Mestrado Integrado em Engenharia Química
Ano lectivo de 2012/2013, 1^o semestre

Exame de Recurso de Física I
6 de Fevereiro de 2013

1. Numa das muitas experiências que foram mostradas nas aulas teóricas desta disciplina o Prof. Walter Lewin andou, ele próprio, num pêndulo.

a Explique, utilizando as equações dinâmicas do pêndulo, qual o objectivo desta experiência. Porque é que o Prof. Walter Lewin andou com o seu corpo numa posição horizontal?

Numa outra experiência uma senhora mostrou um tubo de gás com muitas chamas dentro do qual se propagava uma onda sonora.

b Explique, utilizando as equações do fenómeno de interferência, qual o objectivo desta segunda experiência e descreva as observações.

Um conjunto de três vídeos mostrou o que se passava com uma bola de ping-pong dentro de uma corrente de ar. Num deles o Dr. Carlson utilizou um tubo transparente com um corrente de ar no seu interior.

c Descreva pormenorizadamente o que se observou nesta última demonstração e explique qual o objectivo deste conjunto de experiências?

2.

a Determina, deduzindo e analizando as fórmulas necessárias, o momento de inércia duma esfera metálica oca de raio R e massa m relativamente a um eixo de rotação que passa pelo centro da esfera.

b Uma esfera metálica sólida de raio R e massa m rola numa descida plana sem deslizar. A descida faz um ângulo θ com a horizontal e o momento de inércia da esfera é dado por $\frac{2}{5}mR^2$.
Determine, deduzindo e analizando as fórmulas necessárias, a aceleração da esfera.

3.

a Uma pessoa sopra ar, com uma velocidade de 1.0 m/s, sobre uma folha de papel A4 segurando a folha com os mãos em frente da sua boca. Uma folha A4 tem uma massa de 5.0 gramas e uma área de $\frac{1}{16}$ m^2, a densidade do ar é igual a 1.29 kg/m^3 e a aceleração gravítica a 9.83 m/s^2.
Determine, deduzindo e analizando as fórmulas necessárias, o ângulo da folha de papel com a horizontal.

b Determine, deduzindo e analizando as fórmulas necessárias, o calor necessário para aquecer uma quantidade de 10^{26} moléculas de oxigénio de 20° C para 120° C sem alterar a pressão sobre esta quantidade de moléculas.

4.

a Duas esferas metálicas maciças entram em queda livre ($g = 9.83\,\mathrm{m/s^2}$) ao mesmo tempo. Uma esfera é lançada a partir do centro do sistema de coordenadas ($\hat{x}$ horizontal, $\hat{y}$ vertical) com uma velocidade inicial de $2.0\,(4\hat{x} + 3\hat{y})$ (m/s). Outra esfera entra em queda livre sem velocidade inicial a partir do ponto $2.0\,(4\hat{x} + 3\hat{y})$ (m). É desprezável o efeito da resistência do ar.
Determina, deduzindo e analizando as fórmulas necessárias, em que altura as duas esferas chocam.

b Um rolo de papel, com uma altura de 1.30 m e um fundo com 23.0 cm de raio, está, na posição vertical, sem apoio e em repouso no chão de um carrinho a ser transportado com uma velocidade constante. O centro de massa do rolo encontra-se no meio do rolo. Numa subida o rolo se tomba. Determine, deduzindo e analizando as fórmulas necessárias, o coeficiente de atrito mínimo entre o rolo e o chão e o ângulo da subida.

5. A equação dinâmica de um sistema massa-mola (m massa, C constante da mola) com uma força dissipativa (γ constante de resistência, regime I) é dada por

$$m\frac{d^2u}{dt^2} = -Cu - \gamma\frac{du}{dt}$$

a Quais as unidades das constantes C e γ?

b Mostre que a seguinte expressão para oscilações $u(t)$ em função do tempo t é solução da equação dinâmica referida.

$$u(t) = Ae^{-\frac{1}{2}\frac{t}{\tau}}\sin(\omega t)$$

com

$$\tau = \frac{m}{\gamma} \quad \text{e} \quad \omega = \sqrt{\frac{C}{m} - \frac{\gamma^2}{4m^2}} \quad .$$

c Faça uma representação gráfica da solução $u(t)$ e indica no gráfico as grandezas τ e ω.

