69), (11.70) and (11.71) and the information of table (11.4), are collected in table (11.5).

$\hat{u}$	i_3	$y = \frac{2m}{\sqrt{3}}$	I_-I_+	i(i+1)	i
$\hat{u}_1$	$+1$	$+\frac{2}{3}$	0	2	1
$\hat{u}_2$	0	$+\frac{2}{3}$	2	2	1
$\hat{u}_3$	-1	$+\frac{2}{3}$	2	2	1
$\hat{u}_4$	$+\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{3}{4}$	$\frac{1}{2}$
$\hat{u}_5$	$-\frac{1}{2}$	$-\frac{1}{3}$	1	$\frac{3}{4}$	$\frac{1}{2}$
$\hat{u}_6$	0	$-\frac{4}{3}$	0	0	0

Table 11.5: The isospin parameters i_3 and i for the sextet representation of $SU(3)$ as well as the eigenvalues m for M.

As we notice from table (11.5), the $SU(3)$ sextet contains one isotriplet (*i.e.* the subspace spanned by $\hat{u}_1$, $\hat{u}_2$ and $\hat{u}_3$), one isodoublet (*i.e.* the subspace spanned by $\hat{u}_4$ and $\hat{u}_5$), one isosinglet (*i.e.* the subspace spanned by $\hat{u}_6$). Using the expressions (11.69) for the various basis vectors of the iso-subspaces, one might also notice that the isotriplet contains the indices 1 and 2, the isodoublet one index 3 and the isosinglet two indices 3. From the table of the step operators (11.4) one might notice moreover that I_+ substitutes an index 2 for an index 1, whereas I_- does the reverse.

In the original three quark model, the index 1 is associated with an "up" quark, the index 2 with a "down" quark and the index 3 with a "strange" quark. The isospin step operator I_+ substitutes "up" for "down" and I_- does the reverse. In that picture, the sextet represents the symmetric configuration (not observed in Nature) of two quark states (uu, $\frac{1}{\sqrt{2}}(ud + du)$, dd, $\frac{1}{\sqrt{2}}(us + su)$, $\frac{1}{\sqrt{2}}(ds + sd)$ and ss). The first three states form a triplet with isospin 1, the next two a doublet with isospin $\frac{1}{2}$ and the last an isosinglet.

The quantum number $y = \frac{2m}{\sqrt{3}}$ is called the *hypercharge* of a (multi-)quark state. As one might read from table (11.3) or alternatively from figure (11.1), the "up" and "down" quarks have hypercharge $+\frac{1}{3}$ and the "strange" quark hypercharge $-\frac{2}{3}$. Hypercharge is an additive quantum number, and so the hypercharges of the two quark states of table (11.5) are just the sum of the hypercharges of their constituents.

With the use of the information of the two tables (11.4) and (11.5), the weight diagram for the sextet can be composed. The result is depicted in figure (11.6).

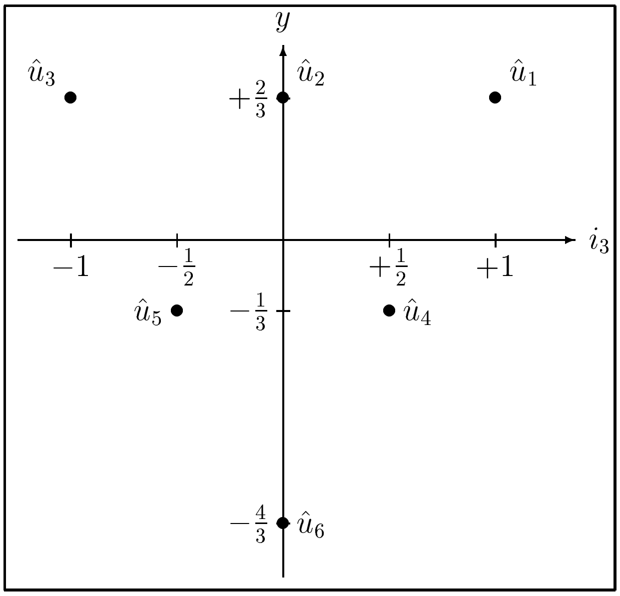


Figure 11.6: The weight diagram for the sextet representation of $SU(3)$.

11.13 The eight-dimensional irrep $D^{(1,1)}$.

In the subspace $V^{(1,1)}$ of the direct product space $V^{(1+1)} = \{3\} \otimes \{3^*\}$ we define as a basis for the traceless tensors (no symmetry in this case) the following linear combinations of the basisvectors (11.50) for $\{3\} \otimes \{3^*\}$:

$$
\begin{array}{llllllll}
\hat{v}_1 = -E_1^3 & , & \hat{v}_2 = -E_2^3 & & , & & \\
\hat{v}_3 = -E_1^2 & , & \hat{v}_4 = \frac{1}{\sqrt{2}}(E_1^1 - E_2^2) & , & \hat{v}_5 = E_2^1 & , \\
\hat{v}_6 = -E_3^2 & , & \hat{v}_7 = E_3^1 & \text{and} & \hat{v}_8 = \frac{1}{\sqrt{6}}(E_1^1 + E_2^2 - 2E_3^3) & .
\end{array}
\tag{11.73}
$$

Besides the normalization constants, we also introduce some minus signs in the definition (11.73) of the basis vectors for the octet representation of $SU(3)$. Those are necessary to get the relative phases right. Exploring expression (11.54), one obtains for the $SU(3)$-induced transformations the relations:

$$
d^{(1,1)}(A_{ij})E_k^\ell = \delta_{jk}E_i^\ell - \delta_{i\ell}E_k^j \ . \tag{11.74}
$$

The transformations under the step operators at the basis vectors (11.73) are collected in table (11.6).

$\hat{v}$	I_+	I_-	K_+	K_-	L_+	L_-
$\hat{v}_1$	0	$\hat{v}_2$	0	$\frac{\hat{v}_4+\sqrt{3}\hat{v}_8}{\sqrt{2}}$	0	$-\hat{v}_3$
$\hat{v}_2$	$\hat{v}_1$	0	0	$\hat{v}_5$	0	$\frac{-\hat{v}_4+\sqrt{3}\hat{v}_8}{\sqrt{2}}$
$\hat{v}_3$	0	$\sqrt{2}\hat{v}_4$	0	$\hat{v}_6$	$-\hat{v}_1$	0
$\hat{v}_4$	$\sqrt{2}\hat{v}_3$	$\sqrt{2}\hat{v}_5$	$\frac{\hat{v}_1}{\sqrt{2}}$	$\frac{\hat{v}_7}{\sqrt{2}}$	$\frac{\hat{v}_5}{\sqrt{2}}$	$\frac{\hat{v}_6}{\sqrt{2}}$
$\hat{v}_5$	$\sqrt{2}\hat{v}_4$	0	$\hat{v}_2$	0	0	$\hat{v}_7$
$\hat{v}_6$	0	$\hat{v}_7$	$\hat{v}_3$	0	$\frac{\hat{v}_4-\sqrt{3}\hat{v}_8}{\sqrt{2}}$	0
$\hat{v}_7$	$\hat{v}_6$	0	$\frac{\hat{v}_4+\sqrt{3}\hat{v}_8}{\sqrt{2}}$	0	$\hat{v}_5$	0
$\hat{v}_8$	0	0	$\frac{3}{2}\hat{v}_1$	$\frac{3}{2}\hat{v}_7$	$\frac{3}{2}\hat{v}_2$	$-\frac{3}{2}\hat{v}_6$

Table 11.6: The action of the step operators on the basis defined in formula (11.73) for the octet representation of $SU(3)$.

Notice from table (11.6) that, because of our choice of the basis (11.73), all matrix elements for $I_\pm$ and $K_\pm$ are positive, as required by the phase conventions mentioned in section (11.2) in the discussion following formula (11.16).

In order to obtain the full structure of the irrep $D^{(1,1)}$, let us also determine the eigenvalues i_3 and m for respectively the operators $I_3 = \frac{1}{2}(A_{11} - A_{22})$ and

$M = \frac{\sqrt{3}}{2}(A_{11} + A_{22})$. The results, with the use of the formulas (11.73) and (11.74) and the information of table (11.6), are joined in figure (11.7).

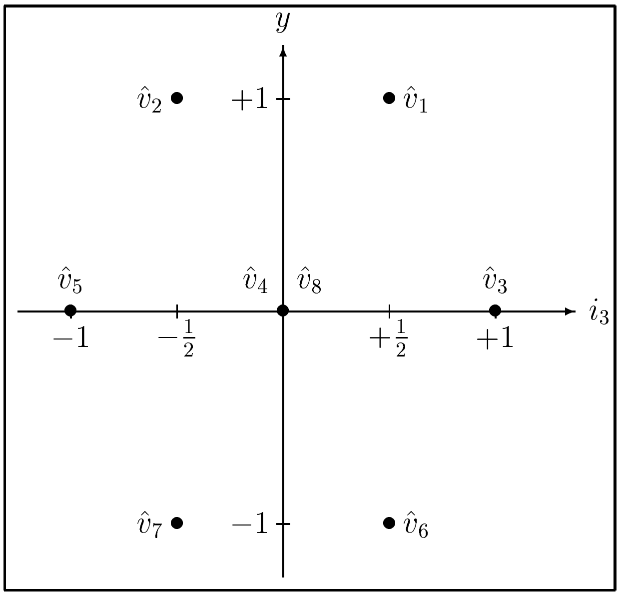


Figure 11.7: The weight diagram for the octet representation of $SU(3)$.

As we notice from figure (11.7), there are two states with the quantum numbers $i_3 = y = 0$. They can only be distinguished by the isospin Casimir operator I^2. Its respective eigenvalues for those two states are given by:

$$I^2 \tfrac{1}{\sqrt{2}}(E_1^1 - E_2^2) = 2\tfrac{1}{\sqrt{2}}(E_1^1 - E_2^2) \quad \text{and} \tag{11.75}$$

$$I^2 \tfrac{1}{\sqrt{6}}(E_1^1 + E_2^2 - 2E_3^3) = 0 \ . \tag{11.76}$$

Consequently, the $SU(3)$ octet contains one isotriplet (*i.e.* the subspace spanned by $\hat{v}_3$, $\hat{v}_4$ and $\hat{v}_5$), two isodoublets (*i.e.* the subspaces spanned by $\hat{v}_1$ and $\hat{v}_2$ and by $\hat{v}_6$ and $\hat{v}_7$), and moreover one isosinglet (*i.e.* the subspace spanned by $\hat{v}_8$).

The covariant index of a tensor is in the original quark model associated with antiquarks. Several quark-antiquark octets have been recognized in Nature: For example the octet of pseudoscalar mesons: A triplet of pions with isospin 1, two different doublets of Kaons, one with a strange antiquark and one with a strange quark, both with isospin $\frac{1}{2}$ and one isosinglet η-particle.

Chapter 12

The classification of semi-simple Lie groups.

In this chapter we present the full classification of all possible semi-simple Lie groups. In the first section we will define the concept *semi-simple*.

12.1 Invariant subgoups and subalgebras.

In formula (1.22) we find the definition for an invariant subgroup, *i.e* "A subgroup S is called *invariant* if for all elements g in the group G holds:

$$gSg^{-1} = S \qquad (1.22)".$$

Let the group G be a Lie group (as defined in section (7.9) at page 90) and generated by the Lie algebra A. When S is a subgroup of G, then it is generated by a subalgebra B of A. If S is moreover an invariant subgroup of G, then B is an invariant subalgebra of A, *i.e.* for all elements a of A holds:

$$[a\,,\;B] = B \quad . \tag{12.1}$$

In order to gain some insight in the above afirmation, we will use the following identity for matrices a and b:

$$e^a e^b e^{-a} = e^{b + [a\,,\;b] + \frac{1}{2!}[a\,,\;[a\,,\;b]] + \frac{1}{3!}[a\,,\;[a\,,\;[a\,,\;b]]] + \cdots} \quad . \tag{12.2}$$

We will proof this identity at the end of this section.

