25. Na mecânica quântica, a barreira de potencial é um exercício modelo para estudar o fenomeno de *tunelamento quântico*, ou *efeito túnel*. A barreira de potencial numa dimensão tem a seguinte forma ($V_0 > 0$):

$$V(x) \; = \; \begin{cases} 0 & \text{para} \quad x < 0 \\ V_0 & \text{para} \quad 0 \leq x \leq a \\ 0 & \text{para} \quad x > a \end{cases}$$

Portanto, para resolver a equação de Schrödinger é preciso considerar as três regiões em separado e depois aplicar condições de fronteira para $x = 0$ e $x = a$ de tal forma que a função da onda e a sua primeira derivada sejam contínuas.

a Mostre que a função da onda para $E = \dfrac{\hbar^2 k_0^2}{2m} = V_0 + \dfrac{\hbar^2 k_1^2}{2m} > V_0$, dada por

$$\psi(x) \; = \; \begin{cases} A_r e^{ik_0 x} + A_\ell e^{-ik_0 x} & \text{para} \quad x < 0 \\ B_r e^{ik_1 x} + B_\ell e^{-ik_1 x} & \text{para} \quad 0 \leq x \leq a \\ C_r e^{ik_0 x} + C_\ell e^{-ik_0 x} & \text{para} \quad x > a \end{cases}$$

A_r, B_r, C_r, A_ℓ, B_ℓ e C_ℓ constantes, é uma solução da equação de Schrödinger.

b Mostre que as condições de fronteira, dadas por

$$\psi(x \downarrow 0) = \psi(x \uparrow 0) \quad \text{e} \quad \frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0) \quad ,$$

resultam nas seguintas equações:

$$A_r + A_\ell = B_r + B_\ell \quad \text{e} \quad ik_0 \left(A_r - A_\ell \right) = ik_1 \left(B_r - B_\ell \right) \quad .$$

c Mostre que as condições de fronteira, dadas por

$$\psi(x \downarrow a) = \psi(x \uparrow a) \quad \text{e} \quad \frac{d\psi}{dx}(x \downarrow a) = \frac{d\psi}{dx}(x \uparrow a) \quad ,$$

resultam nas seguintas equações:

$$B_r e^{ik_1 a} + B_\ell e^{-ik_1 a} = C_r e^{ik_0 a} + C_\ell e^{-ik_0 a}$$

e

$$ik_1 \left(B_r e^{ik_1 a} - B_\ell e^{-ik_1 a} \right) = ik_0 \left(C_r e^{ik_0 a} - C_\ell e^{-ik_0 a} \right) \quad .$$

Solution:

We first consider two solutions, ψ_1 and ψ_2 with eigenenergies $E = E_1 = E_2$, of the general Schrödinger equation $H\psi = E\psi$. It should, in particular, be noticed that the linear combination $A_1\psi_1 + A_2\psi_2$ is another solution of the Schrödinger equation $H\psi = E\psi$, since

$$H\{A_1\psi_1 + A_2\psi_2\} = A_1 H\psi_1 + A_2 H\psi_2 =$$

$$= A_1 E_1 \psi_1 + A_2 E_2 \psi_2 = A_1 E\psi_1 + A_2 E\psi_2 = E\{A_1\psi_1 + A_2\psi_2\} \quad .$$

---

a. For $x < 0$ and $x > a$ the potential vanishes ($V(x) = 0$). Hence the Schrödinger equation to be solved reads

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \quad .$$

Now, for e^{ik_0x} and e^{-ik_0x} we have

$$\frac{d^2}{dx^2}\, e^{\pm ik_0x} = -k_0^2\, e^{\pm ik_0x} \quad .$$

We obtain then in the Schrödinger equation

$$\frac{\hbar^2 k_0^2}{2m} = E \quad .$$

The lefthand side is a positive number and for the righthand side we are informed that $E > V_0 > 0$, which is also a positive number. Hence, k_0 is a normal real number.
Furthermore, since e^{ik_0x} and e^{-ik_0x} are eigensolutions of the Schrödinger equation for the same eigenenergy E, any linear combination, like $A_r e^{ik_0x} + A_\ell e^{-ik_0x}$ and $C_r e^{ik_0x} + C_\ell e^{-ik_0x}$, are automatically also solutions.

For $0 \leq x \leq a$ one has $V(x) = V_0 > 0$. Hence the Schrödinger equation to be solved reads

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V_0\psi(x) = E\psi(x) \quad .$$

Now, for e^{ik_1x} and e^{-ik_1x} we have

$$\frac{d^2}{dx^2}\, e^{\pm ik_1x} = -k_1^2\, e^{\pm ik_1x} \quad .$$

We obtain then in the Schrödinger equation

$$\frac{\hbar^2 k_1^2}{2m} = E - V_0 \quad .$$

The lefthand side is a positive number and for the righthand side we are informed that $E > V_0$, hence $E - V_0$ is also a positive number and so k_1 is a normal real number.
Furthermore, since e^{ik_1x} and e^{-ik_1x} are eigensolutions of the Schrödinger equation for the same eigenenergy E, any linear combination $B_r e^{ik_1x} + B_\ell e^{-ik_1x}$ is automatically also a solution.