d Determine uma solução não nula da equação dinâmica referida para o caso

$$4mC = \gamma^2 \quad .$$

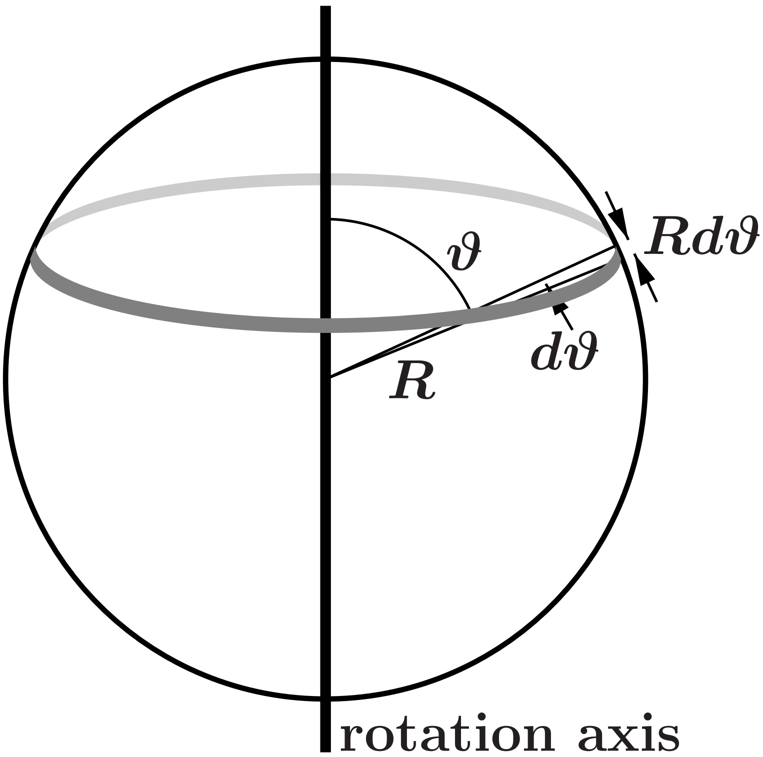
Solutions

Exercício 1

a:

Exercício 2

a: We divide the hollow sphere into small rings, parametrized by the polar angle ϑ and of height given by $Rd\vartheta$, as indicated in the figure below.



The *surface area* of each ring is given by

$$2\pi \times (\text{radius of the ring}) \times Rd\vartheta = 2\pi \times (R\sin(\vartheta)) \times Rd\vartheta = 2\pi R^2 \sin(\vartheta)\, d\vartheta$$

Hence, when σ (in kg/m^2) represents the *surface mass density* of the hollow sphere, each ring has a mass given by

$$(\text{surface area of the ring}) \times \sigma = 2\pi\sigma R^2 \sin(\vartheta)\, d\vartheta$$

The moment of inertia of a ring is given by the product of its mass and its radius squared. Consequently, each ring contributes to the total moment of inertia of the hollow sphere a quantity dI, given by

$$dI = (\text{mass of the ring}) \times (\text{radius of the ring})^2 = 2\pi\sigma R^4 \sin^3(\vartheta)\, d\vartheta$$

Next, we determine the sum (integral) of all rings by varying ϑ from $\vartheta = 0$ to $\vartheta = 180°$:

$$I = \int_{\vartheta=0}^{\vartheta=180°} dI = \int_{\vartheta=0}^{\vartheta=180°} 2\pi\sigma R^4 \sin^3(\vartheta)\, d\vartheta$$

The integral is standard, one obtains

$$
\begin{aligned}
I &= 2\pi\sigma R^4 \left[\tfrac{1}{3}\cos^3(\vartheta) - \cos(\vartheta)\right]_{\vartheta\,=\,0}^{\vartheta\,=\,180^\circ} \\
&= 2\pi\sigma R^4 \left[\left\{\tfrac{1}{3}(-1)^3 - (-1)\right\} - \left\{\tfrac{1}{3}(+1)^3 - (+1)\right\}\right] \\
&= \frac{2}{3} \times \left(4\pi R^2\sigma\right) R^2
\end{aligned}
$$

The *surface area of a sphere* equals $4\pi R^2$. Hence, the mass m of a hollow sphere is given by

$$
m = (\text{surface area of the sphere}) \times (\text{surface mass density}) = 4\pi R^2\sigma
$$