Let b in formula (12.2) represent a generator of the subgroup S (*i.e.* e^b is an element of S) and hence an element of the subalgebra B, and a a generator of the group G (*i.e.* e^a is an element of G) and therefore an element of the algebra A.

1. When B is an invariant subalgebra of A, then according to the definition (12.1) for an invariant subalgebra, we may conclude that the commutators, given by:

$$[a\,,\;b] \quad , \quad [a\,,\;[a\,,\;b]] \quad , \quad [a\,,\;[a\,,\;[a\,,\;b]]] \quad , \ldots$$

are elements of the subalgebra B and therefore that the following sum

$$b + [a\,,\, b] + \tfrac{1}{2!}\,[a\,,\, [a\,,\, b]] + \tfrac{1}{3!}\,[a\,,\, [a\,,\, [a\,,\, b]]] + \cdots \tag{12.3}$$

is an element of B. Consequently, since B represents the generator space of the subgroup S, the exponentiation of (12.3) gives an element of S. Now, because this is true for any element $g = e^{a}$ of the group, we may conclude that S is an invariant subgroup of G.

2. When S is an invariant subgroup of G, then the righthand side of the equation (12.2) is an element of S, according to the definition (1.22) for an invariant subgroup. Consequently, the expression (12.3) has to be an element of the generator space B for S. Here we might introduce a parameter λ and define a matrix field, given by:

$$e^{\lambda a}e^{b}e^{-\lambda a} = e^{b + \lambda\,[a\,,\, b] + \frac{\lambda^2}{2!}\,[a\,,\, [a\,,\, b]] + \cdots} \quad . \tag{12.4}$$

The term $e^{\lambda a}$ represents a field of matrices in the group G and e^{b} an element of the invariant subgroup S. So, the sum, given by:

$$b + \lambda\,[a\,,\, b] + \tfrac{\lambda^2}{2!}\,[a\,,\, [a\,,\, b]] + \cdots$$

must be an element of the subalgebra B for any value of λ. Consequently, we must conclude that

$$[a\,,\, b] \quad , \quad [a\,,\, [a\,,\, b]] \quad , \quad [a\,,\, [a\,,\, [a\,,\, b]]] \quad , \ldots$$

are elements of B and hence that B forms an invariant subalgebra of A.

A group is called *simple* when it has no invariant subgroup and *semi-simple* when it has no Abelian invariant subgroup. The Lie algebra for a semi-simple Lie group has no Abelian invariant subalgebra and is also said to be semi-simple. It can moreover be shown that each semi-simple Lie algebra is the direct sum of simple Lie algebras. Consequently, we may concentrate on the classification of simple Lie algebras.

Proof of the identity (12.2):

Let us for the arbitrary parameter λ, define the following:

$$e^{\lambda c} = e^{a}e^{\lambda b}e^{-a} \quad . \tag{12.5}$$

For $\lambda = 0$ one obtains the identity $\boldsymbol{I}$ of the group G. Now, since G is a Lie group, we may expand the expression (12.5) around the identity, and therefore identify:

$$c = \left.\frac{d}{d\lambda}e^{\lambda c}\right|_{\lambda = 0} = \left.\frac{d}{d\lambda}(e^{a}e^{\lambda b}e^{-a})\right|_{\lambda = 0} = e^{a}be^{-a} \quad . \tag{12.6}$$

When we expand the exponents in the righthand side of formula (12.6), we obtain the expression (12.3) for c which proofs identity (12.2).

12.2 The general structure of the Lie algebra.

Let us consider a p-parameter simple Lie group, generated by a p-dimensional simple Lie-algebra. We select in the Lie algebra a basis which consists of a set of p generators, indicated by:

$$\left\{X_1, \ldots, X_p\right\} \quad . \tag{12.7}$$

Any other generator of the group can be written as a linear combination of the above set of basis generators. The generators $\{X_i\}$ span a Lie algebra and hence satisfy certain commutation relations, the so-called *Lie product*, given by:

$$\left[X_i \, , \, X_j\right] = C_{ijk} X_k \quad . \tag{12.8}$$

The constants $\{C_{ijk}\}$ are called the *structure constants* of the group. Clearly, they depend on the choice and normalization of the basis $\{X_i\}$.

Out of the set of basis generators $\{X_i\}$ one selects a set of generators $\{H_1, \ldots, H_q\}$ which commute amongst each other, *i.e.*

$$\left[H_i \, , \, H_j\right] = 0 \quad . \tag{12.9}$$

When this set contains the largest possible number of commuting generators, it is called the *Cartan subalgebra* of the Lie algebra. For example for $SO(3)$ and for $SU(2)$ we only have one such generator, respectively L_3 and J_3; for $SU(3)$ we have a set of two of such generators, I_3 and M. We assume that the set $\{H_1, \ldots, H_q\}$ is the Cartan algebra of our Lie group. We assume furthermore that from the remaining generators it is not possible to construct a linear combination which commutes with all elements of the Cartan subalgebra.

The dimension of the generator space is p and the dimension of the Cartan subalgebra equals q. Consequently, the dimension of the subalgebra which is spanned by the remaining generators equals $(p-q)$. We will refer to this vector space as the subalgebra of the step operators. It can be shown that:

$$p - q \geq 2q. \tag{12.10}$$

It is allways possible to arrange such a basis $\{E_\alpha\}$ in the subalgebra of the step operators, that for each basis element E_α holds:

$$\left[H_i \, , \, E_\alpha\right] = \alpha_i E_\alpha \quad \text{for real constants} \quad \alpha_i \;\; (i = 1, \ldots, q). \tag{12.11}$$

The constants $\{\alpha_i\}$ depend on the choice of bases $\{H_i\}$ in the Cartan subalgebra and $\{E_\alpha\}$ in the subalgebra of step operators.

For a matrix representation D of the Lie group in a vector space V, it is possible to choose diagonal matrices for the generators of the Cartan subalgebra. We indicate the basis vectors of V by their eigenvalues for $\{d(H_i)\}$, *i.e.*

$$d(H_i) \, |h_1, \ldots, h_q> = h_i \, |h_1, \ldots, h_q> \, , \quad i = 1, \ldots, q. \tag{12.12}$$

Due to the commutation relations (12.11), we find for the basis vectors of the vector space V moreover the following:

$$d(E_\alpha)\,|h_1,\ldots,h_q> = f(\alpha;h_1,\ldots,h_q)\,|h_1+\alpha_1,\ldots,h_q+\alpha_q> \quad , \qquad (12.13)$$

which is the reason that the operators $\{E_\alpha\}$ are called step operators.

The weight diagrams for the matrix representations of the Lie group are q-dimensional in this case, where q is also referred to as the *rank* of the Lie group. $SO(3)$ and $SU(2)$ have rank 1, $SU(3)$ rank 2.

12.3 The step operators.

For each step operator E_α we construct a q-dimensional real root vector $\boldsymbol{\alpha}$. The real components α_i of the root vector $\boldsymbol{\alpha}$ are, using expression (12.11), defined as follows:

$$[\boldsymbol{\alpha}]_i = \alpha_i \quad \text{when} \quad \Big[H_i\, ,\, E_\alpha\Big] = \alpha_i E_\alpha \quad i=1,\ldots,q. \qquad (12.14)$$

Clearly, the components of the root vectors depend on the choice of the bases $\{H_i\}$ in the Cartan subalgebra and $\{E_\alpha\}$ in the subalgebra of step operators.

Let us in the following study the commutator or Lie product of two step operators E_α and E_β, *i.e.*

$$\Big[E_\alpha\, ,\, E_\beta\Big] \quad . \qquad (12.15)$$

For this Lie product we have, using the Jacobi identity and equation (12.11), the following property:

$$\begin{aligned} \Big[H_i\, ,\, \Big[E_\alpha\, ,\, E_\beta\Big]\Big] &= \Big[\Big[H_i\, ,\, E_\alpha\Big]\, ,\, E_\beta\Big] + \Big[E_\alpha\, ,\, \Big[H_i\, ,\, E_\beta\Big]\Big] \\ &= (\alpha_i+\beta_i)\Big[E_\alpha\, ,\, E_\beta\Big] \quad \text{for} \quad i=1,\ldots,q. \end{aligned} \qquad (12.16)$$

From this property we may conclude that in the case $\boldsymbol{\alpha}+\boldsymbol{\beta}\neq 0$, the Lie product (12.15) is a step operator itself, *i.e.*

$$\Big[E_\alpha\, ,\, E_\beta\Big] = N(\alpha,\beta)\; E_{\alpha+\beta} \quad . \qquad (12.17)$$

The constants $N(\alpha,\beta)$ depend on the choice and normalization of the basis in the subalgebra of the step operators.

A special case happens when $\boldsymbol{\beta}=-\boldsymbol{\alpha}$, *i.e.*

$$\Big[H_i\, ,\, \Big[E_\alpha\, ,\, E_{-\alpha}\Big]\Big] = 0 \quad \text{for} \quad i=1,\ldots,q. \qquad (12.18)$$

In that case the Lie product of the two step operators E_α and $E_{-\alpha}$ commutes with all basis generators of the Cartan subalgebra. Which, because the Cartan subalgebra is the largest possible subalgebra of commutating generators of the Lie algebra, leads to the conclusion that the Lie product of E_α and $E_{-\alpha}$ must be a linear combination of $\{H_i\}$, *i.e.*

$$\left[E_\alpha\ ,\ E_{-\alpha}\right] = \alpha^i H_i\ \ . \tag{12.19}$$

The constants $\{\alpha^i\}$ depend on the choice of basis $\{H_i\}$ in the Cartan subalgebra and $\{E_\alpha\}$ in the subalgebra of step operators. However, it is allways possible to arrange such bases for both subalgebras that:

$$\alpha^i = \alpha_i\ \ . \tag{12.20}$$

12.4 The root vectors.

The dimension of the root vector space in which the root vectors are defined, equals q (see formula 12.14). So, because of the relation (12.10), the root vectors are not linearly independent.

The root vectors may be divided into positive and negative root vectors, as follows:

A root vector $\boldsymbol{\alpha}$ is said to be positive (negative) when the first nonzero component of $\boldsymbol{\alpha}$ in the row α_1, α_2, ..., α_q, is positive (negative).

There are $\frac{1}{2}(p-q)$ positive root vectors, which because of relation (12.10), are not linearly independent. Out of this set we select those positive root vectors which cannot be written as the sum of two other positive root vectors. Those positive root vectors are said to be *simple*. There are exactly q simple positive root vectors and linearly independent for any simple Lie group. They form therefore a basis for the vector space in which the root vectors are defined. Below we list some of their properties:

1. When $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two simple positive roots, then $\boldsymbol{\beta}-\boldsymbol{\alpha}$ cannot be a root of any kind. For suppose:

$$\boldsymbol{\beta} - \boldsymbol{\alpha} = \boldsymbol{\gamma}. \tag{12.21}$$

-When $\boldsymbol{\gamma}$ represents a positive root vector, then we find:

$$\boldsymbol{\beta} = \boldsymbol{\alpha} + \boldsymbol{\gamma}.$$

This implies that $\boldsymbol{\beta}$ can be written as the sum of two positive root vectors, which contradicts the definition of $\boldsymbol{\beta}$ as a simple positive root.

-When $\boldsymbol{\gamma}$ in formula (12.21) represents a negative root vector, then, since for each step operator E_α there exists a step operator $E_{-\alpha}$, we have that $-\boldsymbol{\gamma}$ is a positive root vector and we find:

$$\boldsymbol{\alpha} = \boldsymbol{\beta} + (-\boldsymbol{\gamma}).$$

This implies that $\boldsymbol{\alpha}$ can be written as the sum of two positive root vectors, which contradicts the definition of $\boldsymbol{\alpha}$ as a simple positive root.

2. However, there may be root vectors of the form $\boldsymbol{\beta}$+$\boldsymbol{\alpha}$, and indeed, there will generall be strings of roots of the form:

$$\boldsymbol{\beta},\ \boldsymbol{\beta}+\boldsymbol{\alpha},\ \boldsymbol{\beta}+2\boldsymbol{\alpha},\ \dots\ ,\boldsymbol{\beta}+n\boldsymbol{\alpha}, \tag{12.22}$$

which terminate at some integer n (*i.e.* $\boldsymbol{\beta}$+$(n+1)\boldsymbol{\alpha}$ is not a root vector of the Lie group).