---

b. The wave function below $x = 0$ equals

$$\psi(x < 0) = A_r e^{ik_0 x} + A_\ell e^{-ik_0 x} \quad .$$

At $x = 0$ we obtain

$$\psi(x \uparrow 0) = A_r e^{ik_0 0} + A_\ell e^{-ik_0 0} = A_r + A_\ell \quad .$$

The wave function just above $x = 0$ equals

$$\psi(x > 0) = B_r e^{ik_1 x} + B_\ell e^{-ik_1 x} \quad .$$

At $x = 0$ we obtain

$$\psi(x \downarrow 0) = B_r e^{ik_1 0} + B_\ell e^{-ik_1 0} = B_r + B_\ell \quad .$$

For the boundary condition $\psi(x \downarrow 0) = \psi(x \uparrow 0)$ we obtain thus

$$B_r + B_\ell = \psi(x \downarrow 0) = \psi(x \uparrow 0) = A_r + A_\ell \quad .$$

The derivative of the wave function below $x = 0$ equals

$$\frac{d\psi}{dx}(x < 0) = ik_0 A_r e^{ik_0 x} - ik_0 A_\ell e^{-ik_0 x} \quad .$$

At $x = 0$ we obtain

$$\frac{d\psi}{dx}(x \uparrow 0) = ik_0 A_r - ik_0 A_\ell \quad .$$

The derivative of the wave function just above $x = 0$ equals

$$\frac{d\psi}{dx}(x > 0) = ik_1 B_r e^{ik_1 x} - ik_1 B_\ell e^{-ik_1 x} \quad .$$

At $x = 0$ we obtain

$$\frac{d\psi}{dx}(x \downarrow 0) = ik_1 B_r - ik_1 B_\ell \quad .$$

For the boundary condition $\frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0)$ we obtain thus

$$ik_1 B_r - ik_1 B_\ell = \frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0) = ik_0 A_r - ik_0 A_\ell \quad .$$

Note the following property.

$$|A_r|^2 + |A_\ell|^2 = A_r A_r^* + A_\ell A_\ell^* =$$

$$= \frac{1}{2}\left\{A_r A_r^* + A_r A_\ell^* + A_\ell A_r^* + A_\ell A_\ell^* + A_r A_r^* - A_r A_\ell^* - A_\ell A_r^* + A_\ell A_\ell^*\right\}$$

$$= \frac{1}{2}\left\{\left(A_r + A_\ell\right)\left(A_r^* + A_\ell^*\right) + \left(A_r - A_\ell\right)\left(A_r^* - A_\ell^*\right)\right\}$$

$$= \frac{1}{2}\left\{\left(A_r + A_\ell\right)\left(A_r + A_\ell\right)^* + \left(A_r - A_\ell\right)\left(A_r - A_\ell\right)^*\right\}$$

$$= \frac{1}{2}\left\{\left(B_r + B_\ell\right)\left(B_r + B_\ell\right)^* + \frac{k_1}{k_0}\left(B_r - B_\ell\right)\frac{k_1}{k_0}\left(B_r - B_\ell\right)^*\right\} \quad .$$

c. The wave function just below $x = a$ equals

$$\psi(x < a) = B_r e^{ik_1 x} + B_\ell e^{-ik_1 x} \quad .$$

At $x = a$ we obtain

$$\psi(x \uparrow a) = B_r e^{ik_1 a} + B_\ell e^{-ik_1 a} \quad .$$

The wave function above $x = a$ equals

$$\psi(x > a) = C_r e^{ik_0 x} + C_\ell e^{-ik_0 x} \quad .$$

At $x = a$ we obtain

$$\psi(x \downarrow a) = C_r e^{ik_0 a} + C_\ell e^{-ik_0 a} \quad .$$

For the boundary condition $\psi(x \downarrow a) = \psi(x \uparrow a)$ we obtain thus

$$C_r e^{ik_0 a} + C_\ell e^{-ik_0 a} = B_r e^{ik_1 a} + B_\ell e^{-ik_1 a} \quad .$$

The derivative of the wave function just below $x = a$ equals

$$\frac{d\psi}{dx}(x < a) = ik_1 B_r e^{ik_1 x} - ik_1 B_\ell e^{-ik_1 x} \quad .$$

At $x = a$ we obtain

$$\frac{d\psi}{dx}(x \uparrow a) = ik_1 B_r e^{ik_1 a} - ik_1 B_\ell e^{-ik_1 a} \quad .$$

