5. O espectro do átomo de hidrogénio é o conjunto de comprimentos de onda presentes na luz que o átomo de hidrogénio é capaz de emitir quando baixa de nível de energia.

O modelo mais simples do átomo de hidrogénio é representado pelo átomo de Bohr. Neste modelo o espectro de luz é composto de comprimentos de onda discretos, cujos valores são expressos pela fórmula de Rydberg

$$\frac{1}{\lambda_{\text{vac}}} = R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \quad ,$$

onde λ_{vac} é o comprimento de onda da luz emitida no vácuo, R_{H} é a constante de Rydberg para o hidrogénio, e n_1 e n_2 são inteiros tais que $n_1 < n_2$.

Deixando n_1 igual a 1 e fazendo n_2 percorrer valores de 2 a infinito, as linhas de espectro conhecidas como série de Lyman convergem para 91 nm. Da mesma maneira:

n_1	n_2		limite (nm)
1	$2 \to \infty$	Série de Lyman	91
2	$3 \to \infty$	Série de Balmer	365
3	$4 \to \infty$	Série de Paschen	821
4	$5 \to \infty$	Série de Brackett	1459
5	$6 \to \infty$	Série de Pfund	2280
6	$7 \to \infty$	Série de Humphreys	3283

a Determine os valores de n_2 para as linhas da série de Balmer (1885), H_α, H_β, H_γ, etc., que têm respectivamente os seguintes comprimentos de onda (em **nm**): no espectro visível 656.3 (vermelho), 486.1 (azul-verde), 434.1 (azul-violeta) e 410.2 (violeta) e no espectro ultra-violeta 397.0, 388.9, 383.5 e 364.6.

b A mesma pergunta para as linhas da série de Lyman, cujas frequências são (em 10^{15} Hz) 3.238, 3.223, 3.198, 3.158, 3.084, 2.924 e 2.467.

Solution:

First we determine the Rydberg constant for Hydrogen from the information which is given about the series limits. We have collected the results in the following table.

n_1	n_2		limit (nm)	$R_H = n_1^2/\text{limit}$ (1/nm)
1	$2 \to \infty$	Lyman series	91	0.010989
2	$3 \to \infty$	Balmer series	365	0.010959
3	$4 \to \infty$	Paschen series	821	0.010962
4	$5 \to \infty$	Brackett series	1459	0.010966
5	$6 \to \infty$	Pfund series	2280	0.010965
6	$7 \to \infty$	Humphreys series	3283	0.010966
			average	0.010968

The average value is $R_H = 0.010968$ (1/nm) which is different from the value which we find in the literature, namely $R_H = 0.010974$ (1/nm).

a. Since the Balmer series has its limit at 365 nm, the ultraviolet line at 364.6 nm cannot exist. Obviously, there must be some conflict between the limits and the further measurements.
For the other lines from the Balmer series we use the indicated formula

$$\frac{1}{\lambda_{\text{vac}}} = R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \qquad \rightarrow \qquad n_2 = 1 \Big/ \sqrt{\frac{1}{4} - \frac{1}{R_{\text{H}} \lambda_{\text{vac}}}}$$

line (nm)	656.3	486.1	434.1	410.2	397.0	388.9	383.5
n_2	3.00	4.00	5.00	6.01	7.01	8.02	9.03

Those values are indeed the integers 3, 4, ..., 9 which we expected.

b. Here, we do first the conversion to nm, using $\lambda = c/f$ and then determine n_2 by the use of

$$\frac{1}{\lambda} = R_{\text{H}} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \qquad \rightarrow \qquad n_2 = 1 \Big/ \sqrt{\frac{1}{1} - \frac{1}{R_{\text{H}} \lambda}}$$

f (10^{15} Hz)	3.238	3.223	3.198	3.158	3.084	2.924	2.467
λ (nm)	92.588	93.019	93.746	94.934	97.211	102.531	121.524
n_2	8.10	7.10	6.04	5.03	4.01	3.00	2.00

Within some error, those values are the integers 8, 7, ..., 2 which we expected.

6. O termo da energia cinética da equação de Schrödinger é dado por

$$-\frac{\hbar^2}{2m}\bigtriangledown^2\ \psi(\vec{r})\ =\ -\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}\,r\,\psi(\vec{r})\ +\ \frac{L^2}{2mr^2}\,\psi(\vec{r})\quad ,$$

onde o momento angular ao quadrado L^2 é dado por

$$L^2\ =\ -\hbar^2\left\{\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\ +\ \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2}\right\}\quad .$$

