

15. O movimento orbital do electrão dentro do interior do átomo de hidrogénio causa um momento magnético proporcional ao momento angular orbital \vec{L} . Consequentemente, o momento magnético intrínseco do electrão, proporcional ao spin \vec{S} sofre uma força, proporcional com $\vec{L} \cdot \vec{S}$ (*acoplamento spin-orbital*). Portanto, a forma mais completa do Hamiltoniano do átomo de hidrogénio é igual a

$$H' = \frac{p^2}{2\mu_e} - \frac{e^2}{4\pi\epsilon_0 r} + W(r)\vec{L} \cdot \vec{S} \quad ,$$

onde $W(r)$ é uma função da distância entre o protão e o electrão.

Utilizando os resultados dos problemas (13) e (14), mostra que $[H', \vec{S}] \neq 0$, $[H', \vec{L}] \neq 0$, $[H', S^2] = 0$ e $[H', L^2] = 0$ e tira uma conclusão relativamente aos observáveis L_z, S_z, L^2 e S^2 .

Solution:

Let us define (see exercise 12)

$$H = \frac{p^2}{2\mu_e} - \frac{e^2}{4\pi\epsilon_0 r} \quad ,$$

and write

$$H' = H + W(r)\vec{L} \cdot \vec{S} \quad .$$

In exercise (12) we have shown

$$[H, \vec{L}] = H\vec{L} - \vec{L}H = 0 \quad . \quad (1)$$

Furthermore, since H does not contain S or anything on which S operates (see 13), we automatically have

$$[H, \vec{S}] = H\vec{S} - \vec{S}H = 0 \quad . \quad (2)$$

Now, because for any operator A one always has

$$[H', A] = [H + W(r)\vec{L} \cdot \vec{S}, A] = [H, A] + [W(r)\vec{L} \cdot \vec{S}, A] \quad ,$$

we obtain

$$[H', \vec{L}] = [H, \vec{L}] + [W(r)\vec{L} \cdot \vec{S}, \vec{L}] = 0 + [W(r)\vec{L} \cdot \vec{S}, \vec{L}] = [W(r)\vec{L} \cdot \vec{S}, \vec{L}] \quad ,$$

and similar for \vec{S} .

Also, we should recall (see exercise 12)

$$\begin{aligned} [W(r), L_z] &= [W(r), xp_y - yp_x] = \\ &= [W(r), xp_y] - [W(r), yp_x] \end{aligned}$$

$$\begin{aligned}
&= x [W(r), p_y] + [W(r), x] p_y - y [W(r), p_x] - [W(r), y] p_x \\
&= x i\hbar \frac{y}{r} \frac{dW(r)}{dr} + 0 - y i\hbar \frac{x}{r} \frac{dW(r)}{dr} + 0 = 0 \quad ,
\end{aligned}$$

and similar for L_x and L_y . Furthermore, since S does not operate on $W(r)$,

$$[W(r), \vec{S}] = 0 \quad .$$

Hence, the commutators of H' with \vec{L} , \vec{S} , L^2 and S^2 , reduce to

$$[H', \vec{L}] = [W(r)\vec{L} \cdot \vec{S}, \vec{L}] = W(r) [\vec{L} \cdot \vec{S}, \vec{L}] \quad ,$$

and similar for \vec{S} , L^2 and S^2 .

In exercise (14) we have shown

$$[\vec{L} \cdot \vec{S}, \vec{L}] = i\hbar \vec{L} \times \vec{S} \quad ,$$

hence

$$[H', \vec{L}] = iW(r)\hbar \vec{L} \times \vec{S} \quad .$$

Consequently, H' and \vec{L} do not commute. This implies that H' and L_z do not have common eigenstates. Since, suppose that ψ is an eigenstate of H' with $H'\psi = E\psi$. Then,

$$\begin{aligned}
H'L_z\psi &= \{H'L_z - L_zH' + L_zH'\} \psi = \\
&= \{[H', L_z] + L_zH'\} \psi = \{i\hbar [\vec{L} \times \vec{S}]_z + L_zE\} \psi \\
&= E \left\{ i\frac{1}{E}\hbar [\vec{L} \times \vec{S}]_z + L_z \right\} \psi \neq EL_z\psi \quad .
\end{aligned}$$

Or, in common terms, the eigenstates of H' are not of the form $R_{n,\ell}(r)Y_\ell^{\ell_z}(\vartheta, \varphi)$.