The derivative of the wave function above $x = a$ equals

$$\frac{d\psi}{dx}(x > a) = ik_0 C_r e^{ik_0 x} - ik_0 C_\ell e^{-ik_0 x} \quad .$$

At $x = a$ we obtain

$$\frac{d\psi}{dx}(x \downarrow a) = ik_0 C_r e^{ik_0 a} - ik_0 C_\ell e^{-ik_0 a} \quad .$$

For the boundary condition $\frac{d\psi}{dx}(x \downarrow a) = \frac{d\psi}{dx}(x \uparrow a)$ we obtain thus

$$ik_0 C_r e^{ik_0 a} - ik_0 C_\ell e^{-ik_0 a} = ik_1 B_r e^{ik_1 a} - ik_1 B_\ell e^{-ik_1 a} \quad .$$

Note the following property.

$$|A_r|^2 + |A_\ell|^2 = \frac{1}{2}\left\{ (B_r + B_\ell)(B_r + B_\ell)^* + \frac{k_1}{k_0}(B_r - B_\ell)\frac{k_1}{k_0}(B_r - B_\ell)^* \right\} =$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{ (B_r + B_\ell)(B_r^* + B_\ell^*) + \frac{k_1}{k_0}(B_r - B_\ell)\frac{k_1}{k_0}(B_r^* - B_\ell^*) \right\} \\
&= \frac{1}{2}\left\{ \left(B_r e^{ik_1 a} + B_\ell e^{-ik_1 a}\right)\left(B_r^* e^{-ik_1 a} + B_\ell^* e^{ik_1 a}\right) + \right. \\
&\qquad\qquad \left. + \frac{k_1}{k_0}\left(B_r e^{ik_1 a} - B_\ell e^{-ik_1 a}\right)\frac{k_1}{k_0}\left(B_r^* e^{-ik_1 a} - B_\ell^* e^{ik_1 a}\right) \right\} \\
&= \frac{1}{2}\left\{ \left(B_r e^{ik_1 a} + B_\ell e^{-ik_1 a}\right)\left(B_r e^{ik_1 a} + B_\ell e^{-ik_1 a}\right)^* + \right. \\
&\qquad\qquad \left. + \frac{k_1}{k_0}\left(B_r e^{ik_1 a} - B_\ell e^{-ik_1 a}\right)\frac{k_1}{k_0}\left(B_r e^{ik_1 a} - B_\ell e^{-ik_1 a}\right)^* \right\} \\
&= \frac{1}{2}\left\{ \left(C_r e^{ik_0 a} + C_\ell e^{-ik_0 a}\right)\left(C_r e^{ik_0 a} + C_\ell e^{-ik_0 a}\right)^* + \right. \\
&\qquad\qquad \left. + \left(C_r e^{ik_0 a} - C_\ell e^{-ik_0 a}\right)\left(C_r e^{ik_0 a} - C_\ell e^{-ik_0 a}\right)^* \right\} \\
&= \frac{1}{2}\left\{ (C_r + C_\ell)(C_r + C_\ell)^* + (C_r - C_\ell)(C_r - C_\ell)^* \right\} \\
&= |C_r|^2 + |C_\ell|^2 \quad .
\end{aligned}
$$

Hence the sum of the intensities of the right moving (r) and left moving (ℓ) modes are the same, left $|A_r|^2 + |A_\ell|^2$ and right $|C_r|^2 + |C_\ell|^2$ of the barrier.

26. Considere a barreira de potencial numa dimensão do exercício (25).

a Mostre que a função de onda para $E = \dfrac{\hbar^2 k_0^2}{2m} = V_0$, dada por

$$\psi(x) \; = \; \begin{cases} A_r e^{ik_0x} + A_\ell e^{-ik_0x} & \text{para} \quad x < 0 \\ B_1 + B_2 x & \text{para} \quad 0 \leq x \leq a \\ C_r e^{ik_0x} + C_\ell e^{-ik_0x} & \text{para} \quad x > a \end{cases}$$

A_r, B_1, C_r, A_ℓ, B_2 e C_ℓ constantes, é uma solução da equação de Schrödinger.

b Mostre que as condições fronteiras, dadas por

$$\psi(x \downarrow 0) = \psi(x \uparrow 0) \quad \text{e} \quad \frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0) \quad ,$$

resultam nas seguintas equações:

$$A_r + A_\ell = B_1 \quad \text{e} \quad ik_0 \left(A_r - A_\ell \right) = B_2 \quad .$$

c Mostre que as condições fronteiras, dadas por

$$\psi(x \downarrow a) = \psi(x \uparrow a) \quad \text{e} \quad \frac{d\psi}{dx}(x \downarrow a) = \frac{d\psi}{dx}(x \uparrow a) \quad ,$$

resultam nas seguintas equações:

$$B_1 + B_2 a = C_r e^{ik_0a} + C_\ell e^{-ik_0a}$$

e

$$B_2 = ik_0 \left(C_r e^{ik_0a} - C_\ell e^{-ik_0a} \right) \quad .$$

Solution:

a. The cases $x < 0$ and $x > a$ are identical to the corresponding cases of problem (25). So, here we concentrate on the case $0 \leq x \leq a$. For $E = V_0$ the Schrödinger equation to be solved reads