3. The value of the non-negative integer n in formula (12.22) is related to the simple positive root vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, according to

$$n = -2\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} \ . \tag{12.23}$$

Proof:

We first define the scalar product for the root vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ by the expression:

$$\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = \alpha_i\beta_i \ . \tag{12.24}$$

According to formula (12.22) there exists a step operator $E_{\beta\,+\,m\alpha}$ for a non-negative integer $m \leq n$. Let us for this step operator determine the following:

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\beta} + m\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})E_{\beta\,+\,m\alpha} = \sum_i \alpha_i(\beta_i + m\alpha_i)E_{\beta\,+\,m\alpha} \ . \tag{12.25}$$

Using the formulas (12.11) and (12.14), we obtain for this expression:

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\beta} + m\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})E_{\beta\,+\,m\alpha} = \left[\sum_i \alpha_i H_i\,,\,E_{\beta\,+\,m\alpha}\right] \ .$$

Using furthermore the formulas (12.19) and (12.20), we may write this expression as:

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\beta} + m\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})E_{\beta\,+\,m\alpha} = \left[\left[E_\alpha\,,\,E_{-\alpha}\right]\,,\,E_{\beta\,+\,m\alpha}\right] \ .$$

Using finally, once more the Jacobi identity for commutators and moreover the definition of $N(\alpha,\beta)$ as given in formula (12.17), we end up with:

$$\begin{aligned}
&-(\boldsymbol{\alpha}\cdot\boldsymbol{\beta} + m\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})E_{\beta\,+\,m\alpha} = \\
&\quad = \left[E_{\beta\,+\,m\alpha}\,,\,\left[E_\alpha\,,\,E_{-\alpha}\right]\right] \\
&\quad = \left[\left[E_{\beta\,+\,m\alpha}\,,\,E_\alpha\right]\,,\,E_{-\alpha}\right] + \left[E_\alpha\,,\,\left[E_{\beta\,+\,m\alpha}\,,\,E_{-\alpha}\right]\right] \\
&\quad = N(\beta+m\alpha,\alpha)\left[E_{\beta\,+\,m\alpha\,+\,\alpha}\,,\,E_{-\alpha}\right] \\
&\quad\ + N(\beta+m\alpha,-\alpha)\left[E_\alpha\,,\,E_{\beta\,+\,m\alpha\,-\,\alpha}\right] \\
&\quad = \{N(\beta+m\alpha,\alpha)N(\beta+(m+1)\alpha,-\alpha) \\
&\quad\ -N(\beta+m\alpha,-\alpha)N(\beta+(m-1)\alpha,\alpha)\}\,E_{\beta\,+\,m\alpha} \ .
\end{aligned}$$

From which relation we conclude that the following identity holds:

$$-(\boldsymbol{\alpha}\cdot\boldsymbol{\beta}+m\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}) =$$
$$= N(\beta+m\alpha,\alpha)N(\beta+(m+1)\alpha,-\alpha) - N(\beta+m\alpha,-\alpha)N(\beta+(m-1)\alpha,\alpha) \ .$$

When moreover we define:

$$A_m = N(\beta+m\alpha,\alpha)N(\beta+(m+1)\alpha,-\alpha) \ , \tag{12.26}$$

then we end up with the relation:

$$\boldsymbol{\alpha}\cdot\boldsymbol{\beta}+m\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = A_{m-1} - A_m \ . \tag{12.27}$$

This gives a recurrence relation for A_m. Now, since we know that $\boldsymbol{\beta}-\boldsymbol{\alpha}$ is not a root vector (see formula 12.21 and the discussion following that formula), we have from formula (12.17) that $N(\beta,-\alpha)=0$, which leads, by the use of formula (12.26), to $A_{-1}=0$. So, we deduce:

$$\begin{array}{lllll} A_0 &=& A_{-1}-\boldsymbol{\alpha}\cdot\boldsymbol{\beta} &=& -\boldsymbol{\alpha}\cdot\boldsymbol{\beta} \quad , \\ A_1 &=& A_0-\boldsymbol{\alpha}\cdot\boldsymbol{\beta}-\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} &=& -2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}-\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} \quad , \\ A_2 &=& A_1-\boldsymbol{\alpha}\cdot\boldsymbol{\beta}-2\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} &=& -3\boldsymbol{\alpha}\cdot\boldsymbol{\beta}-3\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} \quad ,\ldots \end{array}$$

One ends up with the following relation for A_m:

$$A_m = -(m+1)\boldsymbol{\alpha}\cdot\boldsymbol{\beta} - \tfrac{1}{2}m(m+1)\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} \ . \tag{12.28}$$

The string of root vectors (12.22) is supposed to end at the root vector $\boldsymbol{\beta}+(n+1)\boldsymbol{\alpha}$, so $E_{\beta+(n+1)\alpha}$ does not exist. This implies that $N(\beta+n\alpha,\alpha)=0$ and hence, according to the definition (12.26), $A_n=0$. Consequently, we have

$$0 = A_n = -(n+1)\boldsymbol{\alpha}\cdot\boldsymbol{\beta} - \tfrac{1}{2}n(n+1)\boldsymbol{\alpha}\cdot\boldsymbol{\alpha} \ ,$$

which is solved by equation (12.23).

4. Similarly, there exists a corresponding string of root vectors of the form:

$$\boldsymbol{\alpha},\ \boldsymbol{\alpha}+\boldsymbol{\beta},\ \boldsymbol{\alpha}+2\boldsymbol{\beta},\ \ldots\ ,\boldsymbol{\alpha}+n'\boldsymbol{\beta}, \tag{12.29}$$

which terminates at some integer n' (*i.e.* $\boldsymbol{\alpha}+(n'+1)\boldsymbol{\beta}$ is not a root vector of the Lie group). The value of the non-negative integer n' in formula (12.29) is, in a similar way as given in formula (12.23), related to the simple positive root vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, *i.e.*

$$n' = -2\frac{\boldsymbol{\beta}\cdot\boldsymbol{\alpha}}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}} \ . \tag{12.30}$$

5. This leads, using the formulas (12.23) and (12.30) and Schwarz's inequality for scalar products, to the relation:

$$nn' = 4\frac{(\boldsymbol{\beta}\cdot\boldsymbol{\alpha})^2}{(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})(\boldsymbol{\beta}\cdot\boldsymbol{\beta})} \leq 4 \ . \tag{12.31}$$

So, we find for the the product nn' the possibilities: $nn' = 0$, 1, 2, 3 or 4. However, for $nn' = 4$ one has for the relation (12.31) the solutions $\boldsymbol{\beta}= \pm\boldsymbol{\alpha}$, which solutions are both excluded, since:

$$\left\{ \begin{array}{l} \boldsymbol{\beta} = +\boldsymbol{\alpha} \ \text{ implies: } \boldsymbol{\beta} \ \text{ not simple} \\ \\ \boldsymbol{\beta} = -\boldsymbol{\alpha} \ \text{ implies: } \boldsymbol{\beta} \ \text{ not positive } \ , \end{array} \right.$$

which both contradict the definition of $\boldsymbol{\beta}$ as a simple positive root vector. Consequently, one has for the product nn' the following possibilities:

$$nn' = \ 0 \ , \ \ 1 \ , \ \ 2 \ \text{ and } \ 3 \ . \tag{12.32}$$

6. We define moreover the *relative weight* of two root vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, using equations (12.23) and (12.30), as follows:

$$\omega(\alpha,\beta) = \frac{\boldsymbol{\beta}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} = \frac{n}{n'} \ . \tag{12.33}$$

One might also define an *absolute weight factor* for each root vector, such that the root vector $\boldsymbol{\gamma}$ which has the lowest relative weight with respect to all other root vectors takes absolute weight factor 1, and the absolute weight factors of all other root vectors equal to their relative weights with respect to $\boldsymbol{\gamma}$.

12.5 Dynkin diagrams.

The simple positive root vectors of a simple Lie group and their related products nn' defined in formula (12.31), can be represented diagramatically in a so-called *Dynkin diagram*. Each simple positive root vector of the group is indicated by a point in a Dynkin diagram. Those points are connected by lines, as many as indicated by the product nn' for the two simple positive root vectors represented by the points they connect. So, points which represent simple positive root vectors for which the product nn' of formula (12.31) vanishes are not connected by a line. Points which represent simple positive root vectors for which the product nn' of formula (12.31) equals to 1 are connected by one line. And so on.

Such a diagram has a series of properties related to the structure of a simple Lie group:

1. Only connected diagrams occur. This means that there do not exist simple Lie groups for which the corresponding Dynkin diagram contains parts which are by no line connected to the other parts of the diagram.

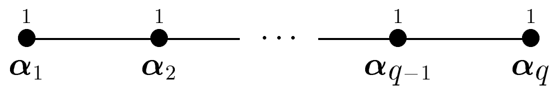
2. There are no loops.

3. There can at most three lines radiate from one point.

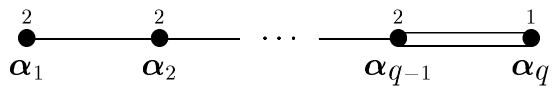
4. There is at most one double-line-connection or one bifurcation in a Dynkin diagram.

The various possible Dynkin diagrams (and hence the various possible simple Lie groups) are grouped into different types. There are A, B, C and D types of Dynkin diagrams and furthermore five exceptional cases: E_6, E_7, E_8, F_4 and G_2. The A-type Dynkin diagrams have only one-line-junctions, the B- and C-type Dynkin diagrams contain one double-line-junction and the D-type has one bifurcation. Below we show the resulting Dynkin diagrams for the various types. The numbers written above each point in the diagram indicate the absolute weight factor of the simple positive root vector represented by that point.

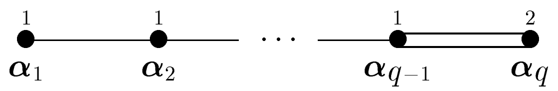
A_q $(q = 1, 2, 3, \ldots)$



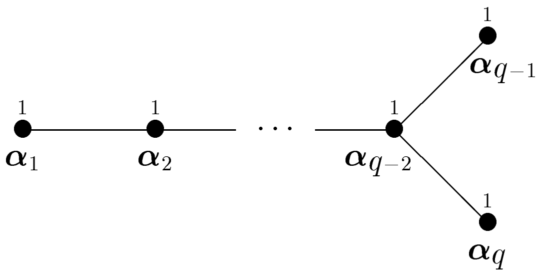
B_q $(q = 2, 3, 4, \ldots)$



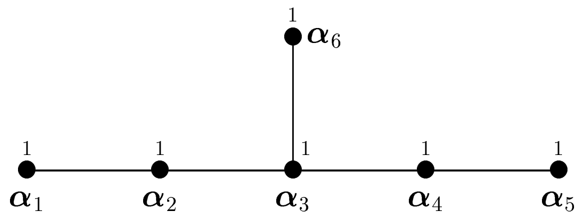
C_q $(q = 3, 4, 5, \ldots)$



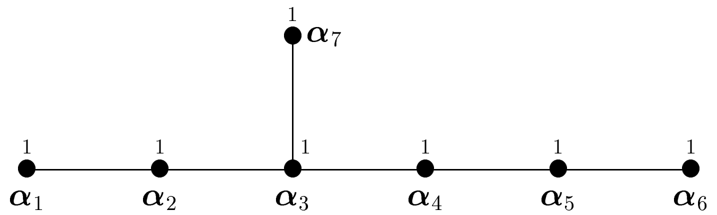
D_q $(q = 4, 5, 6, \ldots)$



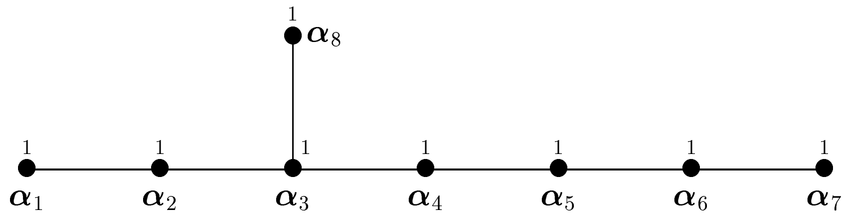
E_6



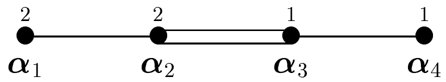
E_7



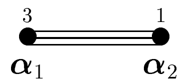
E_8



F_4



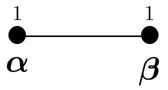
G_2



The diagrams for A_1, B_1 and C_1 are identical, from which it follows that the corresponding Lie algebras are isomorphic. The diagram for B_2 differs from that for C_2 only in the labelling of the simple positive root vectors, so the Lie algebras are also isomorphic. The same is also true for A_3 and D_3. So, all Dynkin diagrams exhibited above correspond to non-isomorphic simple Lie algebras.