In exercise (14) we have shown

$$[\vec{L} \cdot \vec{S}, \vec{S}] = -i\hbar \vec{L} \times \vec{S} \quad ,$$

hence

$$[H', \vec{S}] = -iW(r)\hbar \vec{L} \times \vec{S} \quad .$$

Consequently, H' and \vec{S} do not commute. This implies that H' and S_z do not have common eigenstates.

In exercise (14) we have shown

$$[\vec{L} \cdot \vec{S}, L^2] = 0 \quad ,$$

hence

$$[H', L^2] = 0 \quad .$$

Consequently, H' and L^2 do commute. This implies that H' and L^2 do have common eigenstates. The quantum number ℓ can be used to label the eigenstates of H' .

In exercise (14) we have shown

$$[\vec{L} \cdot \vec{S}, S^2] = 0 \quad ,$$

hence

$$[H', S^2] = 0 \quad .$$

Consequently, H' and S^2 do commute. This implies that H' and S^2 do have common eigenstates. The quantum number s can be used to label the eigenstates of H' .

16. Devido às conclusões do problema (15) é preciso encontrar um novo observável para rotular os estados próprios do Hamiltoniano H' .

Defina-se o *momento angular total* $\vec{J} = \vec{L} + \vec{S}$ do átomo de hidrogénio.

Utilizando os resultados dos problemas (13) e (14), mostra que $[H', \vec{J}] = 0$ e $[H', J^2] = 0$ e tira uma conclusão relativamente aos observáveis J_z e J^2 .

Solution:

We have shown in exercise (14) that $[\vec{L} \cdot \vec{S}, \vec{L}] = i\hbar \vec{L} \times \vec{S}$ and $[\vec{L} \cdot \vec{S}, \vec{S}] = -i\hbar \vec{L} \times \vec{S}$.

This gives

$$[\vec{L} \cdot \vec{S}, \vec{J}] = [\vec{L} \cdot \vec{S}, \vec{L} + \vec{S}] = [\vec{L} \cdot \vec{S}, \vec{L}] + [\vec{L} \cdot \vec{S}, \vec{S}] = i\hbar \vec{L} \times \vec{S} - i\hbar \vec{L} \times \vec{S} = 0.$$

As a consequence J_z and J^2 commute with H' .

We may thus use the quantum numbers j_z and j to label the eigenstates of H' .

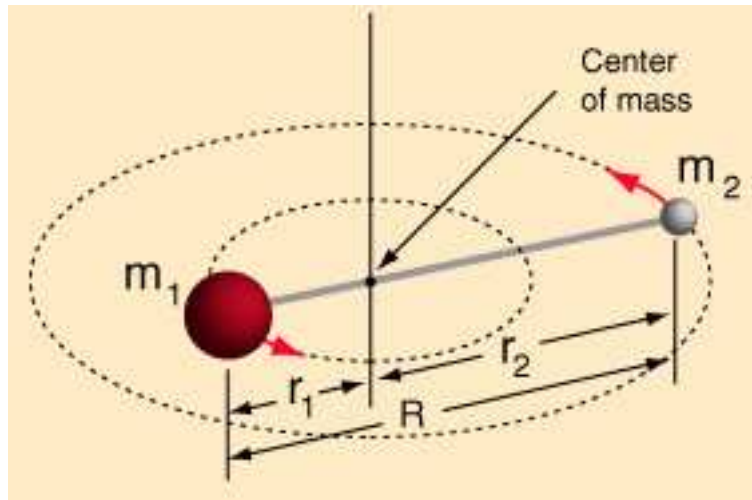
Together with the result of exercise (15), we may then write the eigenstates of H' in the form $R_{n,\ell}(r)\mathcal{Y}_{j,\ell,s}^{j_z}(\vartheta, \varphi)$, where $\mathcal{Y}_{j,\ell,s}^{j_z}(\vartheta, \varphi)$ consists of linear combinations of $Y_\ell^{\ell_z}(\vartheta, \varphi)$ with different values of ℓ_z combined with spin eigenstates $X_s^{s_z}$ with different values of s_z .