For the moment of inertia we find

$$
I = \frac{2}{3} m R^2
$$

b: One may consider a solid sphere as the sum (integral) of hollow spheres with radius r and surface thickness dr. The mass of such hollow sphere is given by

$$
(\text{volume of the hollow sphere}) \times (\text{mass density}) = 4\pi r^2 dr\rho
$$

whereas its contribution dI to the moment of inertia of the solid sphere is given by

$$
dI = \frac{2}{3}(\text{mass of the hollow sphere}) \times (\text{radius of the hollow sphere})^2 = \frac{2}{3} 4\pi r^2 dr\rho r^2
$$

Consequently, the moment of inertia of the solid sphere is given by

$$
I = \int_{r\,=\,0}^{r\,=\,R} dI = \int_{r\,=\,0}^{r\,=\,R} \frac{8}{3}\pi\rho r^4 dr = \frac{8}{3}\pi\rho\frac{1}{5}R^5 = \frac{2}{5} \times \left(\frac{4}{3}\pi R^3\rho\right) R^2
$$

The *volume of a solid sphere* equals $\frac{4}{3}\pi R^3$. Hence, the mass m of a solid sphere is given by

$$
m = (\text{volume of the sphere}) \times (\text{mass density}) = \frac{4}{3}\pi R^3\rho
$$

For the moment of inertia we find

$$
I = \frac{2}{5} m R^2
$$

After rolling downhill a distance s the height of the sphere has changed by $h = s\sin(\theta)$, whereas its velocity is given by $v = ds/dt$. Its gain in kinetic energy is related to its loss in gravitational potential energy by

$$
\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = E_{\text{kin}} = E_{\text{pot}} = mgh
$$

The relation between ω and v is given by

$$
v = \omega R \quad \Longleftrightarrow \quad I\omega^2 = \frac{2}{5}mR^2\left(\frac{v}{R}\right)^2 = \frac{2}{5}mv^2
$$

So we have the relation

$$\frac{1}{2}\left\{1+\frac{2}{5}\right\}mv^2 = E_{\text{kin}} = E_{\text{pot}} = mgh \quad \Longleftrightarrow \quad \frac{7}{10}v^2 = gh$$

We take the time derivative on both sides of the above equation (a stands for the acceleration of the wheel).

$$\frac{7}{5}va = \frac{7}{5}v\frac{dv}{dt} = \frac{7}{10}\frac{dv^2}{dt} = \frac{d\{gh\}}{dt} = \frac{d\{gs\sin(\theta)\}}{dt} = g\sin(\theta)\frac{ds}{dt} = g\sin(\theta)v$$

Hence

$$\frac{7}{5}a = \frac{1}{2}g\sin(\theta) \quad \Longleftrightarrow \quad a = \frac{5}{14}g\sin(\theta)$$

Exercício 3

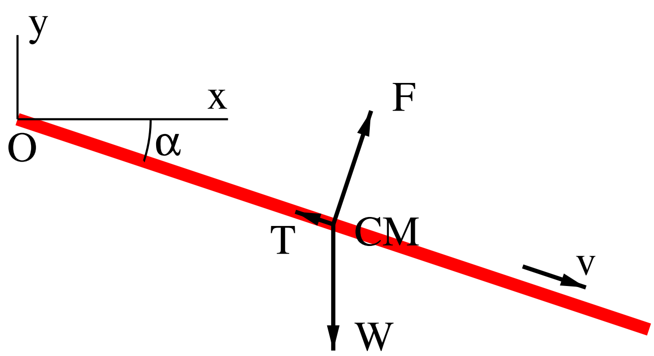
a: First we determine the difference in pressure on the top and on the bottom of the sheet of paper:

$$p_{\text{bottom}} - p_{\text{top}} = \frac{1}{2}\rho_{\text{ar}}v_{\text{ar}}^2 = \frac{1}{2}\left(1.29\ \text{kg/m}^3\right)(1.0m/s)^2 = 0.645\ \text{N/m}^2$$

This results in an upward force, given by

$$F_{\text{upward}} = (p_{\text{bottom}} - p_{\text{top}}) \times (\text{area sheet of paper}) = \left(0.645\ \text{N/m}^2\right)\left(\frac{1}{16}\ \text{m}^2\right) = 0.0403\ \text{N}$$

In the center of mass (CM) we have thus three forces acting: the weight of the sheet of paper (W), the upward force because of the pressure difference (F) and the tension in the sheet of paper (T). That situation is depicted in the figure below