Define ainda o seguinte operador

$$L_z\ =\ -i\,\hbar\,\frac{\partial}{\partial\varphi}\quad .$$

Mostre que as seguintes expressões são simultaneamente funções próprias de L^2 e de L_z e determine os respectivos valores próprios:

a $Y_0^0(\vartheta,\varphi)=\sqrt{1/4\pi}$

b $Y_1^0(\vartheta,\varphi)=\sqrt{3/4\pi}\,\cos(\vartheta)$

c $Y_1^1(\vartheta,\varphi)=-\sqrt{3/8\pi}\,\sin(\vartheta)\,e^{i\varphi}$

d $Y_1^{-1}(\vartheta,\varphi)=\sqrt{3/8\pi}\,\sin(\vartheta)\,e^{-i\varphi}$

e $Y_2^0(\vartheta,\varphi)=\sqrt{5/16\pi}\,\left\{3\cos^2(\vartheta)-1\right\}$

f $Y_2^1(\vartheta,\varphi)=-\sqrt{15/32\pi}\,\sin(2\vartheta)\,e^{i\varphi}$

g $Y_2^{-1}(\vartheta,\varphi)=\sqrt{15/32\pi}\,\sin(2\vartheta)\,e^{-i\varphi}$

h $Y_2^2(\vartheta,\varphi)=\sqrt{15/32\pi}\,\sin^2(\vartheta)\,e^{2i\varphi}$

i $Y_2^{-2}(\vartheta,\varphi)=\sqrt{15/32\pi}\,\sin^2(\vartheta)\,e^{-2i\varphi}$

Solution:

In order to obtain the wanted results, we must study the following derivatives.

$$\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\cos(\vartheta)\ =\ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\ [-\sin(\vartheta)]\ =$$

$$=\ -\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin^2(\vartheta)\ =\ -\frac{1}{\sin(\vartheta)}\ [2\cos(\vartheta)\sin(\vartheta)]\ =\ -2\cos(\vartheta)\quad , \tag{1}$$

$$\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\;\sin(\vartheta) \;=\; \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\;[\cos(\vartheta)] \;=$$

$$=\; \frac{1}{\sin(\vartheta)}\left[\cos^2(\vartheta)-\sin^2(\vartheta)\right] = \frac{1}{\sin(\vartheta)}\left[1-2\sin^2(\vartheta)\right] = \left[\frac{1}{\sin^2(\vartheta)}-2\right]\sin(\vartheta)\;, \tag{2}$$

$$\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\;\left\{3\cos^2(\vartheta)-1\right\} \;=\; \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\;[-6\sin(\vartheta)\cos(\vartheta)] \;=$$

$$=\; -6\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin^2(\vartheta)\cos(\vartheta) \;=\; -6\frac{1}{\sin(\vartheta)}\left[2\sin(\vartheta)\cos^2(\vartheta)-\sin^3(\vartheta)\right]$$

$$=\; -12\cos^2(\vartheta)+6\sin^2(\vartheta) \;=\; -18\cos^2(\vartheta)+6 \quad , \tag{3}$$

$$\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\;\sin(2\vartheta) \;=\; \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\,2\sin(\vartheta)\cos(\vartheta) \;=$$

$$=\; 2\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\;\left[\cos^2(\vartheta)-\sin^2(\vartheta)\right] \;=\; 2\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\;\left[1-2\sin^2(\vartheta)\right]$$

$$=\; 2\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left[\sin(\vartheta)-2\sin^3(\vartheta)\right] \;=\; 2\frac{1}{\sin(\vartheta)}\left[\cos(\vartheta)-6\sin^2(\vartheta)\cos(\vartheta)\right]$$

$$=\; \frac{\sin(2\vartheta)}{\sin^2(\vartheta)}-6\sin(2\vartheta) \quad , \tag{4}$$

$$\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\;\sin^2(\vartheta) \;=\; \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\;[2\sin(\vartheta)\cos(\vartheta)] \;=$$

$$=\; 2\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin^2(\vartheta)\cos(\vartheta) \;=\; 2\frac{1}{\sin(\vartheta)}\left[2\sin(\vartheta)\cos^2(\vartheta)-\sin^3(\vartheta)\right]$$

$$=\; 4\cos^2(\vartheta)-2\sin^2(\vartheta) \;=\; 4-6\sin^2(\vartheta) \quad , \tag{5}$$

and

$$\frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2}\,e^{im\varphi} \;=\; -\,\frac{1}{\sin^2(\vartheta)}\,m^2\,e^{im\varphi}\;. \tag{6}$$

a: $\boldsymbol{Y_0^0(\vartheta,\varphi) = \sqrt{1/4\pi}}$.

Since the derivatives of a constant function are all equal to zero, we find here

$$L^2 Y_0^0(\vartheta,\varphi) \;=\; -\hbar^2 \left\{ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} \;+\; \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\} \sqrt{1/4\pi} \;=\; 0$$

and

$$L_z Y_0^0(\vartheta,\varphi) \;=\; -i\,\hbar\,\frac{\partial}{\partial\varphi}\sqrt{1/4\pi} \;=\; 0 \quad .$$

We find that the eigenvalues of Y_0^0 are zero with respect to the differential operators L^2 and L_z.

b: $\boldsymbol{Y_1^0(\vartheta,\varphi) = \sqrt{3/4\pi}\;\cos(\vartheta)}$.