17. Em virtude de se estar a estudar o espectro rotacional de uma molécula polar diatómica, considere que os dois núcleos atômicos, de massas m_1 e m_2 , giram em volta do seu centro de massa comum.
- a Determine uma relação entre a energia total deste sistema, o momento angular total \vec{K} , a massa reduzida e a distância entre os núcleos.
- b Sabendo que K^2 apenas pode ter valores $\hbar^2 K(K + 1)$, com $K = 0, 1, 2, \dots$, que o espectro rotacional de HCl contém os comprimentos de onda (em μm) 120.3, 96.0, 80.4, 68.9 e 60.4, e ainda que apenas ocorrem transições dipolares com $\Delta K = \pm 1$, determine a distância entre os núcleos de hidrogénio (${}_1H^1$) e cloro (${}_{17}Cl^{35}$) da molécula de HCl .

Solution:

a. The two nuclei, with masses m_1 and m_2 , are supposed to rotate around their common center of mass with a constant angular velocity ω . Let the common center of mass be the center of a coordinate system (x and y) of the plane in which the masses rotate. We indicate the position of mass m_1 by \vec{r}_1 and the position of mass m_2 by \vec{r}_2 . Since, their common center of mass is in the origin of the coordinate system, one has

$$m_2 \vec{r}_2 = -m_1 \vec{r}_1 \quad \text{hence} \quad |\vec{r}_2| = \frac{m_1}{m_2} |\vec{r}_1| \quad .$$



The total moment of inertia is given by

$$\begin{aligned} I &= m_1 |\vec{r}_1|^2 + m_2 |\vec{r}_2|^2 = m_1 |\vec{r}_1|^2 + m_2 \left\{ \frac{m_1}{m_2} |\vec{r}_1| \right\}^2 = \\ &= \left\{ m_1 + \frac{m_1^2}{m_2} \right\} |\vec{r}_1|^2 = \frac{m_1 m_2}{m_1 + m_2} \left(\frac{m_1 + m_2}{m_2} \right)^2 |\vec{r}_1|^2 \\ &= \frac{m_1 m_2}{m_1 + m_2} \left(|\vec{r}_1| + \frac{m_1}{m_2} |\vec{r}_1| \right)^2 = \frac{m_1 m_2}{m_1 + m_2} (|\vec{r}_1| + |\vec{r}_2|)^2 \\ &= \mu R^2 \quad \text{where} \quad R = |\vec{r}_1| + |\vec{r}_2| \quad . \end{aligned}$$

Their velocities are given by

$$|\vec{v}_1| = \omega |\vec{r}_1| \quad \text{and} \quad |\vec{v}_2| = \omega |\vec{r}_2| = \omega \frac{m_1}{m_2} |\vec{r}_1| = \frac{m_1}{m_2} |\vec{v}_1| \quad ,$$

and are pointing in opposite directions, hence

$$\vec{v}_2 = -\frac{m_1}{m_2} \vec{v}_1 \quad .$$

Since the motion of each mass is circular, which implies that \vec{r} and \vec{v} are perpendicular, we have for the angular momentum

$$\begin{aligned} |\vec{K}| &= |m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2| = \\ &= \left| m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \left\{ -\frac{m_1}{m_2} \vec{r}_1 \right\} \times \left\{ -\frac{m_1}{m_2} \vec{v}_1 \right\} \right| \\ &= \left\{ m_1 + \frac{m_1^2}{m_2} \right\} |\vec{r}_1 \times \vec{v}_1| = \left\{ m_1 + \frac{m_1^2}{m_2} \right\} |\vec{r}_1| |\vec{v}_1| = \left\{ m_1 + \frac{m_1^2}{m_2} \right\} |\vec{r}_1| \omega |\vec{r}_1| \\ &= \left\{ m_1 + \frac{m_1^2}{m_2} \right\} \omega |\vec{r}_1|^2 = \omega I \quad \Leftrightarrow \quad \omega = \frac{|\vec{K}|}{I} \quad . \end{aligned}$$