We have chosen an origin (O) at one edge of the sheet. The sheet makes an angle α with the horizontal (x). We read from the figure

$$\vec{W} = \begin{pmatrix} 0 \\ -W \\ 0 \end{pmatrix} \quad , \quad \vec{F} = \begin{pmatrix} F\sin(\alpha) \\ F\cos(\alpha) \\ 0 \end{pmatrix} \quad , \quad \vec{T} = \begin{pmatrix} -T\cos(\alpha) \\ T\sin(\alpha) \\ 0 \end{pmatrix}$$

Hence, equilibrium of the three forces gives

$$0 = \vec{W} + \vec{F} + \vec{T} = \begin{pmatrix} F\sin(\alpha) - T\cos(\alpha) \\ -W + F\cos(\alpha) + T\sin(\alpha) \\ 0 \end{pmatrix}$$

resulting in

$$T = F\tan(\alpha) \quad \text{and} \quad W = F\cos(\alpha) + F\frac{\sin^2(\alpha)}{\cos(\alpha)} = F\frac{1}{\cos(\alpha)}$$

Consequently,

$$\cos(\alpha) = \frac{F}{W}$$

But, we must also have equilibrium in the torque. For the position vector of the center of mass we write

$$\vec{r} = \begin{pmatrix} r\cos(\alpha) \\ -r\sin(\alpha) \\ 0 \end{pmatrix}$$

We then find for the torques

$$\vec{r}\times\vec{W} = \begin{pmatrix} 0 \\ 0 \\ -Wr\cos(\alpha) \end{pmatrix} \quad , \quad \vec{r}\times\vec{F} = \begin{pmatrix} 0 \\ 0 \\ Fr \end{pmatrix} \quad , \quad \vec{r}\times\vec{T} = 0$$

which leads, for $0 = \vec{r}\times\vec{W} + \vec{r}\times\vec{F} + \vec{r}\times\vec{T}$, again to $\cos(\alpha) = F/W$.
Now, we have

$$W = mg = (0.005 \text{ kg}) \times \left(9.83 \text{ m/s}^2\right) = 0.04915 \text{ N}$$

and

$$\cos(\alpha) = \frac{F}{W} = \frac{0.0403 \text{ N}}{0.04915 \text{ N}} = 0.82 \quad \Longleftrightarrow \quad \alpha = 34.9^\circ$$

b: Using the ideal gas law, given by $pV = NkT$ (p pressure, V volume, N number of molecules, k Boltzmann constant, T temperature), we determine the work performed by the gas in passing from temperature T_1 to temperature T_2 while under constant pressure p:

$$p\left(V_2 - V_1\right) = pV_2 - pV_1 = NkT_2 - NkT_1 = Nk\left(T_2 - T_1\right)$$

Furthermore, for the kinetic energy of a quantity of N molecules we found in the lectures the expression $E_{\text{kin}} = \frac{3}{2}NkT$. Hence, for the increase in kinetic energy we find

$$E_{\text{kin, 2}} - E_{\text{kin, 1}} = \frac{3}{2}Nk\left(T_2 - T_1\right)$$

In total one needs thus

$$p\left(V_2 - V_1\right) + E_{\text{kin, 2}} - E_{\text{kin, 1}} = \frac{5}{2}Nk\left(T_2 - T_1\right)$$

of heat for this process. With $N = 10^{26}$, $k = 1.3806503 \times 10^{-23}$ m^{2}kgs^{-2} and $T_2 - T_1 = 100$ K, we obtain for the amount of heat needed

$$3.45 \times 10^5 \text{ J}$$

Exercício 4

a: The expressions which describe the motion of the two spheres are given by

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{g}t^2$$

For one of the spheres, with

$$\vec{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad \vec{v}_0 = \begin{pmatrix} 8 \\ 6 \end{pmatrix} \quad , \quad \vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

