Mestrado Integrado em Engenharia Química
Ano lectivo de 2013/2014, 1^o semestre

Exame de Recurso de Física I
7 de Fevereiro de 2014

Apresente sempre claramente o seu raciocínio e justifique as fórmulas e os símbolos nelas usados.

Para a constante da aceleração gravítica utilizar o valor de $g = 9.83$ m/s^2.

1. Considere uma porta delgada de 2,0 m de altura e 1,0 m de largura feita de um material homogéneo. A densidade da massa superficial da porta é igual a 4,0 kg/m^2. A porta pode girar livremente (sem atrito) em torno das dobradiças estando apoiada pelas mesmas. O eixo de rotação da porta está na vertical. Inicialmente a porta está encostada, em repouso, sendo atingida, perpendicularmente, por uma bola de plasticina de 0,4 kg, que se move na horizontal, com uma velocidade de 6,0 m/s de tal forma que a porta se abre. A bola atinge a porta num ponto a uma distância de 80 cm do eixo de rotação da porta e a uma altura de 1,0 m.

a. Determine o momento de inércia da porta relativamente ao seu eixo de rotação.

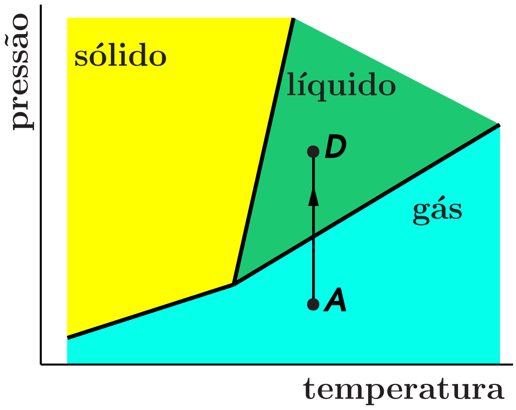
b. Determine a velocidade angular da porta logo depois da colisão, se, neste instante, a bola cair verticalmente para baixo. A última informação significa que a bola transferiu todo o seu momento linear para a porta.

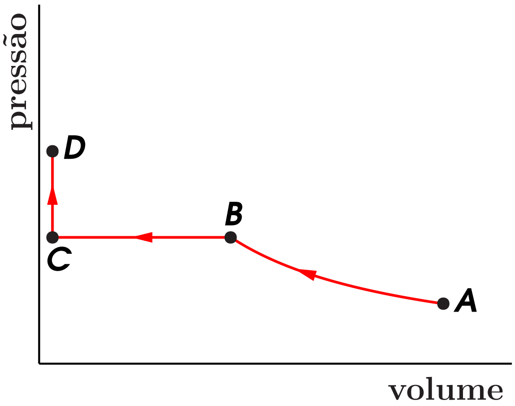
c. Determine a velocidade angular da porta logo depois da colisão, se a bola ficar colada à porta depois da colisão.

2. Considere o tímpano de um ouvido sujeito a dois sinais auditivos com a mesma intensidade mas de frequências diferentes, nomeademente 200,1 Hz e 199,9 Hz. Deduza uma expressão matemática para descrever as oscilações do tímpano e, a partir da fórmula obtida, determine a frequência do sinal resultante e o ritmo da modulação da intensidade. Interprete os resultados obtidos.

3. Um carrinho de carga transporta um barril de petróleo, com uma altura de 1,20 m e uma base com 36 cm de raio. O barril está em repouso no espaço de carga do carrinho, mas não está preso. O centro de massa do barril encontra-se no meio do barril e o coeficiente de atrito entre o barril e o chão do carrinho é igual a 0,65. Numa subida o barril começa a mexer-se. Determine se o barril desliza ou se tomba.

4. Considere uma certa substância dentro de um cilindro fechado por um pistão. Inicialmente a substância encontra-se no estado de gasoso (gás ideal), indicado por A nos diagramas de fase e de *volume-pressão* abaixo. Pelo aumento da pressão sobre o pistão, a substância transforma-se num líquido, por um processo isotérmico, até ao estado indicado por D nos diagramas de fase e de *volume-pressão*.