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\,\psi(x) = \left(E - V_0 \right) \psi(x) = 0 \quad .$$

This equation has two *independent* solutions

$$\psi(x) = \text{constant} \quad \text{and} \quad \psi(x) = x \quad .$$

Hence, the linear combination (B_1, B_2 constants)

$$\psi(x) = B_1 + B_2 x \quad ,$$

is a general solution of the Schrödinger equation in the interval $0 \leq x \leq a$ when $E = V_0 > 0$.

b and **c**. Most of the work is already done in problem (25). Here we concentrate on the solution in the interval $0 \leq x \leq a$. We observe

$$\begin{aligned}
\psi(x \downarrow 0) &= B_1 \\
\psi(x \uparrow a) &= B_1 + B_2 a \\
\frac{d\psi}{dx}(x \downarrow 0) &= B_2 \\
\frac{d\psi}{dx}(x \uparrow a) &= B_2 \quad .
\end{aligned}$$

b. For the boundary condition $\psi(x \downarrow 0) = \psi(x \uparrow 0)$ we obtain then

$$B_1 = \psi(x \downarrow 0) = \psi(x \uparrow 0) = A_r + A_\ell \quad ,$$

whereas, for the boundary condition $\frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0)$ we find now

$$B_2 = \frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0) = ik_0 A_r - ik_0 A_\ell \quad ,$$

c. Similarly, for the boundary condition $\psi(x \downarrow a) = \psi(x \uparrow a)$ we obtain

$$C_r e^{ik_0 a} + C_\ell e^{-ik_0 a} = B_1 + B_2 a \quad ,$$

whereas, for the boundary condition $\frac{d\psi}{dx}(x \downarrow a) = \frac{d\psi}{dx}(x \uparrow a)$ we find

$$ik_0 C_r e^{ik_0 a} - ik_0 C_\ell e^{-ik_0 a} = B_2 \quad .$$

27. Considere a barreira de potencial numa dimensão do exercício (25).

a Mostre que a função da onda para $0 < E = \dfrac{\hbar^2 k_0^2}{2m} = V_0 - \dfrac{\hbar^2 \kappa_1^2}{2m} < V_0$ dada por

$$\psi(x) \;=\; \begin{cases} A_r e^{ik_0 x} + A_\ell e^{-ik_0 x} & \text{para} \quad x < 0 \\ B_r e^{-\kappa_1 x} + B_\ell e^{\kappa_1 x} & \text{para} \quad 0 \leq x \leq a \\ C_r e^{ik_0 x} + C_\ell e^{-ik_0 x} & \text{para} \quad x > a \end{cases}$$

A_r, B_r, C_r, A_ℓ, B_ℓ e C_ℓ constantes, é uma solução da equação de Schrödinger.

b Mostre que as condições de fronteira, dadas por

$$\psi(x \downarrow 0) = \psi(x \uparrow 0) \quad \text{e} \quad \frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0) \quad ,$$

resultam nas seguintas equações:

$$A_r + A_\ell = B_r + B_\ell \quad \text{e} \quad ik_0 \left(A_r - A_\ell\right) = -\kappa_1 \left(B_r - B_\ell\right) \quad .$$

c Mostre que as condições de fronteira, dadas por

$$\psi(x \downarrow a) = \psi(x \uparrow a) \quad \text{e} \quad \frac{d\psi}{dx}(x \downarrow a) = \frac{d\psi}{dx}(x \uparrow a) \quad ,$$

resultam nas seguintas equações:

$$B_r e^{-\kappa_1 a} + B_\ell e^{\kappa_1 a} = C_r e^{ik_0 a} + C_\ell e^{-ik_0 a}$$

e

$$i\kappa_1 \left(B_r e^{-\kappa_1 a} - B_\ell e^{\kappa_1 a}\right) = ik_0 \left(C_r e^{ik_0 a} - C_\ell e^{-ik_0 a}\right) \quad .$$

Solution:

a. The cases $x < 0$ and $x > a$ are identical to the corresponding cases of problem (25). So, here we concentrate on the case $0 \leq x \leq a$. For $0 < E < V_0$ the Schrödinger equation to be solved reads

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\,\psi(x) = \left(E - V_0\right)\psi(x) = -\left(V_0 - E\right)\psi(x) \quad .$$

Note here that $V_0 - E > 0$ when $0 < E < V_0$.
This equation has two *independent* solutions $\left(\text{with } \dfrac{\hbar^2\kappa_1^2}{2m} = V_0 - E\right)$

$$\psi(x) = e^{-\kappa_1 x} \quad \text{and} \quad \psi(x) = e^{\kappa_1 x} \quad .$$