12.6 The Dynkin diagram for $SU(3)$.

In this section we study the Dynkin diagram A_2, given by:



(12.34)

The diagram has two points and hence there are two simple positive root vectors, which are indicated by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Moreover, because the number of simple positive root vectors equals the dimension of the Cartan subalgebra, the rank of the group equals 2 (see section 12.2 for the definition of the rank of a Lie group). The two points are connected by one line, so the product nn' defined in (12.31) equals 1, and therefore $n = n' = 1$. This leads, by the use of formula (12.23), to:

$$\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}} = -\tfrac{1}{2} \quad \text{and} \quad |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = 1 \ \ . \tag{12.35}$$

The latter relation can also be concluded from the absolute weight factors which are indicated in the Dynkin diagram of formula (12.34). As a consequence of the equations (12.35), we find that the angle between the simple positive root vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ equals 120°. In the figure below we show the two root vectors for this case in the corresponding two-dimensional root diagram.

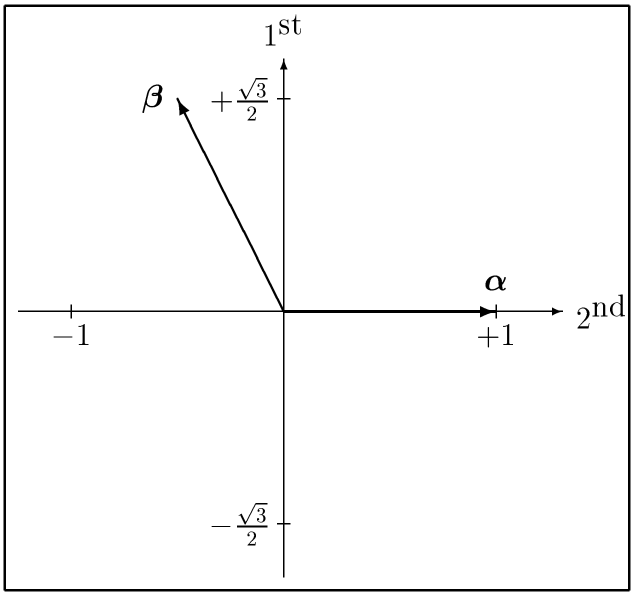


Figure 12.1: The simple positive root vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defined in formula (12.34) in the corresponding two-dimensional root diagram.

There is however a problem with the choice of the first and the second axis in the root diagram. When we take the usual x-axis as the first and the y-axis as the second axis, then the first component of $\boldsymbol{\beta}$ becomes negative, $-\frac{1}{2}$, which contradicts the definition of $\boldsymbol{\beta}$ as a simple positive root. This problem can be solved by selecting the y-axis as the first axis and the x-axis as the second, as indicated in the root diagram of figure (12.1). In that case we find for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the following vectors:

$$\boldsymbol{\alpha} = (0,1) \quad \text{and} \quad \boldsymbol{\beta} = (+\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}) \ . \tag{12.36}$$

The string (12.23) has here length $n = 1$, *i.e.*

$$\boldsymbol{\beta} \ , \quad \boldsymbol{\beta} + \boldsymbol{\alpha} \ .$$

So, $\boldsymbol{\alpha}+\boldsymbol{\beta}$ is the third and also the only non-simple positive root vector. The corresponding negative root vectors are:

$$-\boldsymbol{\alpha} \ , \quad -\boldsymbol{\beta} \ \text{and} \quad -\boldsymbol{\alpha} - \boldsymbol{\beta} \ .$$

In figure (12.2) we show all root vectors in the root diagram for the group corresponding to the Dynkin diagram of formula (12.34).

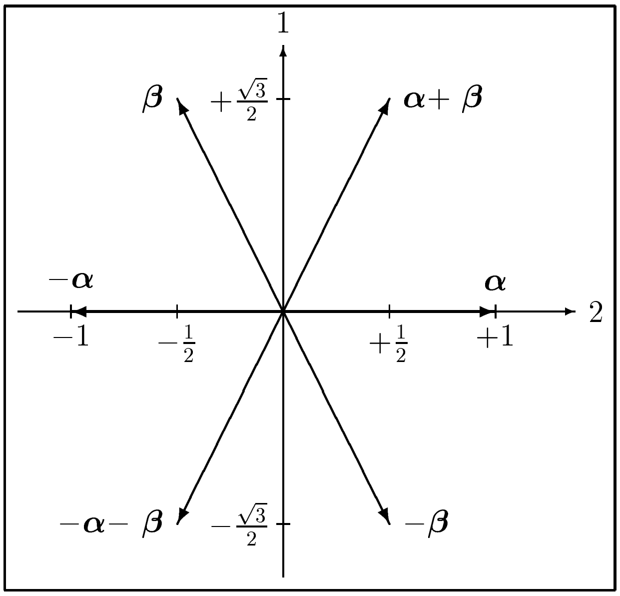


Figure 12.2: All root vectors which follow from the Dynkin diagram given in formula (12.34).

When we compare the above root diagram for the step operators of the Lie group given by formula (12.34) with figure (11.2) at page 150 for the step operators of $SU(3)$, we come to the conclusion that both root diagrams are identical and therefore both Lie algebras isomorphic, as we will see in the following.

As mentioned before, the Cartan subalgebra has two basis elements, say H_1 and H_2. Following formula (12.14) and using the results of formula (12.36), we obtain the commutation relations for H_1 and H_2 with the step operators E_α and E_β, *i.e.*

$$\begin{array}{lllll} [H_1\,,\,E_\alpha] = \alpha_1 E_\alpha = 0 & , & \left[H_1\,,\,E_\beta\right] = \beta_1 E_\beta = +\frac{\sqrt{3}}{2} E_\beta & , \\ [H_2\,,\,E_\alpha] = \alpha_2 E_\alpha = E_\alpha & \text{and} & \left[H_2\,,\,E_\beta\right] = \beta_2 E_\beta = -\frac{1}{2} E_\beta & . \end{array} \tag{12.37}$$

When we compare those relations with the commutation relations for $SU(3)$ given in formula (11.7), then we find the following identifications:

$$H_1 = M \quad , \quad H_2 = I_3 \quad , \quad E_\alpha = I_+ \ \text{ and } \ E_\beta = L_+ \quad .$$

When one also determines the other commutation relations, for example:

$$\begin{aligned} \left[H_1 \, , \, E_{\alpha + \beta}\right] &= (\alpha_1 + \beta_1) \, E_{\alpha + \beta} &= +\tfrac{\sqrt{3}}{2} \, E_{\alpha + \beta} \ \text{ and} \\ \left[H_2 \, , \, E_{\alpha + \beta}\right] &= (\alpha_2 + \beta_2) \, E_{\alpha + \beta} &= +\tfrac{1}{2} \, E_{\alpha + \beta} \qquad , \end{aligned}$$

then one finds moreover:

$$E_{\alpha + \beta} = K_+ \quad , \quad E_{-\alpha} = I_- \quad , \quad E_{-\beta} = L_- \ \text{ and } \ E_{-\alpha - \beta} = K_- \quad .$$

The Lie algebra is now completely known and irreps can be constructed. Notice that all this information is contained in the Dynkin diagram of formula (12.34).

Chapter 13

The orthogonal group in four dimensions.

Rotations in four dimensions may not seem relevant for physics, but incidentely $SO(4)$ is the symmetry group of $(\text{distance})^{-1}$ potentials, like the Newton gravitational potential and the Coulomb potential for the Hydrogen atom. We will study this group in some detail, because it is also closely related to the symmetries of special relativity.

13.1 The generators of $SO(N)$.

The logarithm of any real, unimodular, orthogonal $N \times N$ matrix is, according to the results of chapter 6 (see formulas 6.22 and 6.24), a real, traceless and anti-symmetric $N \times N$ matrix. Moreover, the diagonal elements of an anti-symmetric matrix are zero. Consequently, an anti-symmetric matrix is automatically traceless. An elementary set of real, antisymmetric $N \times N$ matrices is formed by the set of matrices $\mathcal{M}_{ab}$ $(a, b = 1, 2, \ldots N)$, whose matrix elements are defined by:

$$\left(\mathcal{M}_{ab}\right)_{k\ell} = -\delta_{ak}\delta_{b\ell} + \delta_{a\ell}\delta_{bk} \quad , \tag{13.1}$$

i.e. matrix element ab of $\mathcal{M}_{ab}$ $(a \neq b)$ equals -1, matrix element ba of $\mathcal{M}_{ab}$ $(a \neq b)$ equals $+1$ and all other of its matrix elements vanish.

The matrices $\mathcal{M}_{ab}$ are not all independent, because of the following properties:

$$\mathcal{M}_{ba} = -\mathcal{M}_{ab} \quad \text{and} \quad \mathcal{M}_{11} = \mathcal{M}_{22} = \cdots \mathcal{M}_{NN} = 0 \ . \tag{13.2}$$

As a consequence, one ends up with a set of $\frac{1}{2}(N^2 - N)$ independent matrices $\mathcal{M}_{ab}$. This set forms a basis for the space of generators of the group $SO(N)$. Any real, unimodular, orthogonal $N \times N$ matrix $\mathcal{O}$ can be written as the exponent of a linear combination of the matrices $\mathcal{M}_{ab}$, as follows:

$$\mathcal{O}(\alpha) = \exp\left\{ \sum_{i,\,j\,=\,1}^{N} \tfrac{1}{2}\alpha_{ij}\mathcal{M}_{ij} \right\} \quad . \tag{13.3}$$

As a consequence of property (13.2), one may choose for the set of real constants α_{ij} the properties:

$$\alpha_{ji} = -\alpha_{ij} \quad \text{and} \quad \alpha_{11} = \alpha_{22} = \cdots \alpha_{NN} = 0 \ . \tag{13.4}$$

This implies that all real, unimodular, orthogonal $N \times N$ matrices can be parametrized by a set of $\frac{1}{2}(N^2 - N)$ independent parameters. Consequently, $SO(N)$ is a $\frac{1}{2}N(N-1)$ parameter group. $SO(2)$ is thus a one parameter group, $SO(3)$ a three parameter group and $SO(4)$ a six parameter group.