The total rotational kinetic energy of this system is given by

$$\begin{aligned} E &= \frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_2 |\vec{v}_2|^2 = \\ &= \frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_2 \left\{ \frac{m_1}{m_2} |\vec{v}_1| \right\}^2 = \frac{1}{2} \left\{ m_1 + \frac{m_1^2}{m_2} \right\} |\vec{v}_1|^2 \\ &= \frac{1}{2} \left\{ m_1 + \frac{m_1^2}{m_2} \right\} \{ \omega |\vec{r}_1| \}^2 = \frac{1}{2} I \omega^2 = \frac{1}{2} I \left\{ \frac{|\vec{K}|}{I} \right\}^2 = \frac{|\vec{K}|^2}{2I} = \frac{|\vec{K}|^2}{2\mu R^2} \quad . \end{aligned}$$

Hence for a rotational state we have something like

$$H \psi_K = \frac{|\vec{K}|^2}{2\mu R^2} \psi_K = \frac{\hbar^2 K(K+1)}{2\mu R^2} \psi_K \quad .$$

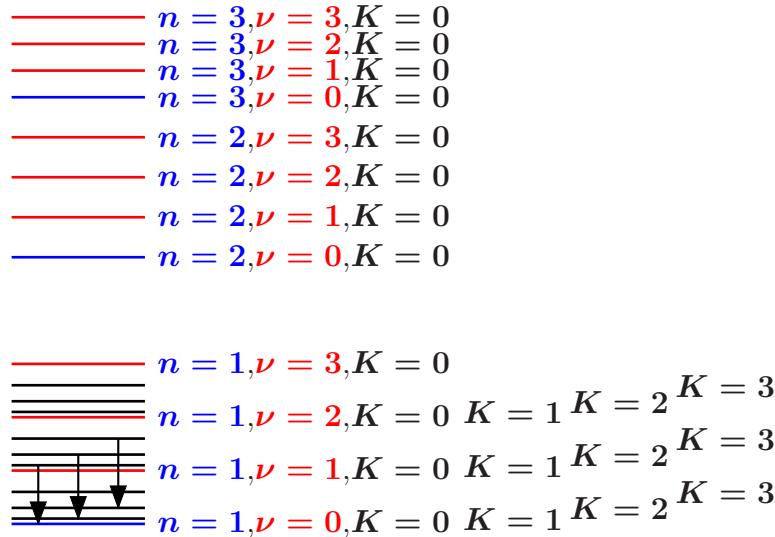
Transition go (to leading order = strong spectral lines) for $\Delta K = \pm 1$. Consequently, the emitted (or absorbed) photons have for $\Delta K = -1$ energy equal to

$$\begin{aligned} \Delta E(\nu, K \rightarrow \nu - 1, K - 1) &= E_\nu + \frac{\hbar^2 K(K+1)}{2\mu R^2} - \left(E_{\nu-1} + \frac{\hbar^2 (K-1)K}{2\mu R^2} \right) = \\ &= \Delta E_{\nu \rightarrow \nu-1} + \frac{\hbar^2}{2\mu R^2} \{ K(K+1) - (K-1)K \} = \Delta E_{\nu \rightarrow \nu-1} + \frac{\hbar^2 K}{\mu R^2} \quad . \end{aligned}$$

We have no knowledge on $\Delta E_{\nu \rightarrow \nu-1} = E_\nu - E_{\nu-1}$, but we can study subsequent spectral lines. For example,

$$\Delta E(\nu, K + 1 \rightarrow \nu - 1, K) - \Delta E(\nu, K \rightarrow \nu - 1, K - 1) = \left(\Delta E_{\nu \rightarrow \nu-1} + \frac{\hbar^2(K+1)}{\mu R^2} \right) - \left(\Delta E_{\nu \rightarrow \nu-1} + \frac{\hbar^2 K}{\mu R^2} \right) = \frac{\hbar^2}{\mu R^2} .$$

In the lectures it was explained how the levels for polar diatomic molecules are supposed to be ordered.