A curva no diagrama de *volume-pressão* corresponde à curva $A \to D$ indicada no diagrama de fase.

a Explique a forma da curva $A \to B$ no diagrama de *volume-pressão*.

b Explique a forma da curva $B \to C$ no diagrama de *volume-pressão*.

c Copie o diagrama de fase para a sua folha e indique na curva $A \to D$ a posição dos estados B e C da substância.

d Explique a forma da curva $C \to D$ no diagrama de *volume-pressão*.

5. Considere um sistema massa-mola com uma força dissipativa cuja equação dinâmica é dada por

$$m\frac{d^2u}{dt^2} + g\frac{du}{dt} + Cu = 0$$

onde m representa a massa do sistema, g a constante de resistência da força dissipativa e C a constante da mola do sistema.

a Mostre que a seguinte expressão para oscilações $u(t)$ em função do tempo t é solução da equação dinâmica referida.

$$u(t) = A\sin(\omega t)\, e^{-\frac{t}{\tau}} \quad \text{com} \quad \tau = \frac{2m}{g} \quad \text{e} \quad \omega = \frac{1}{2m}\sqrt{4mC - g^2} \quad .$$

b Faça uma representação gráfica da solução $u(t)$ e indique nela as grandezas τ e ω.

c Determine uma solução não nula da equação dinâmica referida para o caso $4mC = g^2$ e interprete o resultado obtido.

Solutions

Exercício 1

a: We write σ for the surface density of the door and divide the surface in small vertical strips of thickness Δx. The mass Δm of each strip is then given by

$$\Delta m = \sigma h \Delta x$$

where h represents the height of the door.
The contribution ΔI to the total moment of inertia I of the door of one strip which has distance x from the rotation axis is given by

$$\Delta I(x) = \Delta m x^2 = \sigma h x^2 \Delta x$$

The total moment of inertia I of the door is given by the sum of all partial contributions, which in the limit $\Delta x \downarrow 0$ turns into the integral

$$I = \int_{x=0}^{x=w} \Delta I(x) = \int_{x=0}^{x=w} \sigma h x^2 dx = \tfrac{1}{3}\sigma h w^3$$

where w represents the width of the door.
Now, the mass M of the door is given by

$$M = \text{height} \times \text{width} \times \text{surface density} = hw\sigma = (2{,}0 \text{ m}^2) \times (4{,}0 \text{ kg/m}^2) = 8,0 \text{ kg}$$

Hence, the total moment of inertia I of the door is given by

$$I = \tfrac{1}{3}Mw^2 = \tfrac{1}{3}\,(8{,}0 \text{ kg}) \times (1{,}0 \text{ m})^2 = 2{,}67 \text{ kgm}^2$$

b: The plasticine ball has a linear momentum p given by

$$p = \text{mass ball} \times \text{velocity ball} = (0{,}4 \text{ kg}) \times (6{,}0 \text{ m/s}) = 2,4 \text{ kg m/s}$$

Consequently, its angular momentum with respect to the rotation axis of the door L at the place where it perpendicularly hits the door is given by

$$L = \text{distance from rotation axis} \times \text{linear momentum} = (0{,}80 \text{ m}) \times (2{,}4 \text{ kgm/s}) = 1,92 \text{ kg m}^2\text{/s}$$

After the collision that is transfered to the door. Hence (ω represents the angular velocity of the door),

$$(2{,}67 \text{ kgm}^2)\,\omega = I\omega = L = 1{,}92 \text{ kg m}^2\text{/s}$$

From which we deduce

$$\omega = \frac{L}{I} = \frac{1{,}92 \text{ kg m}^2\text{/s}}{2{,}67 \text{ kgm}^2} = 0{,}72 \text{ rad/s} = 41{,}3^\circ\text{/s}$$

c: Now the total moment of inertia is given by the sum of the moment of inertia of the door and the moment of inertia of the ball:

$$I = (2{,}67 \text{ kgm}^2) + (0{,}40 \text{ kg}) \times (0{,}80 \text{ m})^2 = 2{,}92 \text{ kgm}^2$$