Alternatively, one might define the set of purely imaginary, traceless, Hermitean $N \times N$ matrices M_{ab}, defined by:

$$M_{ab} = i\mathcal{M}_{ab} \ . \tag{13.5}$$

Then, substituting definition (13.5) into expression (13.3) we find that any real, unimodular, orthogonal $N \times N$ matrix $\mathcal{O}$ can be written as the exponent of a linear combination of the matrices M_{ab}, as follows:

$$\mathcal{O}(\alpha) = \exp\left\{ \sum_{i,j=1}^{N} -\frac{i}{2}\alpha_{ij} M_{ij} \right\} \ . \tag{13.6}$$

For the generators M_{ab} of $SO(N)$ one has the following commutation relations:

$$[M_{ab} \ , \ M_{cd}] = i\left\{\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{ad}M_{bc} - \delta_{bc}M_{ad}\right\} \ . \tag{13.7}$$

In order to proof the above relations, we determine one of the matrix elements of the commutator in (13.7), using the definitions (13.1) and (13.5):

$$\begin{aligned}
\left([M_{ab} \ , \ M_{cd}]\right)_{ij} &= \left(M_{ab}M_{cd} - M_{cd}M_{ab}\right)_{ij} \\
&= \left(M_{ab}\right)_{ik}\left(M_{cd}\right)_{kj} - \left(M_{cd}\right)_{ik}\left(M_{ab}\right)_{kj} \\
&= -\left(\delta_{ai}\delta_{bk} - \delta_{ak}\delta_{bi}\right)\left(\delta_{ck}\delta_{dj} - \delta_{cj}\delta_{dk}\right) + \\
&\quad + \left(\delta_{ci}\delta_{dk} - \delta_{ck}\delta_{di}\right)\left(\delta_{ak}\delta_{bj} - \delta_{aj}\delta_{bk}\right) \\
&= \delta_{ac}\left\{\delta_{bi}\delta_{dj} - \delta_{di}\delta_{bj}\right\} + \delta_{bd}\left\{\delta_{ai}\delta_{cj} - \delta_{ci}\delta_{aj}\right\} + \\
&\quad + \delta_{ad}\left\{\delta_{ci}\delta_{bj} - \delta_{bi}\delta_{cj}\right\} + \delta_{bc}\left\{\delta_{di}\delta_{aj} - \delta_{ai}\delta_{dj}\right\} \\
&= \delta_{ac}i\left(M_{bd}\right)_{ij} + \delta_{bd}i\left(M_{ac}\right)_{ij} + \delta_{ad}i\left(M_{cb}\right)_{ij} + \delta_{bc}i\left(M_{da}\right)_{ij} \\
&= i\left\{\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{ad}M_{bc} - \delta_{bc}M_{ad}\right\}_{ij} \ ,
\end{aligned}$$

which shows the equality (13.7) for each matrix element and thus proofs relations (13.7).

13.2 The generators of $SO(4)$.

For the six parameter group $SO(4)$ we define the following six generators:

$$L_1 = M_{23} \;\; , \;\; L_2 = M_{31} \;\; , \;\; L_3 = M_{12} \;\; ,$$
$$K_1 = M_{14} \;\; , \;\; K_2 = M_{24} \;\; , \;\; K_3 = M_{34} \;\; . \tag{13.8}$$

Or, using the property (13.2) of the matrices M_{ab}, in a more compact notation:

$$L_i = \tfrac{1}{2}\epsilon_{ijk}M_{jk} \quad (i,j,k = 1,2,3) \quad \text{and} \quad K_i = M_{i4} \quad (i = 1,2,3) \;\; . \tag{13.9}$$

The related matrices are explicitly given by:

$$L_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -i & \cdot \\ \cdot & i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \; , \; L_2 = \begin{pmatrix} \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \; , \; L_3 = \begin{pmatrix} \cdot & -i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \; , \tag{13.10}$$

$$K_1 = \begin{pmatrix} \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \end{pmatrix} \; , \; K_2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot \end{pmatrix} \; , \; K_3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & i & \cdot \end{pmatrix} \; . \tag{13.11}$$

Using the commutation relations (13.7) for the matrices M_{ab}, we find for the generators $\vec{L}$ and $\vec{K}$ the following commutation relations:

$$[L_i \; , \; L_j] = i\epsilon_{ijk}L_k \;\; , \;\; [L_i \; , \; K_j] = i\epsilon_{ijk}K_k \;\; \text{and} \;\; [K_i \; , \; K_j] = i\epsilon_{ijk}L_k \;\; . \tag{13.12}$$

The simple way to proof the above relations is just to use the explicit form of the commutation relations (13.7) and the definitions (13.8), *i.e.*:

$$[L_1 \; , \; L_2] = [M_{23} \; , \; M_{31}] = -iM_{21} = iM_{12} = iL_3$$

$$[L_2 \; , \; L_3] = [M_{31} \; , \; M_{12}] = -iM_{32} = iM_{23} = iL_1$$

$$[L_3 \; , \; L_1] = [M_{12} \; , \; M_{23}] = -iM_{13} = iM_{31} = iL_2$$

$$[L_1 \; , \; K_1] = [M_{23} \; , \; M_{14}] = 0$$

$$[L_1 \; , \; K_2] = [M_{23} \; , \; M_{24}] = iM_{34} = iK_3$$

$$[L_1 \; , \; K_3] = [M_{23} \; , \; M_{34}] = -iM_{24} = -iK_2$$

$$[L_2\ ,\ K_1] = [M_{31}\ ,\ M_{14}] = -iM_{34} = -iK_3$$

$$[L_2\ ,\ K_2] = [M_{31}\ ,\ M_{24}] = 0$$

$$[L_2\ ,\ K_3] = [M_{31}\ ,\ M_{34}] = iM_{14} = iK_1$$

$$[L_3\ ,\ K_1] = [M_{12}\ ,\ M_{14}] = iM_{24} = iK_2$$

$$[L_3\ ,\ K_2] = [M_{12}\ ,\ M_{24}] = -iM_{14} = iK_1$$

$$[L_3\ ,\ K_3] = [M_{12}\ ,\ M_{34}] = 0$$

$$[K_1\ ,\ K_2] = [M_{14}\ ,\ M_{24}] = iM_{12} = iL_3$$

$$[K_2\ ,\ K_3] = [M_{24}\ ,\ M_{34}] = iM_{23} = iL_1$$

$$[K_3\ ,\ K_1] = [M_{34}\ ,\ M_{14}] = iM_{31} = iL_2$$

When we define the following six parameters:

$$n_1 = \alpha_{23}\ \ ,\ \ n_2 = \alpha_{31}\ \ ,\ \ n_3 = \alpha_{12}\ \ ,\ \ k_1 = \alpha_{14}\ \ ,\ \ k_2 = \alpha_{24}\ \ ,\ \ k_3 = \alpha_{34}\ \ ,$$

then, using the relations (13.2), (13.4), (13.6) and (13.8), we may express any real, unimodular, orthogonal 4×4 matrix $\mathcal{O}$ by the exponent of a linear combination of the matrices $\vec{L}$ and $\vec{K}$, as follows:

$$\mathcal{O}\left(\vec{n},\vec{k}\right) = \exp\left\{-i\vec{n}\cdot\vec{L} - i\vec{k}\cdot\vec{K}\right\} \quad . \tag{13.13}$$

From the commutation relations (13.12) for the generators $\vec{L}$, one might observe that the transformations $\mathcal{O}(\vec{n},0)$ form a subgroup of $SO(4)$ which is equivalent to $SO(3)$. However, since the matrices $\mathcal{O}(\vec{n},0)$ are 4×4, they cannot form an irreducible representation of $SO(3)$, since irreducible representations of $SO(3)$ can only have odd dimensions. The reduction of the 4×4 matrices $\mathcal{O}(\vec{n},0)$ is most conveniently studied by determining the Casimir operator L^2 for the four dimensional representation, *i.e.* $L^2 = (L_1)^2 + (L_2)^2 + (L_3)^2$. Using the explicit expressions (13.10) and (13.11), we obtain for L^2 the following result:

$$L^2 = \begin{pmatrix} 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \quad .$$

We find that the four dimensional representation of the subgroup $SO(3)$ of $SO(4)$ can easily be reduced into two irreducible representations of $SO(3)$: one of dimension three ($L^2 = 2$), and one of dimension one ($L^2 = 0$).

In order to construct irreducible representations of $SO(4)$ it is convenient to introduce yet another set of generators for this group, *i.e.*:

$$\vec{A} = \tfrac{1}{2}\left(\vec{L} + \vec{K}\right) \quad \text{and} \quad \vec{B} = \tfrac{1}{2}\left(\vec{L} - \vec{K}\right) \quad . \tag{13.14}$$

When, using the expression (13.13), we define the following six parameters:

$$\vec{a} = \vec{n} + \vec{k} \quad \text{and} \quad \vec{b} = \vec{n} - \vec{k} \quad ,$$

then we may express any real, unimodular, orthogonal 4×4 matrix $\mathcal{O}$ by the exponent of a linear combination of the matrices $\vec{A}$ and $\vec{B}$, as follows:

$$\mathcal{O}\left(\vec{a}, \vec{b}\,\right) = \exp\left\{-i\vec{a}\cdot\vec{A} - i\vec{b}\cdot\vec{B}\right\} \quad . \tag{13.15}$$

The commutation relations for the generators $\vec{A}$ and $\vec{B}$ can easily be obtained, using the relations (13.12)

$$[A_i\ ,\ A_j] = i\epsilon_{ijk}A_k \ \ ,\ \ [B_i\ ,\ B_j] = i\epsilon_{ijk}B_k \ \text{ and } \ [A_i\ ,\ B_j] = 0 \quad . \tag{13.16}$$

Because of the latter commutation relation and relation (13.15), remembering the Baker-Campbell-Hausdorf relations, we may express any real, unimodular, orthogonal 4×4 matrix $\mathcal{O}$ by the product of two exponents: one of a linear combination of the matrices $\vec{A}$ and one of a linear combination of the matrices $\vec{B}$, as follows:

$$\mathcal{O}\left(\vec{a}, \vec{b}\,\right) = \exp\left\{-i\vec{a}\cdot\vec{A}\right\}\ \exp\left\{-i\vec{b}\cdot\vec{B}\right\} \quad . \tag{13.17}$$

The matrices $\vec{A}$ and $\vec{B}$ are explicitly given by:

$$A_1 = \frac{1}{2}\begin{pmatrix} \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & -i & \cdot \\ \cdot & i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \end{pmatrix} ,\ A_2 = \frac{1}{2}\begin{pmatrix} \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & -i \\ -i & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot \end{pmatrix} ,\ A_3 = \frac{1}{2}\begin{pmatrix} \cdot & -i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & i & \cdot \end{pmatrix} , \tag{13.18}$$

$$B_1 = \frac{1}{2}\begin{pmatrix} \cdot & \cdot & \cdot & i \\ \cdot & \cdot & -i & \cdot \\ \cdot & i & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot \end{pmatrix} ,\ B_2 = \frac{1}{2}\begin{pmatrix} \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & i \\ -i & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot \end{pmatrix} ,\ B_3 = \frac{1}{2}\begin{pmatrix} \cdot & -i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i \\ \cdot & \cdot & -i & \cdot \end{pmatrix} . \tag{13.19}$$

The generator subset formed by the matrices $\vec{A}$ generates a subgroup S_A of $SO(4)$ which, because of the commutation relations (13.16), is equivalent either to $SO(3)$ or to $SU(2)$, and similar for the generator subset formed by the matrices $\vec{B}$. From expressions (13.18) and (13.19) it is easy to determine the Casimir operators A^2 and B^2 for each of the two subspaces of $SO(4)$. One finds $A^2 = \frac{3}{4}$ and $B^2 = \frac{3}{4}$. Consequently, the subgroup S_A of real, unimodular, orthogonal 4×4 matrices $\mathcal{O}(\vec{a}, 0)$ may be reduced to a two-dimensional representation of the group $SU(2)$ as well as the subgroup S_B of real, unimodular, orthogonal 4×4 matrices $\mathcal{O}(0,\vec{b})$. This can be shown more explicitly, using the following similarity transformation S:

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ i & 0 & 0 & i \\ 0 & -1 & -1 & 0 \\ 0 & -i & i & 0 \end{pmatrix} \quad\text{with}\quad S^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & -1 & -i \\ -1 & -i & 0 & 0 \end{pmatrix} \quad .$$

When we apply this basis transformation to the matrices (13.18) and (13.19), we find:

$$S^{-1}\vec{A}S=\frac{\vec{\sigma}}{2}\otimes\mathbf{1} \quad\text{and}\quad S^{-1}\vec{B}S=\mathbf{1}\otimes\frac{\vec{\sigma}}{2} \quad . \tag{13.20}$$

Consequently, for any real, unimodular, orthogonal 4×4 matrix $\mathcal{O}$ we obtain:

$$S^{-1}\mathcal{O}\left(\vec{a},\vec{b}\,\right)S=\exp\left\{-i\vec{a}\cdot\frac{\vec{\sigma}}{2}\right\}\otimes\exp\left\{-i\vec{b}\cdot\frac{\vec{\sigma}}{2}\right\} \quad . \tag{13.21}$$

Each group element $\mathcal{O}\left(\vec{a},\vec{b}\,\right)$ of $SO(4)$ is equivalent to the direct product of two group elements $U\left(\vec{a}\,\right)=\exp\left\{-i\vec{a}\cdot\frac{\vec{\sigma}}{2}\right\}$ and $U\left(\vec{b}\,\right)=\exp\left\{-i\vec{b}\cdot\frac{\vec{\sigma}}{2}\right\}$ of $SU(2)$. Consequently, $SO(4)$ is *homomorphous* with $SU(2)\otimes SU(2)$. However, those two groups are not equal.