Using formula (1), we obtain

$$\begin{aligned} L^2 Y_1^0(\vartheta,\varphi) \;&=\; -\hbar^2 \left\{ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} \;+\; \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\} \sqrt{3/4\pi}\;\cos(\vartheta) \\ &=\; -\hbar^2\,\sqrt{3/4\pi}\;\{-2\cos(\vartheta)\;+\;0\} \;=\; 2\hbar^2 Y_1^0(\vartheta,\varphi) \end{aligned}$$

Furthermore, since Y_1^0 does not depend on φ, we have

$$L_z Y_1^0(\vartheta,\varphi) \;=\; -i\,\hbar\,\frac{\partial}{\partial\varphi}\sqrt{3/4\pi}\;\cos(\vartheta) \;=\; 0 \quad .$$

We find that the eigenvalues of Y_1^0 equal $\hbar^2\ell(\ell+1)$ for $\ell = 1$ with respect to the differential operator L^2, and $m\hbar$ for $m = 0$ with respect to the differential operator L_z.

c: $\boldsymbol{Y_1^1(\vartheta,\varphi) = -\sqrt{3/8\pi}\;\sin(\vartheta)\,e^{i\varphi}}$.

Using formulas (2) and (6), we obtain

$$\begin{aligned} L^2 Y_1^1(\vartheta,\varphi) \;&=\; -\hbar^2 \left\{ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} \;+\; \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\} \left[-\sqrt{3/8\pi}\;\sin(\vartheta)\,e^{i\varphi}\right] \;= \\ &=\; \hbar^2\sqrt{3/8\pi}\left\{ \left[\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\sin(\vartheta)\right] e^{i\varphi} \;+\; \frac{1}{\sin^2(\vartheta)}\sin(\vartheta)\left[\frac{\partial^2}{\partial\varphi^2}e^{i\varphi}\right] \right\} \\ &=\; \hbar^2\sqrt{3/8\pi}\left\{ \left[\frac{1}{\sin^2(\vartheta)} - 2\right]\sin(\vartheta)\,e^{i\varphi} \;+\; \frac{1}{\sin^2(\vartheta)}\sin(\vartheta)\left[-e^{i\varphi}\right] \right\} \\ &=\; -2\hbar^2\sqrt{3/8\pi}\;\sin(\vartheta)\,e^{i\varphi} \;=\; 2\hbar^2 Y_1^1(\vartheta,\varphi) \end{aligned}$$

and

$$\begin{aligned} L_z Y_1^1(\vartheta,\varphi) \;&=\; -i\,\hbar\,\frac{\partial}{\partial\varphi}\left[-\sqrt{3/8\pi}\;\sin(\vartheta)\,e^{i\varphi}\right] \;=\; i\,\hbar\,\sqrt{3/8\pi}\,\sin(\vartheta)\left[\frac{\partial}{\partial\varphi}e^{i\varphi}\right] \\ &=\; i\,\hbar\,\sqrt{3/8\pi}\,\sin(\vartheta)\left[ie^{i\varphi}\right] \;=\; -\hbar\,\sqrt{3/8\pi}\,\sin(\vartheta)\left[e^{i\varphi}\right] \;=\; \hbar\,Y_1^1(\vartheta,\varphi) \quad . \end{aligned}$$

We find that the eigenvalues of Y_1^1 equal $\hbar^2\ell(\ell+1)$ for $\ell = 1$ with respect to the differential operator L^2, and $m\hbar$ for $m = 1$ with respect to the differential operator L_z.

d: $\boldsymbol{Y_1^{-1}(\vartheta,\varphi) = \sqrt{3/8\pi}\ \sin(\vartheta)\, e^{-i\varphi}}$.

Using formulas (2) and (6), we obtain

$$\begin{aligned} L^2 Y_1^{-1}(\vartheta,\varphi) \ &= \ -\hbar^2 \left\{ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} \ + \ \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\} \sqrt{3/8\pi}\ \sin(\vartheta)\, e^{-i\varphi} \ = \\ &= \ -\hbar^2\sqrt{3/8\pi}\left\{ \left[\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\sin(\vartheta) \right] e^{-i\varphi} \ + \ \frac{1}{\sin^2(\vartheta)}\sin(\vartheta)\left[\frac{\partial^2}{\partial\varphi^2} e^{-i\varphi} \right] \right\} \\ &= \ -\hbar^2\sqrt{3/8\pi}\left\{ \left[\frac{1}{\sin^2(\vartheta)} - 2 \right] \sin(\vartheta)\, e^{-i\varphi} \ + \ \frac{1}{\sin^2(\vartheta)}\sin(\vartheta)\left[-e^{-i\varphi} \right] \right\} \\ &= \ 2\hbar^2\sqrt{3/8\pi}\ \sin(\vartheta)\, e^{-i\varphi} \ = \ 2\hbar^2 Y_1^{-1}(\vartheta,\varphi) \end{aligned}$$