The rest is similar. Hence,

$$\omega = \frac{L}{I} = \frac{1{,}92 \text{ kg m}^2/\text{s}}{2{,}92 \text{ kgm}^2} = 0{,}66 \text{ rad/s} = 37{,}7°/\text{s}$$

Exercício 2

The oscillations of the eardrum for each signal separately could respectively be described by

$$u_1(t) = A_1 \sin(\omega_1 t) \quad \text{and} \quad u_2(t) = A_2 \sin(\omega_2 t + \phi)$$

where we allow for a possible phase difference ϕ between the two signals.
The fact that the signals have equal intensity is mathematically expressed by choosing equal amplitudes $A_1 = A_2 = A$ for the oscillations caused to the eardrum by each of the signals.
The resultant signal is given by the sum of the two individual signals, *i.e.*

$$u(t) = u_1(t) + u_2(t)$$

We elaborate on this expression

$$\begin{aligned} u(t) &= A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t + \phi) = A \sin(\omega_1 t) + A \sin(\omega_2 t + \phi) \\ &= = 2A \cos\left(\tfrac{1}{2}(\omega_1 - \omega_2)\ t - \phi/2\right) \sin\left(\tfrac{1}{2}(\omega_1 + \omega_2)\ t + \phi/2\right) \end{aligned}$$

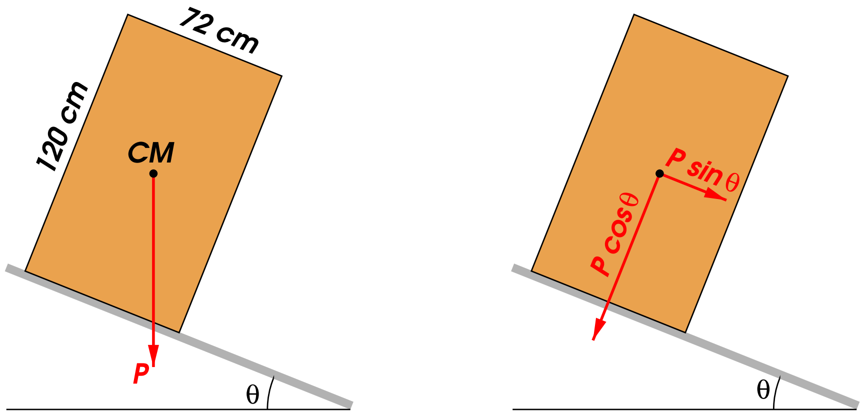
For the case $\omega_1 = 200,1 \times 2\pi$ and $\omega_2 = 199,9 \times 2\pi$, this results to

$$u(t) = 2A\cos(0,2\pi t - \phi/2)\sin(400\pi t + \phi/2) = 2A\cos\left(2\pi\frac{t}{10 \text{ s}} - \phi/2\right)\sin\left(2\pi\frac{t}{0,005 \text{ s}} + \phi/2\right)$$

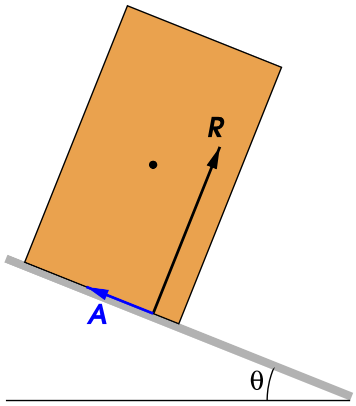
The sine part of the resulting oscillations has a periode of 0,005 seconds, which corresponds to a frequency of 200 Hz, whereas the cosine part of the resulting oscillations has a period of 10 seconds, which corresponds to a frequency of 0,1 Hz.
The eardrum oscillates thus with a frequency of 200 cycles per second, but the intensity of the oscillations is modulated with a rithm of 1 cycle every 10 seconds. The latter result means that every 5 seconds the signal has a maximum of intensity, whereas, in between the maxima, the intensity tends to vanish every 5 seconds.