13.3 The group $SU(2)\otimes SU(2)$.

We represent the direct product $U\left(\vec{a},\vec{b}\,\right)$ of two elements $U\left(\vec{a}\right)$ and $U\left(\vec{b}\,\right)$ of $SU(2)$ in the following abstract way, by:

$$U\left(\vec{a},\vec{b}\,\right)=\left(U\left(\vec{a}\right)\ ,\ U\left(\vec{b}\,\right)\right) \quad . \tag{13.22}$$

The group product is defined in analogy with the product (6.29) of two direct products of two matrices, as follows:

$$U\left(\vec{a},\vec{b}\,\right)U\left(\vec{c},\vec{d}\,\right)=\left(U\left(\vec{a}\right)U\left(\vec{c}\right)\ ,\ U\left(\vec{b}\,\right)U\left(\vec{d}\,\right)\right) \quad . \tag{13.23}$$

It is easy to verify that under the product (13.23) the elements (13.22) form a group, the group $SU(2)\otimes SU(2)$: For example, the identity element $\boldsymbol{I}$ is given by:

$$\boldsymbol{I}=U(0,0)=(\mathbf{1}\ ,\ \mathbf{1}) \quad , \tag{13.24}$$

and the inverse of group element $U\left(\vec{a},\vec{b}\,\right)$ by (compare formula (6.30)):

$$\left[U\left(\vec{a},\vec{b}\,\right)\right]^{-1}=\left(\left[U\left(\vec{a}\right)\right]^{-1}\ ,\ \left[U\left(\vec{b}\,\right)\right]^{-1}\right) \quad . \tag{13.25}$$

The structure of $SU(2)\otimes SU(2)$ is similar to the structure of $SO(4)$ as described in section (13.2). Their algebras are equal. The generators $\vec{A}$ and $\vec{B}$ defined in formula (13.14) and in detail related to $SU(2)\otimes SU(2)$ in formula (13.20), might here, using the operators $\vec{J}$ for $SU(2)$ defined in formula (10.15), be defined by:

$$\vec{A} = \left(\frac{\sigma}{2}, 0\right) = (\vec{J}, 0) \quad \text{and} \quad \vec{B} = \left(0, \frac{\sigma}{2}\right) = (0, \vec{J}) \quad . \tag{13.26}$$

The generators $\vec{A}$ generate the subgroup $S_A = \{(U(\vec{a}), \mathbf{1})\}$ of $SU(2) \otimes SU(2)$, and the generators $\vec{B}$ the subgroup $S_B = \left\{\left(\mathbf{1}, U\left(\vec{b}\right)\right)\right\}$. The group elements of $SU(2) \otimes SU(2)$ can be represented by direct products of matrices. An irreducible representation is fully determined by the eigenvalues of the two Casimir operators A^2 and B^2, or alternatively by j_a and j_b, *i.e.*

$$\begin{aligned} D^{(j_a,\, j_b)}\left(U\left(\vec{a}, \vec{b}\,\right)\right) &= D^{(j_a)}\left(U\left(\vec{a}\right)\right) \otimes D^{(j_b)}\left(U\left(\vec{b}\,\right)\right) \\ &= \left\{e^{-i\vec{a}\,\cdot\, d^{(j_a)}(\vec{J})}\right\} \otimes \left\{e^{-i\vec{b}\,\cdot\, d^{(j_b)}(\vec{J})}\right\} \quad . \end{aligned} \tag{13.27}$$

The dimension of the irreducible representation is given by:

$$\dim\left(D^{(j_a,\, j_b)}\right) = (2j_a + 1)(2j_b + 1) \quad . \tag{13.28}$$

The first few irreps of $SU(2) \otimes SU(2)$ and their dimensions are listed in the table (13.1).

d	(j_a, j_b)	d	(j_a, j_b)
1	(0,0)	9	(1,1), (0,4), (4,0)
2	(0,$\frac{1}{2}$), ($\frac{1}{2}$,0)	10	(0,$\frac{9}{2}$), ($\frac{9}{2}$,0), (2,$\frac{1}{2}$), ($\frac{1}{2}$,2)
3	(0,1), (1,0)	11	(0,5), (5,0)
4	(0,$\frac{3}{2}$), ($\frac{3}{2}$,0), ($\frac{1}{2}$,$\frac{1}{2}$)	12	(0,$\frac{11}{2}$), ($\frac{11}{2}$,0), ($\frac{1}{2}$,$\frac{5}{2}$), ($\frac{5}{2}$,$\frac{1}{2}$), (1,$\frac{3}{2}$), ($\frac{3}{2}$,1)
5	(0,2), (2,0)	13	(0,6), (6,0)
6	(0,$\frac{5}{2}$), ($\frac{5}{2}$,0), (1,$\frac{1}{2}$), ($\frac{1}{2}$,1)	14	(0,$\frac{13}{2}$), ($\frac{13}{2}$,0), (3,$\frac{1}{2}$), ($\frac{1}{2}$,3)
7	(0,3), (3,0)	15	(0,7), (7,0), (1,2), (2,1)
8	(0,$\frac{7}{2}$), ($\frac{7}{2}$,0), ($\frac{1}{2}$,$\frac{3}{2}$), ($\frac{3}{2}$,$\frac{1}{2}$)	16	(0,$\frac{15}{2}$), ($\frac{15}{2}$,0) ($\frac{1}{2}$,$\frac{7}{2}$), ($\frac{7}{2}$,$\frac{1}{2}$), ($\frac{3}{2}$,$\frac{3}{2}$)

Table 13.1: Irreps of $SU(2) \otimes SU(2)$ and their dimensions (d).

As an example, let us study the two dimensional irrep $(0,\frac{1}{2})$ of $SU(2)\otimes SU(2)$. A group element $U\left(\vec{a},\vec{b}\right)$ is, according to formula (13.27), represented by:

$$\begin{aligned} D^{\left(0,\,\frac{1}{2}\right)}\left(U\left(\vec{a},\vec{b}\,\right)\right) &= D^{(0)}\left(U\left(\vec{a}\right)\right)\otimes D^{\left(\frac{1}{2}\right)}\left(U\left(\vec{b}\,\right)\right) \\ &= \left\{e^{0}\right\}\otimes\left\{e^{-i\vec{b}\cdot\frac{\vec{\sigma}}{2}}\right\} \\ &= 1\otimes\left\{e^{-i\vec{b}\cdot\frac{\vec{\sigma}}{2}}\right\} = e^{-i\vec{b}\cdot\frac{\vec{\sigma}}{2}} \quad , \end{aligned} \tag{13.29}$$

which represents a 2×2 matrix.

The two dimensional irrep $(0,\frac{1}{2})$ is not a faithfull representation of $SU(2)\otimes SU(2)$, as one has:

$$D^{\left(0,\,\frac{1}{2}\right)}\left(U\left(\vec{a}_1,\vec{b}\,\right)\right) = D^{\left(0,\,\frac{1}{2}\right)}\left(U\left(\vec{a}_2,\vec{b}\,\right)\right) \quad \text{for all} \quad \vec{a}_1 \ \text{ and } \ \vec{a}_2 \quad .$$

The identity $\boldsymbol{I}=(\mathbf{1},\mathbf{1})$ is, as always, represented by the unit matrix, according to:

$$D^{\left(0,\,\frac{1}{2}\right)}\left(\boldsymbol{I}\right) = D^{(0)}\left(\mathbf{1}\right)\otimes D^{\left(\frac{1}{2}\right)}\left(\mathbf{1}\right) = 1\otimes\mathbf{1} = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \quad .$$

The group element $(-\mathbf{1},-\mathbf{1})$ is represented by the minus unit matrix, according to:

$$D^{\left(0,\,\frac{1}{2}\right)}\left((-\mathbf{1},-\mathbf{1})\right) = D^{(0)}\left(-\mathbf{1}\right)\otimes D^{\left(\frac{1}{2}\right)}\left(-\mathbf{1}\right) = 1\otimes(-\mathbf{1}) = -\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right) \quad .$$

Notice from the above two expressions that for the two dimensional irrep $(0,\frac{1}{2})$ one has:

$$D^{\left(0,\,\frac{1}{2}\right)}\left((\mathbf{1},\mathbf{1})\right) \neq D^{\left(0,\,\frac{1}{2}\right)}\left((-\mathbf{1},-\mathbf{1})\right) \quad .$$

This is not always the case. For example, for the four dimensional irrep $(\frac{1}{2},\frac{1}{2})$ one finds:

$$D^{\left(\frac{1}{2},\,\frac{1}{2}\right)}\left((\mathbf{1},\mathbf{1})\right) = \mathbf{1}\otimes\mathbf{1} = \left(\begin{array}{cccc} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{array}\right) = (-\mathbf{1})\otimes(-\mathbf{1}) = D^{\left(\frac{1}{2},\,\frac{1}{2}\right)}\left((-\mathbf{1},-\mathbf{1})\right) \quad . \tag{13.30}$$

and similar:

$$\begin{aligned} D^{\left(\frac{1}{2},\,\frac{1}{2}\right)}\left(\left(U\left(\vec{a}\right),U\left(\vec{b}\right)\right)\right) &= U\left(\vec{a}\right)\otimes U\left(\vec{b}\right) \\ &= \left\{-U\left(\vec{a}\right)\right\}\otimes\left\{-U\left(\vec{b}\right)\right\} \\ &= D^{\left(\frac{1}{2},\,\frac{1}{2}\right)}\left(\left(-U\left(\vec{a}\right),-U\left(\vec{b}\right)\right)\right) \quad . \end{aligned} \tag{13.31}$$

So, also the four dimensional irrep $(\frac{1}{2},\frac{1}{2})$ is not faithfull, since sets of two elements of $SU(2) \otimes SU(2)$ are represented by the the same matrix.

13.4 The relation between $SO(4)$ and $SU(2) \otimes SU(2)$.

As we have seen in formula (13.21), each group element $\mathcal{O}\left(\vec{a},\vec{b}\right)$ of $SO(4)$ is equivalent to the direct product of two group elements $U(\vec{a}\,) = \exp\left\{-i\vec{a}\cdot\frac{\vec{\sigma}}{2}\right\}$ and $U\left(\vec{b}\,\right) = \exp\left\{-i\vec{b}\cdot\frac{\vec{\sigma}}{2}\right\}$ of $SU(2)$. Consequently, $SO(4)$ is isomorphous to the four dimensional irrep $(\frac{1}{2},\frac{1}{2})$ of $SU(2) \otimes SU(2)$. Which implies that each group element of $SO(4)$ is equivalent to two different group elements of $SU(2) \otimes SU(2)$.