and

$$\begin{aligned} L_z Y_1^{-1}(\vartheta,\varphi) \ &= \ -i\hbar\frac{\partial}{\partial\varphi}\sqrt{3/8\pi}\ \sin(\vartheta)\, e^{-i\varphi} \ = \ -i\hbar\sqrt{3/8\pi}\sin(\vartheta)\left[\frac{\partial}{\partial\varphi} e^{-i\varphi} \right] \\ &= \ -i\hbar\sqrt{3/8\pi}\sin(\vartheta)\left[-ie^{-i\varphi} \right] \ = \ -\hbar\sqrt{3/8\pi}\sin(\vartheta)\left[e^{-i\varphi} \right] \ = \ -\hbar\, Y_1^{-1}(\vartheta,\varphi) \quad . \end{aligned}$$

We find that the eigenvalues of Y_1^{-1} equal $\hbar^2\ell(\ell+1)$ for $\ell = 1$ with respect to the differential operator L^2, and $m\hbar$ for $m = -1$ with respect to the differential operator L_z.

e: $\boldsymbol{Y_2^0(\vartheta,\varphi) = \sqrt{5/16\pi}\ \{3\cos^2(\vartheta) - 1\}}$.

Using formulas (3) and (6), we obtain

$$\begin{aligned} L^2 Y_2^0(\vartheta,\varphi) \ &= \ -\hbar^2 \left\{ \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} \ + \ \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2} \right\} \sqrt{5/16\pi}\ \left\{ 3\cos^2(\vartheta) - 1 \right\} \ = \\ &= \ -\hbar^2\sqrt{5/16\pi}\left\{ \left[\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\left\{ 3\cos^2(\vartheta) - 1 \right\} \right] \ + \ 0 \right\} \\ &= \ -\hbar^2\sqrt{5/16\pi}\ \left\{ -18\cos^2(\vartheta) + 6 \right\} \ = \ 6\hbar^2\sqrt{5/16\pi}\ \left\{ 3\cos^2(\vartheta) - 1 \right\} \ = \ 6\hbar^2 Y_2^0(\vartheta,\varphi) \end{aligned}$$

Furthermore, since Y_2^0 does not depend on φ, we have

$$L_z Y_2^0(\vartheta,\varphi) \ = \ -i\hbar\frac{\partial}{\partial\varphi}\sqrt{5/16\pi}\ \left\{ 3\cos^2(\vartheta) - 1 \right\} \ = \ 0 \quad .$$

We find that the eigenvalues of Y_2^0 equal $\hbar^2\ell(\ell+1)$ for $\ell = 2$ with respect to the differential operator L^2, and $m\hbar$ for $m = 0$ with respect to the differential operator L_z.

f: $Y_2^1(\vartheta,\varphi) = -\sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{i\varphi}$.

Using formulas (4) and (6), we obtain

$$L^2 Y_2^1(\vartheta,\varphi) \;=\; -\hbar^2 \left\{ \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial\vartheta} \sin(\vartheta) \frac{\partial}{\partial\vartheta} \;+\; \frac{1}{\sin^2(\vartheta)} \frac{\partial^2}{\partial\varphi^2} \right\} \left[-\sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{i\varphi} \right] \;=$$

$$= \hbar^2 \sqrt{15/32\pi} \left\{ \left[\frac{1}{\sin(\vartheta)} \frac{\partial}{\partial\vartheta} \sin(\vartheta) \frac{\partial}{\partial\vartheta} \sin(2\vartheta) \right] e^{i\varphi} \;+\; \frac{1}{\sin^2(\vartheta)} \sin(2\vartheta) \left[\frac{\partial^2}{\partial\varphi^2}\, e^{i\varphi} \right] \right\}$$

$$= \hbar^2 \sqrt{15/32\pi} \left\{ \left[\frac{\sin(2\vartheta)}{\sin^2(\vartheta)} - 6\sin(2\vartheta) \right] e^{i\varphi} \;+\; \frac{1}{\sin^2(\vartheta)} \sin(2\vartheta) \left[-e^{i\varphi} \right] \right\}$$

$$= -6\hbar^2 \sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{i\varphi} \;=\; 6\hbar^2 Y_2^1(\vartheta,\varphi)$$

and

$$L_z Y_2^1(\vartheta,\varphi) \;=\; -i\,\hbar \frac{\partial}{\partial\varphi} \left[-\sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{i\varphi} \right] = \; i\,\hbar \sqrt{15/32\pi}\, \sin(2\vartheta) \left[\frac{\partial}{\partial\varphi} e^{i\varphi} \right] \;=$$

$$= \; i\,\hbar \sqrt{15/32\pi}\, \sin(2\vartheta) \left[i e^{i\varphi} \right] \;=\; -\hbar \sqrt{15/32\pi}\, \sin(2\vartheta) \left[e^{i\varphi} \right] \;=\; \hbar\, Y_2^1(\vartheta,\varphi) \quad .$$