Exercício 3

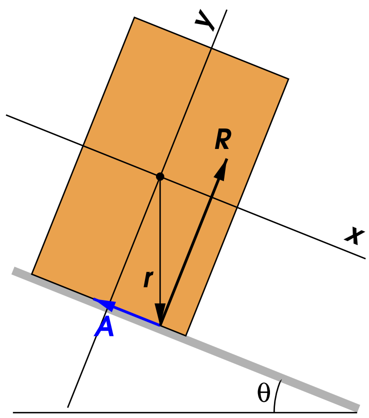
Let us first study the forces and torques which work at the cylindrical box (barrel) when the ship makes an angle θ with the horizontal.



The weight P acts in the center of mass (CM) of the barrel. In the righthand-side figure, we show the components of P in the directions parallel and perpendicular to the deck of the ship. The parallel component has magnitude $P\sin(\theta)$, whereas the perpendicular component has magnitude $P\cos(\theta)$.



The reaction force of the deck of the ship is perpendicular to the surface of the deck and acts vertically below the center of mass. Its magnitude equals the magnitude of the perpendicular component of P. Hence, $|R| = P\cos(\theta)$. The friction force A is parallel to the surface. Its magnitude equals the magnitude of the parallel component of P. Hence, $|R| = P\sin(\theta)$.



The perpendicular component of P and the reaction force R result in a torque which would rotate the barrel counterclockwise, whereas the parallel component of P and the friction force give a torque which would rotate the barrel clockwise.
When we take the coordinate system as indicated in the above figure, we calculate for the respective torques:

$$\vec{r} \times \vec{R} = \begin{pmatrix} (50\text{ cm})\tan(\theta) \\ 50\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ P\cos(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (50\text{ cm})P\sin(\theta) \end{pmatrix}$$

and

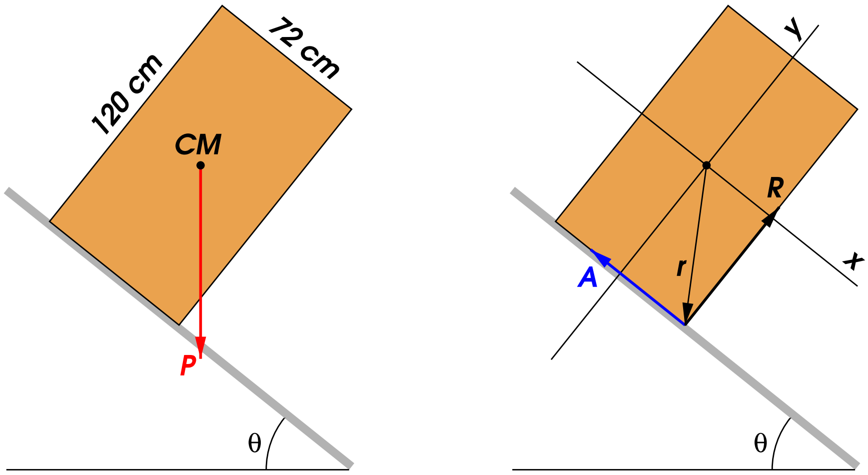
$$\vec{r} \times \vec{A} = \begin{pmatrix} (50\text{ cm})\tan(\theta) \\ 50\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} P\sin(\theta) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(50\text{ cm})P\sin(\theta) \end{pmatrix}$$

The torque $\vec{r} \times \vec{R}$ is in the positive z direction because the rotation direction is counterclockwise. The torque $\vec{r} \times \vec{A}$ is in the negative z direction because the rotation direction is clockwise.
The sum of the torques $\vec{r} \times \vec{R} + \vec{r} \times \vec{A}$ equals zero. Hence there is no net torque in this case. The only movement which the barrel could eventually make is sliding. That depends on the maximum possible friction, given by

$$A_{\text{max}} = 0.75\ |R|$$

If $|A| \leq A_{\text{max}}$ then nothing will happen. But, if the angle is such that $P\sin(\theta) > A_{\text{max}}$, then the barrel starts sliding because, in that case, $|A| = A_{\text{max}} < P\sin(\theta)$, hence the forces are not in equilibrium.