Hence, the linear combination (B_r, B_ℓ constants)

$$\psi(x) = B_r e^{-\kappa_1 x} + B_\ell e^{\kappa_1 x} \quad ,$$

is a general solution of the Schrödinger equation in the interval $0 \leq x \leq a$ when $0 < E < V_0$.

b and **c**. Most of the work is already done in problem (25). Here we concentrate on the solution in the interval $0 \leq x \leq a$. We observe

$$\begin{aligned}
\psi(x \downarrow 0) &= B_r + B_\ell \\
\psi(x \uparrow a) &= B_r e^{-\kappa_1 a} + B_\ell e^{\kappa_1 a} \\
\frac{d\psi}{dx}(x \downarrow 0) &= -\kappa_1 B_r + \kappa_1 B_\ell \\
\frac{d\psi}{dx}(x \uparrow a) &= -\kappa_1 B_r e^{-\kappa_1 a} + \kappa_1 B_\ell e^{\kappa_1 a} \quad .
\end{aligned}$$

b. For the boundary condition $\psi(x \downarrow 0) = \psi(x \uparrow 0)$ we obtain then

$$B_r + B_\ell = \psi(x \downarrow 0) = \psi(x \uparrow 0) = A_r + A_\ell \quad ,$$

whereas, for the boundary condition $\frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0)$ we find now

$$-\kappa_1 B_r + \kappa_1 B_\ell = \frac{d\psi}{dx}(x \downarrow 0) = \frac{d\psi}{dx}(x \uparrow 0) = ik_0 A_r - ik_0 A_\ell \quad ,$$

c. Similarly, for the boundary condition $\psi(x \downarrow a) = \psi(x \uparrow a)$ we obtain

$$C_r e^{ik_0 a} + C_\ell e^{-ik_0 a} = B_r e^{-\kappa_1 a} + B_\ell e^{\kappa_1 a} \quad ,$$

whereas, for the boundary condition $\frac{d\psi}{dx}(x \downarrow a) = \frac{d\psi}{dx}(x \uparrow a)$ we find

$$ik_0 C_r e^{ik_0 a} - ik_0 C_\ell e^{-ik_0 a} = -\kappa_1 B_r e^{-\kappa_1 a} + \kappa_1 B_\ell e^{\kappa_1 a} \quad .$$

Note, that all the above could have been obtained by substituting k_1 of problem (25) by $i\kappa_1$. This follows from

$$\frac{\hbar^2 k_1^2}{2m} = E - V_0 \quad \text{and} \quad \frac{\hbar^2 \kappa_1^2}{2m} = V_0 - E \quad .$$

The lefthand expression is positive for $E > V_0 > 0$ (problem 25), whereas the righthand expression is positive for $0 < E < V_0$ (the present problem).

28. Considere mais uma vez a barreira de potencial numa dimensão do exercício (25). Pretendemos agora estudar uma onda que vem de $x = -\infty$ e que se desloca no sentido positivo. A onda é suposta ser reflectida em parte (em $x = 0$) e na outra parte ser transmitida (em $x = a$) pela barreira de potencial. Esta situação é representada por uma função de onda para $E = \frac{\hbar^2 k_0^2}{2m} = V_0 + \frac{\hbar^2 k_1^2}{2m} > V_0$, dada por

$$\psi(x) \; = \; \left\{ \begin{array}{rll} e^{ik_0 x} + r\, e^{-ik_0 x} & \text{para} & x < 0 \\ B_r e^{ik_1 x} + B_\ell e^{-ik_1 x} & \text{para} & 0 \leq x \leq a \\ t\, e^{ik_0 x} & \text{para} & x > a \end{array} \right.$$

idêntica à função de onda do exercício (25), mas com $A_r = 1$, $A_\ell = r$ (coeficiente de reflexão), $C_r = t$ (coeficiente de transmissão) e $C_\ell = 0$ (não há sinal que se propague no sentido negativo para $x > a$, mas apenas um sinal transmitido que se propaga no sentido pósitivo).

a Usando as condições

$$A_r + A_\ell = B_r + B_\ell \quad , \quad ik_0 \left(A_r - A_\ell\right) = ik_1 \left(B_r - B_\ell\right) \quad ,$$

$$B_r e^{ik_1 a} + B_\ell e^{-ik_1 a} = C_r e^{ik_0 a} + C_\ell e^{-ik_0 a}$$

e

$$ik_1 \left(B_r e^{ik_1 a} - B_\ell e^{-ik_1 a} \right) = ik_0 \left(C_r e^{ik_0 a} - C_\ell e^{-ik_0 a} \right)$$

obtidas no exercício (25), mostre que

$$(1 + r) \cos\left(k_1 a\right) + \frac{k_0}{k_1}(1 - r) i \sin\left(k_1 a\right) \;\; = \;\; t\, e^{ik_0 a}$$