one obtains

$$\vec{r}_1(t) = \begin{pmatrix} 8t \\ 6t - \frac{1}{2}gt^2 \end{pmatrix}$$

For the other sphere one has, similarly,

$$\vec{r}_2(t) = \begin{pmatrix} 8 \\ 6 - \frac{1}{2}gt^2 \end{pmatrix}$$

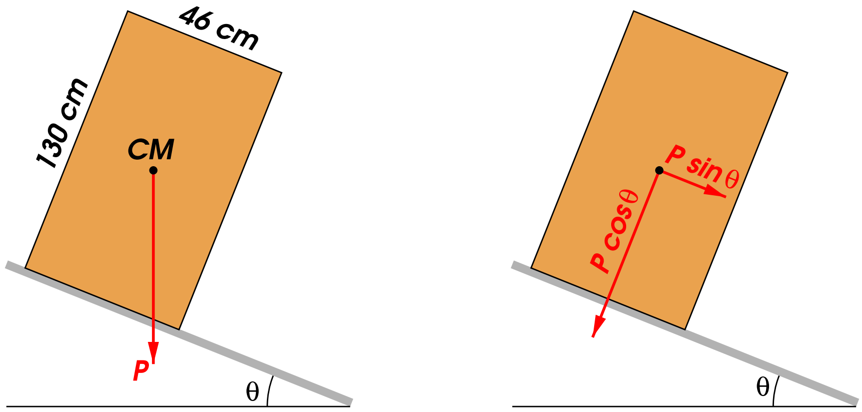
In order to collide one must have $\vec{r}_1(\tau) = \vec{r}_2(\tau)$ for a certain instant of time $t = \tau$. Hence, we must solve

$$\begin{pmatrix} 8\tau \\ 6\tau - \frac{1}{2}g\tau^2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 - \frac{1}{2}g\tau^2 \end{pmatrix}$$

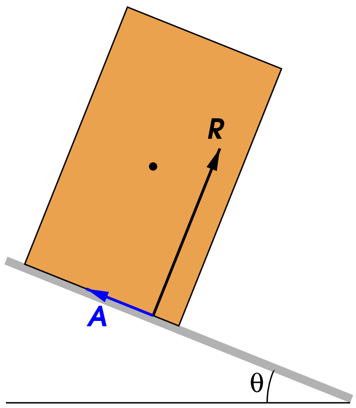
From the x component we obtain $\tau = 1$ s, which, indeed, also solves for the y component. The position where they collide is thus given by either $\vec{r}_1(1\text{ s})$ or $\vec{r}_2(1\text{ s})$. We find

$$\vec{r}_2(1\text{ s}) = \begin{pmatrix} 8 \\ 6 - \frac{1}{2}g \end{pmatrix} = \begin{pmatrix} 8 \\ 6 - \frac{1}{2}\left(9.83\text{ m/s}^2\right) \end{pmatrix} = \begin{pmatrix} 8\text{ m} \\ 1.085\text{ m} \end{pmatrix}$$

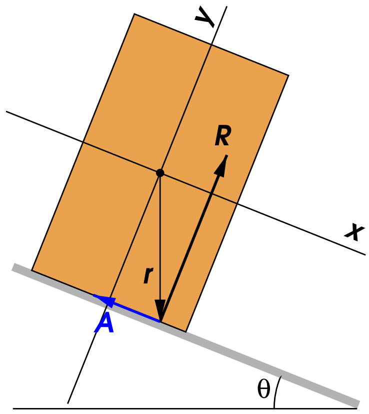
b: Let us first study the forces and torques which work at the cylindrical paper roll when the truck makes an angle θ with the horizontal.



The weight P acts in the center of mass (CM) of the paper roll. In the righthand-side figure, we show the components of P in the directions parallel and perpendicular to the floor of the truck. The parallel component has magnitude $P\sin(\theta)$, whereas the perpendicular component has magnitude $P\cos(\theta)$.



The reaction force of the floor of the truck is perpendicular to the surface of the floor and acts vertically below the center of mass. Its magnitude equals the magnitude of the perpendicular component of P. Hence, $|R| = P\cos(\theta)$. The friction force A is parallel to the surface. Its magnitude equals the magnitude of the parallel component of P. Hence, $|R| = P\sin(\theta)$.