In the first place one has the electronic levels (n) of the outer electron(s). They have level spacings of the order of eV's. Then come the vibrational levels (ν) with spacings of tenths of eV's and finally the rotational levels (K) with hundredths of eV's.

Schematically this looks like depicted in the above figure, where we also depicted some of the possible transitions for the case $\Delta K = -1$.

From the wave lengths one may determine the energy differences ΔE between the various levels by using

$$\Delta E = hf = \frac{hc}{\lambda} ,$$

where $hc = 1.239842 \text{ eV}\mu\text{m}$.

We obtain

i	λ (μm)	ΔE (eV)	$\Delta E(i) - \Delta E(i-1)$ (eV)
1	120.3	0.01031	-
2	96.0	0.01292	0.00261
3	80.4	0.01542	0.00251
4	68.9	0.01800	0.00257
5	60.4	0.02053	0.00253

For the average difference $\Delta E(i) - \Delta E(i - 1)$ we find 0.00256 eV.

When we assume that the above value corresponds to $\frac{\hbar^2}{\mu R^2}$, then we may estimate the distance of the H and the Cl nuclei.

We still need to determine μ .

In the tables we find masses 1.007825 and 34.96885 atomic units for H and Cl, respectively, and, furthermore, 931.494 MeV/c² for 1 atomic unit. Hence

$$\mu c^2 = \frac{1.007825 \times 34.96885}{1.007825 + 34.96885} \times 931.494 \text{ MeV} = 912.485 \text{ MeV} \quad .$$

We obtain then for the distance of the H and the Cl nuclei the value

$$R = \sqrt{\frac{\hbar^2 c^2}{\mu c^2 \times 0.00256 \text{ eV}}} = 1.29 \times 10^{-10} \text{ m} \quad .$$

18. Oscilações pequenas de um sistema de duas partículas, ligadas por uma mola (constante de mola C), cujos movimentos estão restritos a uma dimensão (coordenada r), são classicamente descritas pela equação dinâmica $\mu\ddot{r} = -C(r - r_0)$, onde r_0 representa a distância de equilíbrio do sistema das duas partículas e μ a sua massa reduzida.

a Mostre que o Hamiltoniano deste sistema é dado por

$$H = \frac{1}{2} \left\{ \frac{p^2}{\mu} + \mu\omega^2 x^2 \right\} ,$$

com $x = r - r_0$ e $\omega = \sqrt{C/\mu}$.

b A correspondente equação de onda na Mecânica Quântica obtém-se pela substituição $p \rightarrow -i\hbar\partial/\partial x$ e é dada por

$$H = \frac{1}{2\mu} \left\{ -\hbar^2 \frac{\partial^2}{\partial x^2} + (\mu\omega x)^2 \right\} . \quad (3)$$

Mostre que a função de onda, dada por

$$\psi_0(x) = \left[\frac{\mu\omega}{\pi\hbar} \right]^{1/4} e^{-\frac{1}{2}\mu\omega x^2/\hbar} , \quad (4)$$

representa uma solução normalizada da equação de Schrödinger $H\psi = E\psi$ para $E = E_0 = \hbar\omega/2$.

c Os operadores de criação e aniquilação são respectivamente definidos por

$$a^\dagger = \sqrt{\frac{\mu\omega}{2\hbar}} \left(x - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \quad \text{e} \quad a = \sqrt{\frac{\mu\omega}{2\hbar}} \left(x + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) . \quad (5)$$

Mostre (i) $a\psi_0 = 0$, (ii) $H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$, (iii) $[a, a^\dagger] = 1$, (iv) $[H, a^\dagger] = \hbar\omega a^\dagger$ e (v) $[H, a] = -\hbar\omega a$.

d Mostre que, se ψ é uma solução própria com valor próprio E da equação $H\psi = E\psi$, então $a^\dagger\psi$ é uma solução própria com valor próprio $E + \hbar\omega$ da mesma equação.