$$ik_1 \left\{ (1 + r) i \sin\left(k_1 a\right) + \frac{k_0}{k_1}(1 - r) \cos\left(k_1 a\right) \right\} \;\; = \;\; ik_0 \left(t\, e^{ik_0 a} \right) \quad .$$

b Mostre que

$$r = \frac{\left(k_0^2 - k_1^2\right) \sin\left(k_1 a\right)}{\left(k_0^2 + k_1^2\right) \sin\left(k_1 a\right) + 2ik_0 k_1 \cos\left(k_1 a\right)} \quad ,$$

e

$$t = \frac{2k_0 k_1 \left\{ \sin\left(k_0 a\right) + i \cos\left(k_0 a\right) \right\}}{\left(k_0^2 + k_1^2\right)\, \sin\left(k_1 a\right) + 2ik_0 k_1\, \cos\left(k_1 a\right)} \quad .$$

c A intensidade relativa T do sinal transmitido é igual ao quadrado do módulo da coeficiente de transmissão t. Mostre que

$$T(E > V_0 > 0) = \frac{1}{1 + \frac{V_0^2}{4E(E - V_0)} \sin^2(k_1 a)} \quad \text{e} \quad T + R = |t|^2 + |r|^2 = 1 \quad .$$

d O caso $0 < E < V_0$ obtém-se da alínea **c** através da substituição $k_1 \to i\kappa_1$. Mostre que

$$T(0 < E < V_0) = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(\kappa_1 a)} \quad .$$

Solution:

Here, we consider

$$A_r = 1 \quad , \quad A_\ell = r \quad , \quad C_r = t \quad \text{and} \quad C_\ell = 0 \quad .$$

A_r is the amplitude for $\exp(ik_0 x)$ which represents a wave coming from $x = -\infty$ and moving to the right (subscript r) towards $x = +\infty$. At the barrier ($x = 0$) part of the wave is reflected, represented by $\exp(-ik_0 x)$ (with amplitude $A_\ell = r$), moving back to the left (subscript ℓ) towards $x = -\infty$.

Similarly, $C_r = t$ is the amplitude for $\exp(ik_0 x)$ which represents the part of the wave which passes (amplitude t for transmission) through the barrier ($x > a$) and moves to the right (subscript r) towards $x = +\infty$. No wave is supposed to come from $x = +\infty$ moving towards the barrier. Consequently, the amplitude for $\exp(-ik_0 x)$ is supposed to be zero for $x > a$, *i.e.* $C_\ell = 0$.

The equations for the amplitudes in problem (25) reduce to

$$1 + r = B_r + B_\ell \tag{1}$$

$$ik_0 (1 - r) = ik_1 (B_r - B_\ell) \tag{2}$$

$$B_r e^{ik_1 a} + B_\ell e^{-ik_1 a} = t\, e^{ik_0 a} \tag{3}$$

$$ik_1 \left(B_r e^{ik_1 a} - B_\ell e^{-ik_1 a} \right) = ik_0 \left(t\, e^{ik_0 a} \right) \quad . \tag{4}$$

a. Using (1) and (2), we express B_r and B_ℓ in the amplitudes r and t.

$$B_r = \frac{1}{2} \left\{ 1 + r + \frac{k_0}{k_1}(1 - r) \right\} = \frac{1}{2k_1} \left\{ (k_0 + k_1) - (k_0 - k_1)\, r \right\}$$

$$B_\ell = \frac{1}{2} \left\{ 1 + r - \frac{k_0}{k_1}(1 - r) \right\} = \frac{1}{2k_1} \left\{ -(k_0 - k_1) + (k_0 + k_1)\, r \right\} \quad ,$$

whereas, from Eqs. (3) and (4) we obtain

$$(k_0 - k_1)\, B_r e^{ik_1 a} + (k_0 + k_1)\, B_\ell e^{-ik_1 a} = 0 \quad .$$

Hence, by substitution

$$(k_0 - k_1)\left\{(k_0 + k_1) - (k_0 - k_1)\ r\right\} e^{ik_1 a} + (k_0 + k_1)\left\{-(k_0 - k_1) + (k_0 + k_1)\ r\right\} e^{-ik_1 a} = 0 \quad ,$$

or

$$\left\{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}\right\} r = \left(k_0^2 - k_1^2\right)\left\{e^{ik_1 a} - e^{-ik_1 a}\right\} = \left(k_0^2 - k_1^2\right) 2i \sin(k_1 a) .$$

Which gives

$$r = \frac{\left(k_0^2 - k_1^2\right) 2i \sin(k_1 a)}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}} \quad .$$

From this expression we may determine expressions for B_r and B_ℓ.