The perpendicular component of P and the reaction force R result in a torque which would rotate the paper roll counterclockwise, whereas the parallel component of P and the friction force give a torque which would rotate the paper roll clockwise.
When we take the coordinate system as indicated in the above figure, we calculate for the respective torques:

$$\vec{r} \times \vec{R} = \begin{pmatrix} (65\text{ cm})\tan(\theta) \\ 65\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ P\cos(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (65\text{ cm})P\sin(\theta) \end{pmatrix}$$

and

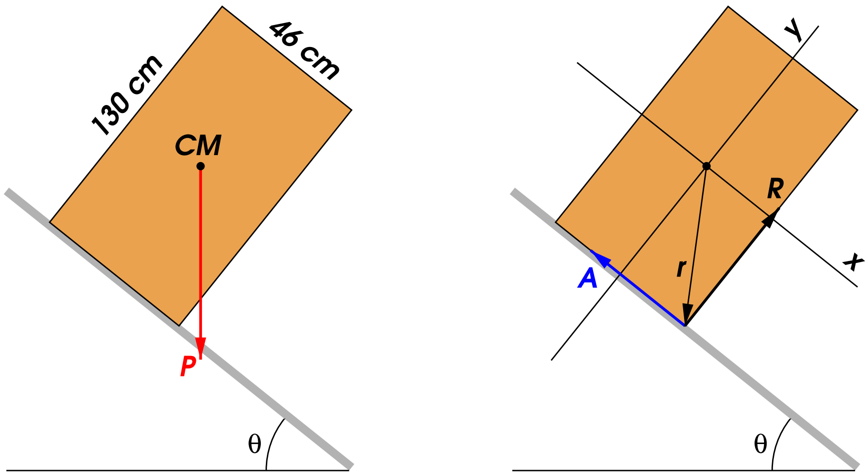
$$\vec{r} \times \vec{A} = \begin{pmatrix} (65\text{ cm})\tan(\theta) \\ 65\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} P\sin(\theta) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(65\text{ cm})P\sin(\theta) \end{pmatrix}$$

The torque $\vec{r} \times \vec{R}$ is in the positive z direction because the rotation direction is counterclockwise.
The torque $\vec{r} \times \vec{A}$ is in the negative z direction because the rotation direction is clockwise.
The sum of the torques $\vec{r} \times \vec{R} + \vec{r} \times \vec{A}$ equals zero. Hence there is no net torque in this case. The only movement which the paper roll could eventually make is sliding. That depends on the maximum possible friction, given by A_{max}.

If $|A| \leq A_{\text{max}}$ then nothing will happen.

But, if the angle is such that $P\sin(\theta) > A_{\max}$, then the paper roll starts sliding because, in that case, $|A| = A_{\max} < P\sin(\theta)$, hence the forces are not in equilibrium.

When the angle increases and the paper roll does not start sliding, we may imagine that we could arrive at the situation shown in the figures below.



However, here the reaction force cannot come below the center of mass of the paper roll. It can at most act in the border of the paper roll. Consequently, the application vector $\vec{r}$ is now different. When we take the coordinate system as indicated in the above figures, we calculate for the respective torques:

$$\vec{r} \times \vec{R} = \begin{pmatrix} 23 \text{ cm} \\ 65 \text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ P\cos(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (23 \text{ cm})P\cos(\theta) \end{pmatrix}$$

and

$$\vec{r} \times \vec{A} = \begin{pmatrix} 23 \text{ cm} \\ 65 \text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} P\sin(\theta) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(65 \text{ cm})P\sin(\theta) \end{pmatrix}$$

Those torques can only be in equilibrium if

$$(23 \text{ cm})P\cos(\theta) = (65 \text{ cm})P\sin(\theta) \quad \Longleftrightarrow \quad \tan(\theta) = \frac{23 \text{ cm}}{65 \text{ cm}} = 0.35385$$

For $\tan(\theta) = 0.35385$ the center of mass of the paper roll is just vertically above the lower right border edge of the paper roll. Hence, when in that situation the friction force A is still smaller than the maximum friction force $A_{\max}$, the paper roll can only tumble.

When $\tan(\theta) = 0.35385$ and $|A| < A_{\max}$, one has furthermore that

$$\text{static friction coefficient} = \frac{A_{\max}}{|R|} > \frac{|A|}{|R|} = \frac{P\sin(\theta)}{P\cos(\theta)} = \tan(\theta) = 0.35385$$

Hence the criterium for tumbling is

$$\text{static friction coefficient} \; > \; 0.35385$$

where 0.35385 stems from the dimensions of the paper roll.
The angle θ equals

$$\theta = \arctan(0.35385) = 19.5^\circ$$

Exercício 5

a:

$$m\frac{d^2u}{dt^2} = -Cu - \gamma\frac{du}{dt}$$

The units of the mass m are kilograms (**kg**), of u are meters (**m**) and of t are seconds (**s**). Hence the units of $m\frac{d^2u}{dt^2}$ are $\boldsymbol{kg\frac{m}{s^2} = N}$. Consequently, the units of Cu and $\gamma\frac{du}{dt}$ are **N**, whereas the units of C are $\boldsymbol{\frac{N}{m} = \frac{kg}{s^2}}$ and the units of γ are $\boldsymbol{\frac{N}{m/s} = \frac{kg}{s}}$.