The set of group elements of $SU(2) \otimes SU(2)$ indicated by $\{(\mathbf{1},\mathbf{1})\ ,\ (-\mathbf{1},-\mathbf{1})\}$ is called the kernel of the homomorphism of $SU(2) \otimes SU(2)$ onto $SO(4)$. When we divide $SU(2) \otimes SU(2)$ by this kernel, which means that when we either consider group element $\left(U(\vec{a}), U\left(\vec{b}\,\right)\right)$ or group element $\left(-U(\vec{a}), -U\left(\vec{b}\,\right)\right)$, then we obtain the following *isomorphism* for $SO(4)$:

$$
SO(4) = \frac{SU(2) \otimes SU(2)}{\{(\mathbf{1},\mathbf{1})\ ,\ (-\mathbf{1},-\mathbf{1})\}} \quad . \tag{13.32}
$$

13.5 Irreps for $SO(4)$.

Let us compare the above relation (13.32) with the relation between $SO(3)$ and $SU(2)$. All irreps for $SO(3)$ can be obtained from the irreps of $SU(2)$, provided that we eliminate those irreps of $SU(2)$ for which $D(\mathbf{1}) \neq D(-\mathbf{1})$, *i.e.* those with even dimensions. For $SO(4)$ the situation is similar: From all irreps of $SU(2)\otimes SU(2)$ we just must eliminate those for which:

$$
D((\mathbf{1},\mathbf{1})) \ \neq\ D((-\mathbf{1},-\mathbf{1})) \quad . \tag{13.33}
$$

When we denote the eigenvalues of A^2 by $2j_a+1$ and of B^2 by $2j_b+1$, then, as we have seen in formula (13.27), a general irrep for $SU(2) \otimes SU(2)$ may be written in the form

$$
D^{(j_a,\,j_b)}\left(\left(U(\vec{a}), U\left(\vec{b}\,\right)\right)\right) = D^{(j_a)}\left(U(\vec{a}\,)\right) \otimes D^{(j_b)}\left(U\left(\vec{b}\,\right)\right) \quad . \tag{13.34}
$$

Now, for the element $-\mathbf{1}$ of $SU(2)$, one has:

$$
D^{(j)}(-\mathbf{1}) \ = \left\{\begin{array}{ll} D^{(j)}(\mathbf{1}) & ,\quad j = 0,1,2,3,\ldots \\ & \\ -D^{(j)}(\mathbf{1}) & ,\quad j = \frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots \end{array}\right. \quad . \tag{13.35}
$$

Consequently, using relation (13.34), one obtains for the representation of the comparable group elements of $SU(2) \otimes SU(2)$ the following:

$$D^{(j_a,\, j_b)}((-\mathbf{1},-\mathbf{1})) = \begin{cases} D^{(j_a,\, j_b)}((\mathbf{1},\mathbf{1})) & , \quad j_a+j_b = 0,1,2,3,\ldots \\ \\ -D^{(j_a,\, j_b)}((\mathbf{1},\mathbf{1})) & , \quad j_a+j_b = \frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots \end{cases} \quad . \tag{13.36}$$

Now, since for $SO(4)$ we must eliminate those irreps of $SU(2)\otimes SU(2)$ for which relation (13.33) holds, we find as a result from the above formula (13.36) that:

$$j_a + j_b = \text{ integer } \quad \Leftrightarrow \quad \begin{cases} \text{either } \; j_a \text{ and } j_b \text{ both integer} \\ \\ \text{or } \; j_a \text{ and } j_b \text{ both half-integer} \end{cases} \quad . \tag{13.37}$$

The dimension of an irrep of $SO(4)$ is given by:

$$\dim\left(D^{(j_a,\, j_b)}\right) = (2j_a+1)(2j_b+1) \quad . \tag{13.38}$$

In table (13.2), we indicate the first few irreps of $SO(4)$ and their dimension:

dimension	(j_a, j_b)	dimension	(j_a, j_b)
1	(0,0)	9	(1,1), (0,4), (4,0)
2	-	10	-
3	(0,1), (1,0)	11	(0,5), (5,0)
4	$(\frac{1}{2},\frac{1}{2})$	12	$(\frac{1}{2},\frac{5}{2})$, $(\frac{5}{2},\frac{1}{2})$
5	(0,2), (2,0)	13	(0,6), (6,0)
6	-	14	-
7	(0,3), (3,0)	15	(0,7), (7,0), (1,2), (2,1)
8	$(\frac{1}{2},\frac{3}{2})$, $(\frac{3}{2},\frac{1}{2})$	16	$(\frac{1}{2},\frac{7}{2})$, $(\frac{7}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{3}{2})$

Table 13.2: Irreps of $SO(4)$ and their dimension.

From this table, we notice that for some dimensions (2, 6 ,10, 14, ...) $SO(4)$ has no corresponding irreps. Whereas for other dimensions more than one inequivalent irrep exist. There are, for instance, two three dimensional irreps, namely ($j_a = 0$, $j_b = 1$) and ($j_a = 1$, $j_b = 0$). Let us, as an example, show that those two irreps are not equivalent:

For the first irrep ($j_a = 0$, $j_b = 1$) we have the representations:

$$d^{(0,1)}(\vec{A}) = 0 \quad \text{and} \quad d^{(0,1)}(\vec{B}) = \vec{L} \quad , \tag{13.39}$$

where $\vec{L}$ represents the three 3×3 matrices given in formula (9.24). For the group element $\mathcal{O}\left(\vec{a},\vec{b}\,\right)$ of $SO(4)$ we obtain then the representation:

$$
\begin{aligned}
D^{(0,1)}\left(\mathcal{O}\left(\vec{a},\vec{b}\,\right)\right) &= \exp\left\{-i\vec{a}\cdot d^{(0,1)}(\vec{A})\right\} \otimes \exp\left\{-i\vec{b}\cdot d^{(0,1)}(\vec{B})\right\} \\
&= 1\otimes e^{-i\vec{b}\cdot\vec{L}} = e^{-i\vec{b}\cdot\vec{L}} \quad , \qquad (13.40)
\end{aligned}
$$

which represents a 3×3 matrix.

Similarly, for the other irrep ($j_a=1$, $j_b=0$) we have the representations:

$$
d^{(1,0)}(\vec{A}) = \vec{L} \quad \text{and} \quad d^{(1,0)}(\vec{B}) = 0 \quad , \qquad (13.41)
$$

leading for the representation of the group element $\mathcal{O}\left(\vec{a},\vec{b}\,\right)$ of $SO(4)$ to the expression:

$$
D^{(1,0)}\left(\mathcal{O}\left(\vec{a},\vec{b}\,\right)\right) = e^{-i\vec{a}\cdot\vec{L}} \quad . \qquad (13.42)
$$

Now, representations ($j_a=0$, $j_b=1$) and ($j_a=1$, $j_b=0$) are equivalent when there exists a similarity transformation S, such that for all possible $\vec{a}$ and $\vec{b}$ yields:

$$
S^{-1}\,D^{(0,1)}\left(\mathcal{O}\left(\vec{a},\vec{b}\,\right)\right)\,S = D^{(1,0)}\left(\mathcal{O}\left(\vec{a},\vec{b}\,\right)\right) \quad , \qquad (13.43)
$$

which, using the expressions (13.40) and (13.42), amounts in finding a basis transformation S such that:

$$
S^{-1}\,e^{-i\vec{b}\cdot\vec{L}}\,S = e^{-i\vec{a}\cdot\vec{L}} \quad , \qquad (13.44)
$$

or at the level of the algebra, such that:

$$
S^{-1}\,\vec{b}\cdot\vec{L}\,S = \vec{a}\cdot\vec{L} \quad \text{for all} \quad \vec{a} \quad \text{and} \quad \vec{b} \quad . \qquad (13.45)
$$

This relation must hold for arbitrary and independent $\vec{a}$ and $\vec{b}$, which is impossible for one single transformation S. Consequently, the two irreps ($j_a=0$, $j_b=1$) and ($j_a=1$, $j_b=0$) are not equivalent.

13.6 Weight diagrams for $SO(4)$.

Let us for the irreducible representation $D^{(j_a,\,j_b)}$, select basis vectors $\{|m_a,m_b>\}$ which are simultaneously eigenvectors for $d^{(j_a,\,j_b)}\,(A_3)$ and $d^{(j_a,\,j_b)}\,(B_3)$ with eigenvalues m_a and m_b respectively, *i.e.*:

$$
\begin{aligned}
d^{(j_a,\,j_b)}\,(A_3)\;|m_a,m_b> &= m_a\,|m_a,m_b> \quad \text{and} \\
d^{(j_a,\,j_b)}\,(B_3)\;|m_a,m_b> &= m_b\,|m_a,m_b> \quad . \qquad (13.46)
\end{aligned}
$$

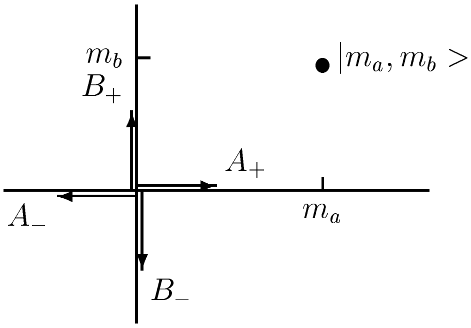
One might, in analogy with $SO(3)$, also define the step operators $A_\pm = A_1 \pm iA_2$ for which:

$$d^{(j_a,\, j_b)}\,(A_\pm)\ \ |m_a, m_b> = \sqrt{j_a(j_a+1) - m_a(m_a \pm 1)}\ |m_a \pm 1, m_b> \quad . \qquad (13.47)$$

Similarly, we might define the step operators $B_\pm = B_1 \pm iB_2$ for which:

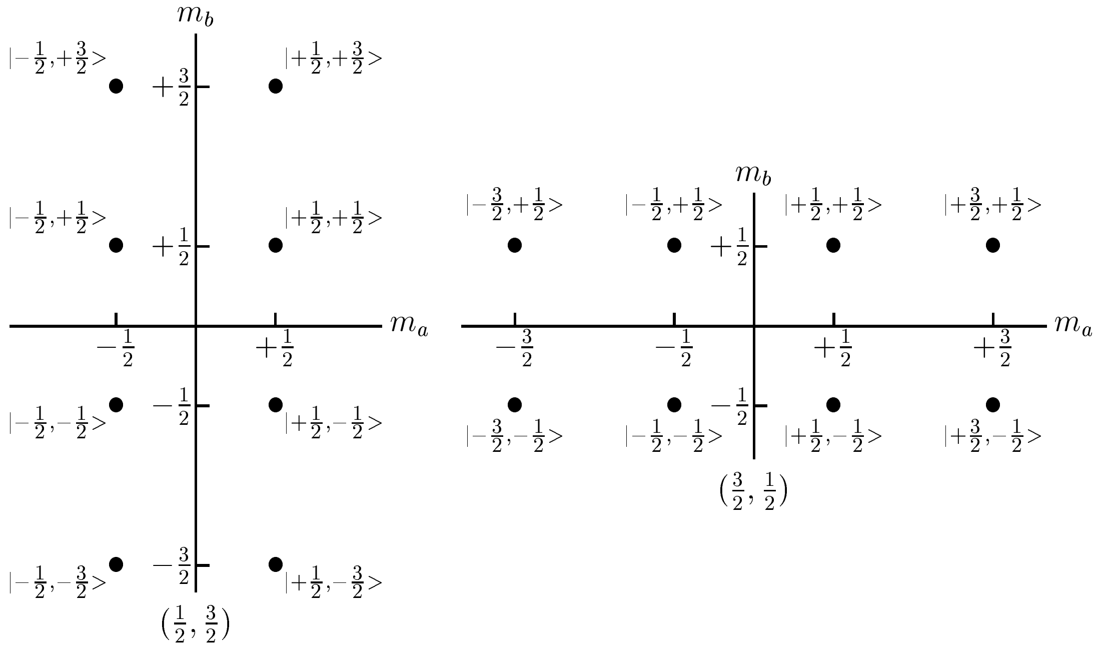
$$d^{(j_a,\, j_b)}\,(B_\pm)\ \ |m_a, m_b> = \sqrt{j_a(j_a+1) - m_b(m_b \pm 1)}\ |m_a, m_b \pm 1> \quad . \qquad (13.48)$$

Weight diagrams for $SO(4)$ have consequently two dimensions. Let us select the horizontal axis to indicate m_a and the vertical axis to indicate m_b. Each basis vector $|m_a, m_b>$ is represented by a point as shown in the figure below.



In this figure are also indicated the directions in which act the various step operators.