We find that the eigenvalues of Y_2^1 equal $\hbar^2\ell(\ell+1)$ for $\ell = 2$ with respect to the differential operator L^2, and $m\hbar$ for $m = 1$ with respect to the differential operator L_z.

g: $Y_2^{-1}(\vartheta,\varphi) = \sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{-i\varphi}$.

Using formulas (4) and (6), we obtain

$$L^2 Y_2^{-1}(\vartheta,\varphi) \;=\; -\hbar^2 \left\{ \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial\vartheta} \sin(\vartheta) \frac{\partial}{\partial\vartheta} \;+\; \frac{1}{\sin^2(\vartheta)} \frac{\partial^2}{\partial\varphi^2} \right\} \left[\sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{-i\varphi} \right] \;=$$

$$= -\hbar^2 \sqrt{15/32\pi} \left\{ \left[\frac{1}{\sin(\vartheta)} \frac{\partial}{\partial\vartheta} \sin(\vartheta) \frac{\partial}{\partial\vartheta} \sin(2\vartheta) \right] e^{-i\varphi} \;+\; \frac{1}{\sin^2(\vartheta)} \sin(2\vartheta) \left[\frac{\partial^2}{\partial\varphi^2}\, e^{-i\varphi} \right] \right\}$$

$$= -\hbar^2 \sqrt{15/32\pi} \left\{ \left[\frac{\sin(2\vartheta)}{\sin^2(\vartheta)} - 6\sin(2\vartheta) \right] e^{-i\varphi} \;+\; \frac{1}{\sin^2(\vartheta)} \sin(2\vartheta) \left[-e^{-i\varphi} \right] \right\}$$

$$= 6\hbar^2 \sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{-i\varphi} \;=\; 6\hbar^2 Y_2^{-1}(\vartheta,\varphi)$$

and

$$L_z Y_2^{-1}(\vartheta,\varphi) \;=\; -i\,\hbar \frac{\partial}{\partial\varphi} \left[\sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{-i\varphi} \right] = \; -i\,\hbar \sqrt{15/32\pi}\, \sin(2\vartheta) \left[\frac{\partial}{\partial\varphi} e^{-i\varphi} \right] \;=$$

$$= \; -i\,\hbar \sqrt{15/32\pi}\, \sin(2\vartheta) \left[-i e^{-i\varphi} \right] \;=\; -\hbar \sqrt{15/32\pi}\, \sin(2\vartheta) \left[e^{-i\varphi} \right] \;=\; -\hbar\, Y_2^{-1}(\vartheta,\varphi) \quad .$$

We find that the eigenvalues of Y_2^{-1} equal $\hbar^2\ell(\ell+1)$ for $\ell = 2$ with respect to the differential operator L^2, and $m\hbar$ for $m = -1$ with respect to the differential operator L_z.

h: $\boldsymbol{Y_2^2(\vartheta,\varphi) = \sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{2i\varphi}}$.

Using formulas (5) and (6), we obtain

$$\begin{aligned}
L^2Y_2^2(\vartheta,\varphi) &= -\hbar^2 \left\{\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2}\right\}\left[\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{2i\varphi}\right] = \\
&= -\hbar^2\sqrt{15/32\pi}\left\{\left[\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\sin^2(\vartheta)\right]e^{2i\varphi} + \frac{1}{\sin^2(\vartheta)}\sin^2(\vartheta)\left[\frac{\partial^2}{\partial\varphi^2}e^{2i\varphi}\right]\right\} \\
&= -\hbar^2\sqrt{15/32\pi}\left\{\left[4-6\sin^2(\vartheta)\right]e^{2i\varphi} + \left[-4e^{2i\varphi}\right]\right\} \\
&= 6\hbar^2\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{2i\varphi} = 6\hbar^2 Y_2^2(\vartheta,\varphi)
\end{aligned}$$

and

$$\begin{aligned}
L_zY_2^2(\vartheta,\varphi) &= -i\hbar\frac{\partial}{\partial\varphi}\left[\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{2i\varphi}\right] = -i\hbar\sqrt{15/32\pi}\sin^2(\vartheta)\left[\frac{\partial}{\partial\varphi}e^{2i\varphi}\right] = \\
&= -i\hbar\sqrt{15/32\pi}\sin^2(\vartheta)\left[2ie^{2i\varphi}\right] = 2\hbar\sqrt{15/32\pi}\sin^2(\vartheta)\left[e^{2i\varphi}\right] = 2\hbar\, Y_2^2(\vartheta,\varphi) \quad .
\end{aligned}$$

We find that the eigenvalues of Y_2^2 equal $\hbar^2\ell(\ell+1)$ for $\ell = 2$ with respect to the differential operator L^2, and $m\hbar$ for $m = 2$ with respect to the differential operator L_z.

i: $\boldsymbol{Y_2^{-2}(\vartheta,\varphi) = \sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{-2i\varphi}}$.