Solution:

a. From exercise (17) we have learned that in the center-of-mass system of a two-body system

$$\vec{p}_2 = m_2 \vec{v}_2 = -m_2 \frac{m_1}{m_2} \vec{v}_1 = -m_1 \vec{v}_1 = -\vec{p}_1 .$$

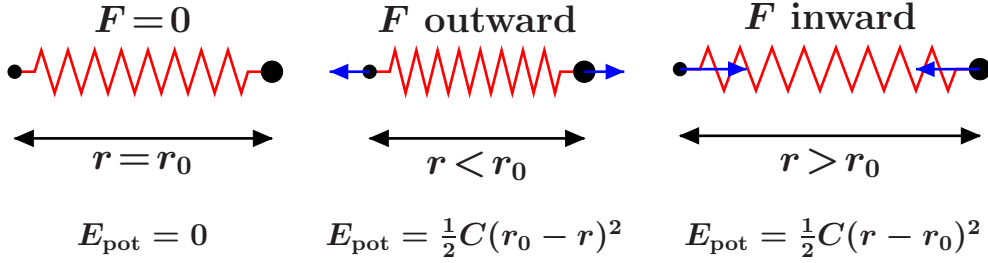
As a consequence, we obtain for the kinetic energy of a two-body system

$$E_{\text{kin}} = \frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_2 |\vec{v}_2|^2 = \frac{|\vec{p}_1|^2}{2m_1} + \frac{|\vec{p}_2|^2}{2m_2} = \frac{1}{2} \frac{m_1 + m_2}{m_1 m_2} |\vec{p}_1|^2 = \frac{|\vec{p}_1|^2}{2\mu} .$$

When we define $\vec{p} = \vec{p}_1 = -\vec{p}_2$, then we obtain thus

$$E_{\text{kin}} = \frac{|\vec{p}|^2}{2\mu} = \frac{p^2}{2\mu} .$$

For the potential energy of the spring we have the following results from classical mechanics and Hooke's law.



With the substitution $x = r - r_0$, also defining $\omega = \sqrt{C/\mu}$ we find for the total energy

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} = \frac{p^2}{2\mu} + \frac{1}{2}Cx^2 = \frac{1}{2} \left\{ \frac{p^2}{\mu} + \mu\omega^2 x^2 \right\} .$$

b. We first define $\mathcal{N}_0 = \left[\frac{\mu\omega}{\pi\hbar} \right]^{1/4}$ and $\alpha = \mu\omega/\hbar$, to obtain for the wave function of equation (4), the expression

$$\psi_0(x) = \mathcal{N}_0 e^{-\frac{1}{2}\alpha x^2} .$$

The second derivative follows readily

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \psi_0(x) &= \frac{\partial^2}{\partial x^2} \mathcal{N}_0 e^{-\frac{1}{2}\alpha x^2} = \\ &= \mathcal{N}_0 \frac{\partial}{\partial x} \left\{ -\alpha x e^{-\frac{1}{2}\alpha x^2} \right\} = \mathcal{N}_0 \left\{ \left(\frac{\partial}{\partial x} (-\alpha x) \right) e^{-\frac{1}{2}\alpha x^2} - \alpha x \left(\frac{\partial}{\partial x} e^{-\frac{1}{2}\alpha x^2} \right) \right\} \\ &= \mathcal{N}_0 \left\{ (-\alpha) e^{-\frac{1}{2}\alpha x^2} - \alpha x \left(-\alpha x e^{-\frac{1}{2}\alpha x^2} \right) \right\} = \mathcal{N}_0 \left\{ -\alpha + \alpha^2 x^2 \right\} e^{-\frac{1}{2}\alpha x^2} \\ &= \left\{ -\alpha + \alpha^2 x^2 \right\} \psi_0(x) . \end{aligned}$$

Consequently

$$\hbar^2 \frac{\partial^2}{\partial x^2} \psi_0(x) = \left\{ -\hbar\mu\omega + \mu^2\omega^2 x^2 \right\} \psi_0(x) .$$

Or, also

$$\frac{1}{2\mu} \left(-\hbar^2 \frac{\partial^2}{\partial x^2} + (\mu\omega x)^2 \right) \psi_0(x) = \frac{1}{2\mu} \left\{ \hbar\mu\omega - \mu^2\omega^2 x^2 + (\mu\omega x)^2 \right\} \psi_0(x) = \frac{1}{2} \hbar\omega \psi_0(x) .$$

c.