$$2k_1 B_r e^{ik_1 a} = \left\{(k_0 + k_1) - (k_0 - k_1)\ r\right\} e^{ik_1 a} =$$

$$= (k_0 + k_1) e^{ik_1 a} - (k_0 - k_1) \frac{\left(k_0^2 - k_1^2\right)\left\{e^{ik_1 a} - e^{-ik_1 a}\right\}}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}} e^{ik_1 a}$$

$$= \frac{(k_0 + k_1)(k_0 - k_1)^2 e^{2ik_1 a} - (k_0 + k_1)(k_0 + k_1)^2 - (k_0 - k_1)\left(k_0^2 - k_1^2\right)\left\{e^{2ik_1 a} - 1\right\}}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}}$$

$$= \frac{(k_0 + k_1)(k_0 - k_1)^2 e^{2ik_1 a} - (k_0 + k_1)^3 - (k_0 + k_1)(k_0 - k_1)^2\left\{e^{2ik_1 a} - 1\right\}}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}}$$

$$= \frac{-(k_0 + k_1)^3 + (k_0 + k_1)(k_0 - k_1)^2}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}}$$

$$= \frac{-4(k_0 + k_1) k_0 k_1}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}}$$

and similarly

$$2k_1 B_\ell e^{-ik_1 a} = \left\{-(k_0 - k_1) + (k_0 + k_1)\ r\right\} e^{-ik_1 a} =$$

$$= -(k_0 - k_1) e^{-ik_1 a} + (k_0 + k_1) \frac{\left(k_0^2 - k_1^2\right)\left\{e^{ik_1 a} - e^{-ik_1 a}\right\}}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}} e^{-ik_1 a}$$

$$= \frac{-(k_0 - k_1)^3 + (k_0 - k_1)(k_0 + k_1)^2 e^{-2ik_1 a} + (k_0 + k_1)^2 (k_0 - k_1)\left\{1 - e^{-2ik_1 a}\right\}}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}}$$

$$= \frac{-(k_0 - k_1)^3 + (k_0 + k_1)^2 (k_0 - k_1)}{(k_0 - k_1)^2 e^{ik_1 a} - (k_0 + k_1)^2 e^{-ik_1 a}}$$

$$= \frac{4\left(k_0 - k_1\right) k_0 k_1}{\left(k_0 - k_1\right)^2 e^{ik_1 a} - \left(k_0 + k_1\right)^2 e^{-ik_1 a}}$$

Furthermore, from relation (3) we have

$$2k_1 t\, e^{ik_0 a} = 2k_1 B_r e^{ik_1 a} + 2k_1 B_\ell e^{-ik_1 a} =$$

$$= \frac{-4\left(k_0 + k_1\right) k_0 k_1}{\left(k_0 - k_1\right)^2 e^{ik_1 a} - \left(k_0 + k_1\right)^2 e^{-ik_1 a}} + \frac{4\left(k_0 - k_1\right) k_0 k_1}{\left(k_0 - k_1\right)^2 e^{ik_1 a} - \left(k_0 + k_1\right)^2 e^{-ik_1 a}}$$

$$= \frac{-8 k_0 k_1^2}{\left(k_0 - k_1\right)^2 e^{ik_1 a} - \left(k_0 + k_1\right)^2 e^{-ik_1 a}} \quad .$$

Hence,

$$t = \frac{-4 k_0 k_1 e^{-ik_0 a}}{\left(k_0 - k_1\right)^2 e^{ik_1 a} - \left(k_0 + k_1\right)^2 e^{-ik_1 a}} \quad .$$

---

b. We determine

$$\left(k_0 - k_1\right)^2 \, e^{ik_1 a} - \left(k_0 + k_1\right)^2 \, e^{-ik_1 a} =$$

$$= \left(k_0^2 + k_1^2 - 2k_0 k_1\right) \, e^{ik_1 a} - \left(k_0^2 + k_1^2 + 2k_0 k_1\right) \, e^{-ik_1 a}$$

$$= \left(k_0^2 + k_1^2\right) \left\{ e^{ik_1 a} - e^{-ik_1 a} \right\} - 2k_0 k_1 \left\{ e^{ik_1 a} + e^{-ik_1 a} \right\}$$

$$= 2i \left(k_0^2 + k_1^2\right) \sin\left(k_1 a\right) - 4 k_0 k_1 \cos\left(k_1 a\right) \quad .$$