When the angle increases and the barrel does not start sliding, we may imagine that we could arrive at the situation shown in the figures below.



However, here the reaction force cannot come below the center of mass of the barrel. It can at most act in the border of the barrel. Consequently, the application vector $\vec{r}$ is now different.
When we take the coordinate system as indicated in the above figures, we calculate for the respective torques:

$$\vec{r} \times \vec{R} = \begin{pmatrix} 30\text{ cm} \\ 50\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ P\cos(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (30\text{ cm})P\cos(\theta) \end{pmatrix}$$

and

$$\vec{r} \times \vec{A} = \begin{pmatrix} 30\text{ cm} \\ 50\text{ cm} \\ 0 \end{pmatrix} \times \begin{pmatrix} P\sin(\theta) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(50\text{ cm})P\sin(\theta) \end{pmatrix}$$

Those torques can only be in equilibrium if

$$(30\text{ cm})P\cos(\theta) = (50\text{ cm})P\sin(\theta) \quad \Longleftrightarrow \quad \tan(\theta) = \frac{30\text{ cm}}{50\text{ cm}} = 0.6$$

For $\tan(\theta) = 0.6$ the center of mass of the barrel is just vertically above the lower right border edge of the barrel. Hence, when in that situation the friction force A is still smaller than the

maximum friction force $A_{\max}$, the barrel can only tumble.
When $\tan(\theta) = 0.6$ and $|A| < A_{\max}$, one has furthermore that

$$0.75 = \frac{A_{\max}}{|R|} > \frac{|A|}{|R|} = \frac{P\sin(\theta)}{P\cos(\theta)} = \tan(\theta) = 0.6$$

Hence the criterium for tumbling is

$$\text{static friction coefficient} \;>\; 0.6$$

where 0.6 stems from the dimensions of the barrel.

Exercício 4

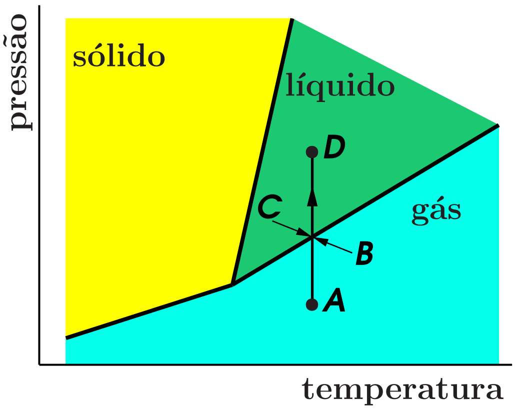
a: Since the process is isothermic one has the ideal-gas relation (Boyle):

$$PV = nRT = \text{constant} = X \quad \Longleftrightarrow \quad P(V) = \frac{X}{V}$$

The pressure P is inversely proportional to V.

b: When the substance condensates the pressure remains constant until no more gas or vapor is left. The volume reduces since gas takes more space than liquid.

c: In the phase diagram B and C represent the same state, because the pressure is the same in B as in C.



Nevertheless, in B one has only gas, in C only liquid.

d: One may increase the pressure, but it does only marginally effect the volume of a liquid. Hence, the volume remains constant in the process $C \to D$.

Exercício 5

a: If

$$u(t) = Ae^{-\frac{t}{\tau}} \sin(\omega t)$$

then

$$\frac{du}{dt} = Ae^{-\frac{t}{\tau}} \left\{ -\frac{1}{\tau} \sin(\omega t) + \omega \cos(\omega t) \right\}$$

and

$$\frac{d^2u}{dt^2} = Ae^{-\frac{t}{\tau}} \left\{ \frac{1}{\tau^2} \sin(\omega t) - \frac{2}{\tau}\omega \cos(\omega t) - \omega^2 \sin(\omega t) \right\}$$