(i)

$$\begin{aligned}
a\psi_0 &= \sqrt{\frac{\mu\omega}{2\hbar}} \left(x + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \mathcal{N}_0 e^{-\frac{1}{2}\alpha x^2} = \mathcal{N}_0 \sqrt{\frac{\mu\omega}{2\hbar}} \left(x e^{-\frac{1}{2}\alpha x^2} + \frac{1}{\alpha} \frac{\partial}{\partial x} e^{-\frac{1}{2}\alpha x^2} \right) = \\
&= \mathcal{N}_0 \sqrt{\frac{\mu\omega}{2\hbar}} \left(x e^{-\frac{1}{2}\alpha x^2} + \frac{1}{\alpha} \left\{ -\alpha x e^{-\frac{1}{2}\alpha x^2} \right\} \right) = \mathcal{N}_0 \sqrt{\frac{\mu\omega}{2\hbar}} \left(x e^{-\frac{1}{2}\alpha x^2} - x e^{-\frac{1}{2}\alpha x^2} \right) = 0
\end{aligned}$$

(ii)

$$\begin{aligned}
\hbar\omega \left(a^\dagger a + \frac{1}{2} \right) &= \hbar\omega \sqrt{\frac{\mu\omega}{2\hbar}} \left(x - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \sqrt{\frac{\mu\omega}{2\hbar}} \left(x + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) + \frac{1}{2} \hbar\omega = \\
&= \hbar\omega \frac{\mu\omega}{2\hbar} \left(x - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \left(x + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) + \frac{1}{2} \hbar\omega \\
&= \hbar\omega \frac{\mu\omega}{2\hbar} \left\{ x^2 + \frac{\hbar x}{\mu\omega} \frac{\partial}{\partial x} - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} x - \left(\frac{\hbar}{\mu\omega} \right)^2 \frac{\partial^2}{\partial x^2} \right\} + \frac{1}{2} \hbar\omega \\
&= \hbar\omega \frac{\mu\omega}{2\hbar} \left\{ x^2 + \frac{\hbar}{\mu\omega} \left[x \frac{\partial}{\partial x} - \frac{\partial}{\partial x} x \right] - \left(\frac{\hbar}{\mu\omega} \right)^2 \frac{\partial^2}{\partial x^2} \right\} + \frac{1}{2} \hbar\omega \\
&= \frac{\hbar\omega}{2} \left\{ \frac{\mu\omega}{\hbar} x^2 + \left[x \frac{\partial}{\partial x} - \left(\frac{\partial x}{\partial x} + x \frac{\partial}{\partial x} \right) \right] - \frac{\hbar}{\mu\omega} \frac{\partial^2}{\partial x^2} \right\} + \frac{1}{2} \hbar\omega \\
&= \frac{\hbar\omega}{2} \left\{ \frac{\mu\omega}{\hbar} x^2 + \left[x \frac{\partial}{\partial x} - \left(1 + x \frac{\partial}{\partial x} \right) \right] - \frac{\hbar}{\mu\omega} \frac{\partial^2}{\partial x^2} \right\} + \frac{1}{2} \hbar\omega \\
&= \frac{\hbar\omega}{2} \left\{ \frac{\mu\omega}{\hbar} x^2 - 1 - \frac{\hbar}{\mu\omega} \frac{\partial^2}{\partial x^2} \right\} + \frac{1}{2} \hbar\omega = \frac{1}{2} \mu\omega^2 x^2 - \frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x^2} = H
\end{aligned}$$