As an example we show in the figure below the weight diagrams for the two eight dimensional irreps of $SO(4)$ indicated by $(\frac{1}{2},\frac{3}{2})$ and $(\frac{3}{2},\frac{1}{2})$.



13.7 Invariant subgroups of $SO(4)$.

In this section we show that the sets of group elements $S_A = \{\mathcal{O}(\vec{a}, 0)\}$ and $S_B = \left\{\mathcal{O}(0, \vec{b})\right\}$ are invariant subgroups of $SO(4)$. Let us, using the expression (13.21), represent group element $\mathcal{O}(\vec{a}, \vec{b})$ of $SO(4)$ by:

$$\mathcal{O}\left(\vec{a}, \vec{b}\right) = S\left\{U\left(\vec{a}\,\right) \otimes U\left(\vec{b}\,\right)\right\} S^{-1} \quad . \tag{13.49}$$

According to the definition of an invariant subgroup of a group, which is given in formula (1.24), we must concentrate on the following expression:

$$\left[\mathcal{O}\left(\vec{\alpha}, \vec{\beta}\,\right)\right]^{-1} \mathcal{O}\left(\vec{a}, \vec{b}\,\right) \left[\mathcal{O}\left(\vec{\alpha}, \vec{\beta}\,\right)\right] \quad . \tag{13.50}$$

Inserting the above expression (13.49) and using formula (6.30), we obtain:

$$S\left\{\left[U\left(\vec{\alpha}\,\right)\right]^{-1} \otimes \left[U\left(\vec{\beta}\,\right)\right]^{-1}\right\} S^{-1} S\left\{U\left(\vec{a}\,\right) \otimes U\left(\vec{b}\,\right)\right\} S^{-1} S\left\{U\left(\vec{\alpha}\,\right) \otimes U\left(\vec{\beta}\,\right)\right\} S^{-1} \quad .$$

Next, using the property (6.29) for the product of direct products of matrices, we obtain:

$$S\left(\left\{\left[U\left(\vec{\alpha}\,\right)\right]^{-1} U\left(\vec{a}\,\right) U\left(\vec{\alpha}\,\right)\right\} \otimes \left\{\left[U\left(\vec{\beta}\,\right)\right]^{-1} U\left(\vec{b}\,\right) U\left(\vec{\beta}\,\right)\right\}\right) S^{-1} \quad .$$

In case that the group element $\mathcal{O}(\vec{a}, \vec{b})$ belongs to one of the two subgroups S_A or S_B, we find in particular:

$$\begin{cases} S\left(\left\{\left[U\left(\vec{\alpha}\,\right)\right]^{-1} U\left(\vec{a}\,\right) U\left(\vec{\alpha}\,\right)\right\} \otimes \mathbf{1}\right) S^{-1} & \text{for} \quad \vec{b} = 0, \\ S\left(\mathbf{1} \otimes \left\{\left[U\left(\vec{\beta}\,\right)\right]^{-1} U\left(\vec{b}\,\right) U\left(\vec{\beta}\,\right)\right\}\right) S^{-1} & \text{for} \quad \vec{a} = 0. \end{cases}$$

This can be written in terms of the transformed vectors $\vec{a}\,'$ and $\vec{b}\,'$, according to:

$$\begin{cases} S\left(U\left(\vec{a}\,'\right) \otimes \mathbf{1}\right) S^{-1} & \text{for} \quad \vec{b} = 0, \\ S\left(\mathbf{1} \otimes U\left(\vec{b}\,'\right)\right) S^{-1} & \text{for} \quad \vec{a} = 0. \end{cases} \tag{13.51}$$

So, we find that under the transformation (13.50) each subgroup element $\mathcal{O}(\vec{a}, 0)$ of S_A is transformed into another (or the same) subgroup element of S_A and similar for S_B. It implies that for an arbitrary element $\mathcal{O}$ of $SO(4)$ we have:

$$\mathcal{O}^{-1}\, S_A\, \mathcal{O} \;=\; S_A \quad \text{and} \quad \mathcal{O}^{-1}\, S_B\, \mathcal{O} \;=\; S_B \quad . \tag{13.52}$$

Consequently, S_A and S_B are invariant subgroups of $SO(4)$. From the commutation relations (13.16) for the generators $\vec{A}$ of S_A and $\vec{B}$ of S_B, we conclude that those subgroups are moreover not Abelian.

One may show in general that $SO(4)$ has non-trivial invariant subgroups, but no non-trivial Abelian invariant subgroups. Such a group is called *semi-simple*.

13.8 The Hydrogen atom

As an application of the representations of $SO(4)$ we study in this section the Hydrogen atom.

1. Classically

Classically, the equations of motion of a charged spinless particle in a central atractive Coulomb field (or a massive particle in a gravitational field) can be derived from the following Lagrangian:

$$\mathcal{L}\left(\vec{r},\dot{\vec{r}}\right) = \tfrac{1}{2}\dot{\vec{r}}^{\,2} + \frac{1}{r} \quad . \tag{13.53}$$

The equations of motion are given by:

$$\ddot{\vec{r}} = -\frac{\vec{r}}{r^3} \quad . \tag{13.54}$$

Associated with the Lagrangian (13.53) are the following two conserved quantities:

The orbital angular momentum	$\vec{L} = \vec{r}\times\dot{\vec{r}}$	$\frac{d\vec{L}}{dt} = 0$
The Runge-Lenz vector	$\vec{M} = -\dot{\vec{r}}\times\vec{L} + \frac{\vec{r}}{r}$	$\frac{d\vec{M}}{dt} = 0$

Using the equations of motion (13.54), it is easy to show that the above quantities are conserved:

$$\begin{aligned}
\frac{d\vec{L}}{dt} &= \dot{\vec{r}}\times\dot{\vec{r}} + \vec{r}\times\ddot{\vec{r}} = 0 + \vec{r}\times\left(-\frac{\vec{r}}{r^3}\right) = 0. \\
\frac{d\vec{M}}{dt} &= -\ddot{\vec{r}}\times\vec{L} - \dot{\vec{r}}\times\dot{\vec{L}} + \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^3}\left(\vec{r}\cdot\dot{\vec{r}}\right) \\
&= \frac{\vec{r}}{r^3}\times\vec{L} - 0 + \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^3}\left(\vec{r}\cdot\dot{\vec{r}}\right) \\
&= \frac{\vec{r}}{r^3}\times\left(\vec{r}\times\dot{\vec{r}}\right) + \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^3}\left(\vec{r}\cdot\dot{\vec{r}}\right) \\
&= \frac{1}{r^3}\left\{\vec{r}\left(\vec{r}\cdot\dot{\vec{r}}\right) - \dot{\vec{r}}\left(\vec{r}\cdot\vec{r}\right)\right\} + \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^3}\left(\vec{r}\cdot\dot{\vec{r}}\right) = 0
\end{aligned}$$

The Runge-Lenz vector for a bound state solution of the equations of motion (elipse) points from the force center towards the perihelion of the elipse. The modulus of $\vec{M}$ equals the excentricity of the orbit.

2. Quantum Mechanically

In Quantum Mechanics, the equations of motion are given by the following Hamiltonian:

$$\mathcal{H} = \dot{\vec{r}} \cdot \vec{p} - \mathcal{L}\left(\vec{r}, \dot{\vec{r}}\right) = \tfrac{1}{2}\vec{p}^{\,2} - \frac{1}{r} \quad \text{with} \quad \vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \dot{\vec{r}} \quad . \tag{13.55}$$

In comparison with the classical case, one may derive from this Hamiltonian the following constants of motion:

$$\vec{L} = \vec{r} \times \vec{p} \quad \text{and} \quad \vec{M} = \tfrac{1}{2}\left\{\vec{L} \times \vec{p} - \vec{p} \times \vec{L}\right\} + \frac{\vec{r}}{r} \quad . \tag{13.56}$$

It may be verified that both $\vec{L}$ and $\vec{M}$ commute with the Hamiltonian, *i.e.*

$$[H\ ,\ \vec{L}] = [H\ ,\ \vec{M}] = 0 \quad . \tag{13.57}$$

Furthermore, one has the expression:

$$\vec{L} \cdot \vec{M} = \vec{M} \cdot \vec{L} = 0; \quad . \tag{13.58}$$

When we define next the following operators:

$$\vec{A} = \tfrac{1}{2}\left(\vec{L} + \vec{M}\right) \quad \text{and} \quad \vec{B} = \tfrac{1}{2}\left(\vec{L} - \vec{M}\right) \quad , \tag{13.59}$$

then we may verify for $\vec{A}$ and $\vec{B}$ the same commutation relations as those given in formula (13.16). Moreover, using the commutation relations (13.57), it may be clear that $\vec{A}$ and $\vec{B}$ commute with the Hamiltonian (13.55), and using the identity (13.58), it follows that the Casimir operators A^2 and B^2 satisfy the relation:

$$A^2 - B^2 = \vec{L} \cdot \vec{M} = 0 \quad . \tag{13.60}$$

The group associated with the operators $\vec{A}$ and $\vec{B}$ is evidently $SO(4)$. Consequently, we may expect that the eigenstates of the Hamiltonian (13.55) can be grouped into irreps of $SO(4)$.

Eigenstates of the Schrödinger equation may then be chosen such that they are simultaneously eigenstates of $d(A_3)$ and $d(B_3)$, *i.e.*:

$$\begin{aligned} \mathcal{H}\ |n, m_a, m_b > &\sim -\frac{1}{n^2}\ |n, m_a, m_b > \quad , \\ d(A_3)\ |n, m_a, m_b > &= m_a\ |n, m_a, m_b > \quad \text{and} \\ d(B_3)\ |n, m_a, m_b > &= m_b\ |n, m_a, m_b > \quad . \end{aligned} \tag{13.61}$$

For the Casimir operators A^2 and B^2 one finds:

$$\begin{aligned} A^2\ |n, m_a, m_b > &= j_a(j_a + 1)\ |n, m_a, m_b > \quad \text{and} \\ B^2\ |n, m_a, m_b > &= j_b(j_b + 1)\ |n, m_a, m_b > \quad . \end{aligned} \tag{13.62}$$

However, because of the relation (13.60), one has an extra condition for the possible irreps of $SO(4)$ into which the eigenstates of the Hamiltonian (13.55) can be grouped, *i.e.*

$$j_a(j_a+1) - j_b(j_b+1) = 0 \quad \Leftrightarrow \quad j_a = j_b \quad . \tag{13.63}$$

Eigenstates which belong to the same irrep (j_a, j_a) of $SO(4)$ must have the same energy eigenvalues, because of the commutation relations (13.57). Consequently, the degeneracy of an energy level is given by the dimension of the associated irrep. For the permitted irreps of $SO(4)$ we obtain dimensions 1, 4, 9, 16, ... , which can easily be verified to be in agreement with the degeneracies of the solutions of the Schrödinger equation for Hydrogen. In order to see that, we might select a basis for the eigenstates of the Schrödinger equation which are simultaneously eigenstates of L^2 and $d(L_3)$, *i.e.*:

$$\begin{aligned}
\mathcal{H}\,|n,\ell,\ell_z> \;&\sim\; -\frac{1}{n^2}\,|n,\ell,\ell_z> \quad , \quad n \;=\; 1,\ 2,\ 3,\ \ldots \\
L^2\,|n,\ell,\ell_z> \;&=\; \ell(\ell+1)\,|n,\ell,\ell_z> \quad , \quad \ell < n \ , \quad \text{and} \\
d(L_3)\,|n,\ell,\ell_z> \;&=\; \ell_z\,|n,\ell,\ell_z> \quad , \quad |\ell_z| \leq \ell \quad .
\end{aligned} \tag{13.64}$$

Below we compare the degeneracies for the two different bases, (13.61) and (13.64), for the first few levels of Hydrogen:

j_a	$(2j_a+1)^2$	energy level, n	$\ell < n$	number of states
0	1	1	0	1
$\frac{1}{2}$	4	2	0,1	1+3=4
1	9	3	0,1,2	1+3+5=9
$\frac{3}{2}$	16	4	0,1,2,3	1+3+5+7=16
2	25	5	0,1,2,3,4	1+3+5+7+9=25