Using formulas (5) and (6), we obtain

$$\begin{aligned}
L^2Y_2^2(\vartheta,\varphi) &= -\hbar^2 \left\{\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2}\right\}\left[\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{-2i\varphi}\right] = \\
&= -\hbar^2\sqrt{15/32\pi}\left\{\left[\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\sin^2(\vartheta)\right]e^{-2i\varphi} + \frac{1}{\sin^2(\vartheta)}\sin^2(\vartheta)\left[\frac{\partial^2}{\partial\varphi^2}e^{-2i\varphi}\right]\right\} \\
&= -\hbar^2\sqrt{15/32\pi}\left\{\left[4-6\sin^2(\vartheta)\right]e^{-2i\varphi} + \left[-4e^{-2i\varphi}\right]\right\} \\
&= 6\hbar^2\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{-2i\varphi} = 6\hbar^2 Y_2^2(\vartheta,\varphi)
\end{aligned}$$

and

$$\begin{aligned}
L_zY_2^{-2}(\vartheta,\varphi) &= -i\hbar\frac{\partial}{\partial\varphi}\left[\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{-2i\varphi}\right] = -i\hbar\sqrt{15/32\pi}\sin^2(\vartheta)\left[\frac{\partial}{\partial\varphi}e^{-2i\varphi}\right] = \\
&= -i\hbar\sqrt{15/32\pi}\sin^2(\vartheta)\left[-2ie^{-2i\varphi}\right] = -2\hbar\sqrt{15/32\pi}\sin^2(\vartheta)\left[e^{-2i\varphi}\right] = \hbar\, Y_2^{-2}(\vartheta,\varphi) \quad .
\end{aligned}$$

We find that the eigenvalues of Y_2^{-2} equal $\hbar^2\ell(\ell+1)$ for $\ell = 2$ with respect to the differential operator L^2, and $m\hbar$ for $m = -2$ with respect to the differential operator L_z.

In the following table we collect the results of the preceding calculus.

	angular momentum eigenvalue L^2		magnetic quantum number eigenvalue L_z	
Y_ℓ^m	ℓ	$\ell(\ell+1)\hbar^2$	m	$m\hbar$
Y_0^0	0	0	0	0
Y_1^1	1	$2\hbar^2$	1	$\hbar$
Y_1^0	1	$2\hbar^2$	0	0
Y_1^{-1}	1	$2\hbar^2$	-1	$-\hbar$
Y_2^2	2	$6\hbar^2$	2	$2\hbar$
Y_2^1	2	$6\hbar^2$	1	$\hbar$
Y_2^0	2	$6\hbar^2$	0	0
Y_2^{-1}	2	$6\hbar^2$	-1	$-\hbar$
Y_2^{-2}	2	$6\hbar^2$	-2	$-2\hbar$

7. Os operadores de subida L_+ e de descida L_- são dados por

$$L_+ \ = \ \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \frac{\partial}{\partial\varphi} \right\} \quad \text{e} \quad L_- \ = \ \hbar\, e^{-i\varphi} \left\{ -\frac{\partial}{\partial\vartheta} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \frac{\partial}{\partial\varphi} \right\} \quad .$$

Determine $L_+ Y_\ell^m$ e $L_- Y_\ell^m$ para cada uma das funções dadas no exercício (6).

Solution:

a: $\boldsymbol{Y_0^0(\vartheta,\varphi) = \sqrt{1/4\pi}}$.

Since the derivatives of a constant function are all equal to zero, we find here

$$L_+ Y_0^0(\vartheta,\varphi) \ = \ \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \frac{\partial}{\partial\varphi} \right\} \sqrt{1/4\pi} \ = \ 0$$

and

$$L_- Y_0^0(\vartheta,\varphi) \ = \ \hbar\, e^{-i\varphi} \left\{ -\ \frac{\partial}{\partial\vartheta} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \frac{\partial}{\partial\varphi} \right\} \sqrt{1/4\pi} \ = \ 0$$

We find $L_+ Y_0^0 = 0$ and $L_- Y_0^0 = 0$

b: $\boldsymbol{Y_1^0(\vartheta,\varphi) = \sqrt{3/4\pi}\ \cos(\vartheta)}$.