Consequently,

$$\begin{aligned} r &= \frac{(k_0^2 - k_1^2)\, 2i \sin\left(k_1 a\right)}{2i\,(k_0^2 + k_1^2) \sin\left(k_1 a\right) - 4 k_0 k_1 \cos\left(k_1 a\right)} \\ &= \frac{(k_0^2 - k_1^2) \sin\left(k_1 a\right)}{(k_0^2 + k_1^2) \sin\left(k_1 a\right) + 2i k_0 k_1 \cos\left(k_1 a\right)} \quad , \end{aligned}$$

and

$$\begin{aligned} t &= \frac{-4 k_0 k_1 e^{-ik_0 a}}{2i\,(k_0^2 + k_1^2) \sin\left(k_1 a\right) - 4 k_0 k_1 \cos\left(k_1 a\right)} \\ &= \frac{2i k_0 k_1 e^{-ik_0 a}}{(k_0^2 + k_1^2) \sin\left(k_1 a\right) + 2i k_0 k_1 \cos\left(k_1 a\right)} \quad . \end{aligned}$$

c. The intensity T of the transmitted signal is obtained from the transmission amplitude by

$$T = |t|^2 = \frac{k_0^2 k_1^2}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)}$$

$$= \frac{k_0^2 k_1^2}{k_0^2 k_1^2 + \frac{1}{4}\left(k_0^2 - k_1^2\right)^2 \sin^2(k_1 a)} = \frac{1}{1 + \frac{\left(k_0^2 - k_1^2\right)^2}{4k_0^2 k_1^2} \sin^2(k_1 a)}$$

$$= \frac{1}{1 + \frac{V_0^2}{4E(E - V_0)} \sin^2(k_1 a)} \quad .$$

This formula works also for the other cases as we will see below.

But, first let us determine the intensity of the reflected signal $R = |r|^2$.

We obtained before

$$r = \frac{(k_0^2 - k_1^2)\sin(k_1 a)}{(k_0^2 + k_1^2)\sin(k_1 a) + 2ik_0 k_1 \cos(k_1 a)} \quad .$$

This can also be written in the form

$$r = \frac{-\frac{i}{2}(k_0^2 - k_1^2)\sin(k_1 a)}{k_0 k_1 \cos(k_1 a) - \frac{i}{2}(k_0^2 + k_1^2)\sin(k_1 a)} \quad .$$

Consequently R takes the form

$$R = |r|^2 = \frac{\frac{1}{4}\left(k_0^2 - k_1^2\right)^2 \sin^2(k_1 a)}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)} \quad .$$

If we add this expression to

$$T = \frac{k_0^2 k_1^2}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)} \quad ,$$

then we obtain

$$R+T = \frac{k_0^2 k_1^2 + \frac{1}{4}\left(k_0^2 - k_1^2\right)^2 \sin^2(k_1 a)}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)} = \frac{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)} = 1 \quad .$$

The sum of the intensities of the reflected and the transmitted signals, $R+T$, equals the initial intensity $|A_r|^2 = 1$.

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d. For $0 < E < V_0$, we just perform the substitution $k_1 \rightarrow i\kappa_1$ and also observe

$$\sin(ix) = \frac{1}{2i}\left(e^{-x} - e^{x}\right) = i\sinh(x) \quad .$$

We obtain

$$T = \frac{1}{1 - \dfrac{V_0^2}{4E(E-V_0)}\sinh^2(\kappa_1 a)} = \frac{1}{1 + \dfrac{V_0^2}{4E(V_0-E)}\sinh^2(\kappa_1 a)} \quad .$$

For the case $E = V_0$ we take the limit $k_1 \downarrow 0$.

$$\frac{\left(k_0^2 - k_1^2\right)^2}{4k_0^2 k_1^2}\sin^2(k_1 a) = \frac{\left(k_0^2 - k_1^2\right)^2 a^2}{4k_0^2}\left(\frac{\sin(k_1 a)}{k_1 a}\right)^2 \underset{k_1 \downarrow 0}{\longrightarrow} \frac{k_0^2 a^2}{4} = \frac{mV_0 a^2}{2\hbar^2} \quad .$$

Consequently,

$$T = \frac{1}{1 + \dfrac{mV_0 a^2}{2\hbar^2}} \quad .$$

In figure 1 we show how the transmission T develops with the energy E of the initial wave.

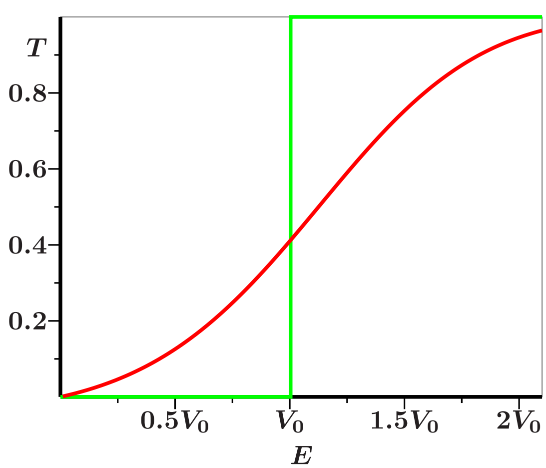

Figure 1: A typical curve for the development of the transmission T when E is varried. For $0 < E < V_0$ we observe that some signal passes the barrier ($T > 0$). That is the phenomenon of quantum tunneling. The green line indicates the classical situation: Zero transmission untill the energy is large enough. For large enough energy ($E > V_0$) the transmission is 100%.