(iii)

$$\begin{aligned}
[a, a^\dagger] &= \left[\sqrt{\frac{\mu\omega}{2\hbar}} \left(x + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right), \sqrt{\frac{\mu\omega}{2\hbar}} \left(x - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \right] = \\
&= \frac{\mu\omega}{2\hbar} \left\{ \left(x + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \left(x - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) - \left(x - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \left(x + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} \right) \right\} \\
&= \frac{\mu\omega}{2\hbar} \left\{ \left(x^2 - \frac{\hbar x}{\mu\omega} \frac{\partial}{\partial x} + \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} x - \left(\frac{\hbar}{\mu\omega} \right)^2 \frac{\partial^2}{\partial x^2} \right) - \left(x^2 + \frac{\hbar x}{\mu\omega} \frac{\partial}{\partial x} - \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} x - \left(\frac{\hbar}{\mu\omega} \right)^2 \frac{\partial^2}{\partial x^2} \right) \right\} \\
&= \frac{\mu\omega}{2\hbar} \left\{ -2 \frac{\hbar x}{\mu\omega} \frac{\partial}{\partial x} + 2 \frac{\hbar}{\mu\omega} \frac{\partial}{\partial x} x \right\} = -x \frac{\partial}{\partial x} + \frac{\partial}{\partial x} x = -x \frac{\partial}{\partial x} + \left(\frac{\partial x}{\partial x} + x \frac{\partial}{\partial x} \right) = 1
\end{aligned}$$

(iv)

$$\begin{aligned} [H, a^\dagger] &= \left[\hbar\omega \left(a^\dagger a + \frac{1}{2} \right), a^\dagger \right] = \hbar\omega \left\{ [a^\dagger a, a^\dagger] + \left[\frac{1}{2}, a^\dagger \right] \right\} = \\ &= \hbar\omega [a^\dagger a, a^\dagger] = \hbar\omega \left\{ a^\dagger [a, a^\dagger] + [a^\dagger, a^\dagger] a \right\} = \hbar\omega a^\dagger [a, a^\dagger] = \hbar\omega a^\dagger \end{aligned}$$

(v)

$$\begin{aligned} [H, a] &= \left[\hbar\omega \left(a^\dagger a + \frac{1}{2} \right), a \right] = \hbar\omega \left\{ [a^\dagger a, a] + \left[\frac{1}{2}, a \right] \right\} = \hbar\omega [a^\dagger a, a] = \\ &= \hbar\omega \left\{ a^\dagger [a, a] + [a^\dagger, a] a \right\} = \hbar\omega [a^\dagger, a] a = -\hbar\omega [a, a^\dagger] a = -\hbar\omega a \end{aligned}$$

d.

$$\begin{aligned} H(a^\dagger\psi) &= \{Ha^\dagger\} \psi = \{Ha^\dagger - a^\dagger H + a^\dagger H\} \psi = \\ &= \{[H, a^\dagger] + a^\dagger H\} \psi = \{\hbar\omega a^\dagger + a^\dagger H\} \psi = \hbar\omega a^\dagger\psi + a^\dagger H\psi \\ &= \hbar\omega a^\dagger\psi + a^\dagger E\psi = (E + \hbar\omega)(a^\dagger\psi) \end{aligned}$$

Note that similarly

$$\begin{aligned} H(a\psi) &= \{Ha\} \psi = \{[H, a] + aH\} \psi = \\ &= \{-\hbar\omega a + aE\} \psi = -\hbar\omega a\psi + aE\psi = (E - \hbar\omega)(a\psi) \end{aligned}$$

Furthermore, since $a\psi_0 = 0$, it can be shown that ψ_0 is the ground state of the quantum harmonic oscillator. All other states can be created with a^\dagger . For example, we obtain all eigenstates of $H\psi_\nu = E_\nu\psi_\nu$, with $E_\nu = \left(\frac{1}{2} + \nu\right)\hbar\omega$, for $\nu = 0, 1, 2, \dots$, by choosing $\psi_\nu \propto (a^\dagger)^\nu \psi_0$ (proportionality constants: X_1, X_2, X_3, \dots).

eigenstate of H	eigenenergy
ψ_0	$\frac{1}{2}\hbar\omega$ (b)
$a^\dagger\psi_0 = X_1\psi_1$	$\frac{3}{2}\hbar\omega$ (d)
$(a^\dagger)^2\psi_0 = X_1a^\dagger\psi_1 = X_1X_2\psi_2$	$\frac{5}{2}\hbar\omega$ (d)
\vdots	\vdots
$(a^\dagger)^\nu\psi_0 = X_1X_2\dots X_\nu\psi_\nu$	$\left(\frac{1}{2} + \nu\right)\hbar\omega$ (d)