Hence the dynamical equation is given by

$$mAe^{-\frac{t}{\tau}} \left\{ \frac{1}{\tau^2} \sin(\omega t) - \frac{2}{\tau}\omega \cos(\omega t) - \omega^2 \sin(\omega t) \right\} = m\frac{d^2u}{dt^2} =$$

$$= -Cu - g\frac{du}{dt} = -CAe^{-\frac{t}{\tau}} \sin(\omega t) - gAe^{-\frac{t}{\tau}} \left\{ -\frac{1}{\tau} \sin(\omega t) + \omega \cos(\omega t) \right\}$$

which is equivalent to

$$Ae^{-\frac{t}{\tau}} \left[\sin(\omega t) \left\{ \frac{m}{\tau^2} - m\omega^2 + C - \frac{g}{\tau} \right\} + \omega \cos(\omega t) \left\{ -\frac{2m}{\tau} + g \right\} \right] = 0$$

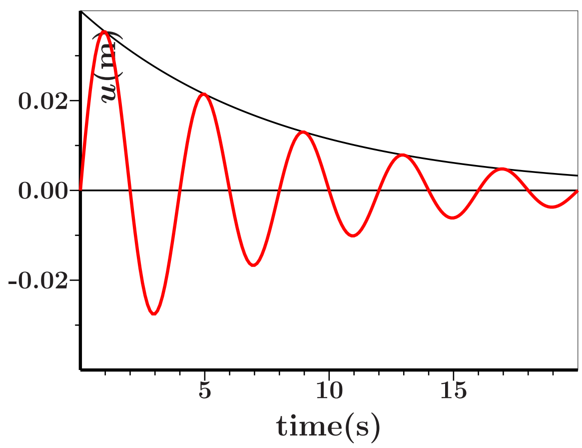
The coefficients of $\sin(\omega t)$ and $\omega \cos(\omega t)$ vanish respectively for

$$\omega^2 = \frac{\frac{m}{\tau^2} + C - \frac{g}{\tau}}{m} \quad \text{and} \quad g = \frac{2m}{\tau}$$

which relations lead to

$$\tau = \frac{2m}{g} \quad \text{and} \quad \omega^2 = \frac{\frac{g^2}{4m} + C - \frac{g^2}{2m}}{m} = \frac{C}{m} - \frac{g^2}{4m^2} = \frac{1}{4m^2}\left(4mC - g^2\right)$$

b: For the figure below we have chosen $A = 4$ cm, $\omega = 90°/\text{s}$ and $\tau = 8$ s.



Note that one complete oscillation takes 4 seconds, because $4\omega = 360°$. Note, moreover, that the amplitude equals 4 cm only at $t = 0$. Note, furthermore, that each 8 seconds the signal amplitude reduces by a factor $\exp(-1) = 0.37$.

c: It is important to note that solutions exist of the form

$$u(t) = Ae^{-\frac{t}{\tau}} \sin(\omega t + \varphi_0)$$

for an arbitrary initial phase φ_0.

Then it is simple to construct a solution for $\omega = 0$, namely

$$u(t) = Ae^{-\frac{t}{\tau}} \sin(\varphi_0) \quad ,$$

with an amplitude at $t = 0$ given by $A \sin(\varphi_0)$.

The proof is simple:

With $C = \frac{g^2}{4m}$ one has for the dynamic equation of the system

$$m\frac{d^2u}{dt^2} + g\frac{du}{dt} + \frac{g^2}{4m}u = 0 \quad .$$

For $u(t) = e^{-\frac{t}{\tau}}$ this gives

$$m\frac{1}{\tau^2}e^{-\frac{t}{\tau}} - g\frac{1}{\tau}e^{-\frac{t}{\tau}} + \frac{g^2}{4m}e^{-\frac{t}{\tau}} = 0 \quad ,$$

or

$$m\left(\frac{1}{\tau} - \frac{g}{2m}\right)^2 e^{-\frac{t}{\tau}} = 0 \quad .$$

This is indeed solved by $\tau = \frac{2m}{g}$.