$$L_+ Y_1^0(\vartheta,\varphi) \ = \ \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \frac{\partial}{\partial\varphi} \right\} \sqrt{3/4\pi}\ \cos(\vartheta) \ =$$

$$= \ \hbar \sqrt{3/4\pi}\, e^{i\varphi} \ \{ -\sin(\vartheta) \ + \ 0 \} \ = \ -\hbar \sqrt{2} \sqrt{3/8\pi}\ \sin(\vartheta)\, e^{i\varphi} \ = \ \hbar \sqrt{2}\, Y_1^1(\vartheta,\varphi)$$

and

$$L_- Y_1^0(\vartheta,\varphi) \ = \ \hbar\, e^{-i\varphi} \left\{ -\frac{\partial}{\partial\vartheta} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \frac{\partial}{\partial\varphi} \right\} \sqrt{3/4\pi}\ \cos(\vartheta) \ =$$

$$= \ \hbar \sqrt{3/4\pi}\, e^{-i\varphi} \ \{ \sin(\vartheta) \ + \ 0 \} \ = \ \hbar \sqrt{2} \sqrt{3/8\pi}\ \sin(\vartheta)\, e^{-i\varphi} \ = \ \hbar \sqrt{2}\, Y_1^{-1}(\vartheta,\varphi).$$

We find $L_+ Y_1^0 = \hbar \sqrt{2}\, Y_1^1$ and $L_- Y_1^0 = \hbar \sqrt{2}\, Y_1^{-1}$.

c: $\boldsymbol{Y_1^1(\vartheta,\varphi) = -\sqrt{3/8\pi}\ \sin(\vartheta)\, e^{i\varphi}}$.

$$L_+ Y_1^1(\vartheta,\varphi) \ = \ \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \frac{\partial}{\partial\varphi} \right\} \left[-\sqrt{3/8\pi}\ \sin(\vartheta)\, e^{i\varphi} \right] \ =$$

$$= \ -\hbar \sqrt{3/8\pi}\, e^{i\varphi} \left\{ \cos(\vartheta)\, e^{i\varphi} \ + \ i\, \frac{\cos(\vartheta)}{\sin(\vartheta)} \ \left[\sin(\vartheta)\, i\, e^{i\varphi} \right] \right\} \ = \ 0$$

and

$$L_- Y_1^1(\vartheta,\varphi) \;=\; \hbar\, e^{-i\varphi} \left\{ -\frac{\partial}{\partial\vartheta} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\, \frac{\partial}{\partial\varphi} \right\} \left[-\sqrt{3/8\pi}\, \sin(\vartheta)\, e^{i\varphi} \right] \;=$$

$$=\; -\hbar\, \sqrt{3/8\pi}\, e^{-i\varphi} \left\{ -\; \cos(\vartheta)\, e^{i\varphi} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\; \left[\sin(\vartheta)\, i\, e^{i\varphi} \right] \right\}$$

$$=\; \hbar\, 2\sqrt{3/8\pi}\, \cos(\vartheta) =\; \hbar\, \sqrt{2}\, Y_1^0(\vartheta,\varphi).$$

We find $L_+ Y_1^1 = 0$ and $L_- Y_1^1 = \hbar\, \sqrt{2}\, Y_1^0$.

d: $\boldsymbol{Y_1^{-1}(\vartheta,\varphi) = \sqrt{3/8\pi}\ \sin(\vartheta)\, e^{-i\varphi}}$.

$$L_+ Y_1^{-1}(\vartheta,\varphi) \;=\; \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\, \frac{\partial}{\partial\varphi} \right\} \left[\sqrt{3/8\pi}\, \sin(\vartheta)\, e^{-i\varphi} \right] \;=$$

$$=\; \hbar\, \sqrt{3/8\pi}\, e^{i\varphi} \left\{ \cos(\vartheta)\, e^{-i\varphi} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\; \left[\sin(\vartheta)\, (-i)\, e^{-i\varphi} \right] \right\}$$

$$=\; \hbar\, 2\sqrt{3/8\pi}\, \cos(\vartheta) =\; \hbar\, \sqrt{2}\, Y_1^0(\vartheta,\varphi)$$

and

$$L_- Y_1^{-1}(\vartheta,\varphi) \;=\; \hbar\, e^{-i\varphi} \left\{ -\frac{\partial}{\partial\vartheta} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\, \frac{\partial}{\partial\varphi} \right\} \left[\sqrt{3/8\pi}\, \sin(\vartheta)\, e^{-i\varphi} \right] \;=$$

$$=\; \hbar\, \sqrt{3/8\pi}\, e^{-i\varphi} \left\{ -\; \cos(\vartheta)\, e^{-i\varphi} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\; \left[\sin(\vartheta)\, (-i)\, e^{-i\varphi} \right] \right\} \;=\; 0.$$

We find $L_+ Y_1^{-1} = \hbar\, \sqrt{2}\, Y_1^0$ and $L_- Y_1^{-1} = 0$.