b: If

$$u(t) = Ae^{-\frac{1}{2}\frac{t}{\tau}}\sin(\omega t)$$

then

$$\frac{du}{dt} = Ae^{-\frac{1}{2}\frac{t}{\tau}}\left\{-\frac{1}{2\tau}\sin(\omega t) + \omega\cos(\omega t)\right\}$$

and

$$\frac{d^2u}{dt^2} = Ae^{-\frac{1}{2}\frac{t}{\tau}}\left\{\frac{1}{4\tau^2}\sin(\omega t) - \frac{1}{\tau}\omega\cos(\omega t) - \omega^2\sin(\omega t)\right\}$$

Hence the dynamical equation is given by

$$mAe^{-\frac{1}{2}\frac{t}{\tau}}\left\{\frac{1}{4\tau^2}\sin(\omega t) - \frac{1}{\tau}\omega\cos(\omega t) - \omega^2\sin(\omega t)\right\} = m\frac{d^2u}{dt^2} =$$

$$= -Cu - \gamma\frac{du}{dt} = -CAe^{-\frac{1}{2}\frac{t}{\tau}}\sin(\omega t) - \gamma Ae^{-\frac{1}{2}\frac{t}{\tau}}\left\{-\frac{1}{2\tau}\sin(\omega t) + \omega\cos(\omega t)\right\}$$

which is equivalent to

$$Ae^{-\frac{1}{2}\frac{t}{\tau}}\left[\sin(\omega t)\left\{\frac{m}{4\tau^2} - m\omega^2 + C - \frac{\gamma}{2\tau}\right\} + \omega\cos(\omega t)\left\{-\frac{m}{\tau} + \gamma\right\}\right] = 0$$

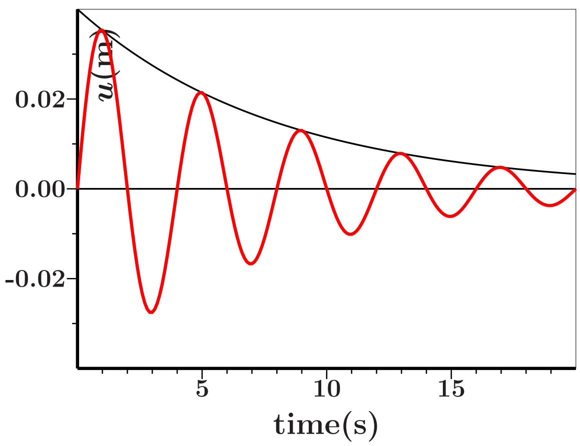
The coefficients of $\sin(\omega t)$ and $\omega\cos(\omega t)$ vanish respectively for

$$\omega^2 = \frac{\frac{m}{4\tau^2} + C - \frac{\gamma}{2\tau}}{m} \quad \text{and} \quad \gamma = \frac{m}{\tau}$$

which relations lead to

$$\tau = \frac{m}{\gamma} \quad \text{and} \quad \omega^2 = \frac{\frac{\gamma^2}{4m} + C - \frac{\gamma^2}{2m}}{m} = \frac{C}{m} - \frac{\gamma^2}{4m^2}$$

c: For the figure below we have chosen $A = 4$ cm, $\omega = 90^\circ$/s and $\tau = 4$ s.



Note that one complete oscillation takes 4 seconds, because $4\omega = 360°$. Note, moreover, that the amplitude equals 4 cm only at $t = 0$. Note, furthermore, that each 4 seconds the signal amplitude reduces by a factor $\exp(-1/2) = 0.6$.

d: It is important to note that solutions exist of the form

$$u(t) = Ae^{-\frac{1}{2}\frac{t}{\tau}} \sin\left(\omega t + \varphi_0\right)$$

for an arbitrary initial phase φ_0.

Then it is simple to construct a solution for $\omega = 0$, namely

$$u(t) = Ae^{-\frac{1}{2}\frac{t}{\tau}} \sin\left(\varphi_0\right) \quad ,$$

with an amplitude at $t = 0$ given by $A \sin\left(\varphi_0\right)$.