e: $\boldsymbol{Y_2^0(\vartheta,\varphi) = \sqrt{5/16\pi}\ \{3\cos^2(\vartheta) - 1\}}$.

$$L_+ Y_1^0(\vartheta,\varphi) \;=\; \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\, \frac{\partial}{\partial\varphi} \right\} \left[\sqrt{5/16\pi}\, \left\{ 3\cos^2(\vartheta) - 1 \right\} \right] \;=$$

$$=\; \hbar\, \sqrt{5/16\pi}\, e^{i\varphi}\, \{-6\sin(\vartheta)\cos(\vartheta) \;+\; 0\} \;=\; -\hbar\, \sqrt{6}\, \sqrt{15/32\pi}\, \sin(2\vartheta)\, e^{i\varphi} \;=\; \hbar\, \sqrt{6}\, Y_2^1(\vartheta,\varphi)$$

and

$$L_- Y_1^0(\vartheta,\varphi) \;=\; \hbar\, e^{-i\varphi} \left\{ -\frac{\partial}{\partial\vartheta} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\, \frac{\partial}{\partial\varphi} \right\} \left[\sqrt{5/16\pi}\, \left\{ 3\cos^2(\vartheta) - 1 \right\} \right] \;=$$

$$=\; \hbar\, \sqrt{5/16\pi}\, e^{-i\varphi}\, \{6\sin(\vartheta)\cos(\vartheta) \;+\; 0\} \;=\; \hbar\, \sqrt{6}\, \sqrt{15/32\pi}\, \sin(2\vartheta)\, e^{-i\varphi} \;=\; \hbar\, \sqrt{6}\, Y_2^{-1}(\vartheta,\varphi)$$

We find $L_+Y_2^0 = \hbar\sqrt{6}\,Y_2^1$ and $L_-Y_2^0 = \hbar\sqrt{6}\,Y_2^{-1}$.

f: $\boldsymbol{Y_2^1(\vartheta,\varphi) = -\sqrt{15/32\pi}\,\sin(2\vartheta)\,e^{i\varphi}}$.

Let us first determine the derivative of $\sin(2\vartheta)$.

$$\frac{\partial}{\partial\vartheta}\sin(2\vartheta) \;=\; 2\cos(2\vartheta) \;=$$

$$=\; 2\left[\cos^2(\vartheta) - \sin^2(\vartheta)\right] \;=\; 2\left[2\cos^2(\vartheta) - 1\right] \;=\; 2\left[1 - 2\sin^2(\vartheta)\right] \quad .$$

Using either of the final results, we obtain

$$L_+Y_2^1(\vartheta,\varphi) \;=\; \hbar\,e^{i\varphi}\left\{\frac{\partial}{\partial\vartheta} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{\partial}{\partial\varphi}\right\}\left[-\sqrt{15/32\pi}\,\sin(2\vartheta)\,e^{i\varphi}\right] \;=$$

$$=\; -\hbar\sqrt{15/32\pi}\,e^{i\varphi}\left\{2\left[\cos^2(\vartheta) - \sin^2(\vartheta)\right]\,e^{i\varphi} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\sin(2\vartheta)\,\left[i\,e^{i\varphi}\right]\right\}$$

$$=\; -\hbar\sqrt{15/32\pi}\,e^{i\varphi}\left\{2\left[\cos^2(\vartheta) - \sin^2(\vartheta)\right]\,e^{i\varphi} \;-\; \frac{\cos(\vartheta)}{\sin(\vartheta)}\,2\sin(\vartheta)\cos(\vartheta)\,e^{i\varphi}\right\}$$

$$=\; 2\hbar\sqrt{15/32\pi}\,\sin^2(\vartheta)\,e^{2i\varphi} \;=\; 2\hbar\,Y_2^2(\vartheta,\varphi)$$

and

$$L_-Y_2^1(\vartheta,\varphi) \;=\; \hbar\,e^{-i\varphi}\left\{-\frac{\partial}{\partial\vartheta} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{\partial}{\partial\varphi}\right\}\left[-\sqrt{15/32\pi}\,\sin(2\vartheta)\,e^{i\varphi}\right] \;=$$

$$=\; -\hbar\sqrt{15/32\pi}\,e^{-i\varphi}\left\{-2\left[2\cos^2(\vartheta) - 1\right]\,e^{i\varphi} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\sin(2\vartheta)\,\left[i\,e^{i\varphi}\right]\right\}$$

$$=\; -\hbar\sqrt{15/32\pi}\,e^{-i\varphi}\left\{-2\left[2\cos^2(\vartheta) - 1\right]\,e^{i\varphi} \;-\; \frac{\cos(\vartheta)}{\sin(\vartheta)}\,2\sin(\vartheta)\cos(\vartheta)\,e^{i\varphi}\right\}$$

$$=\; 2\hbar\sqrt{15/32\pi}\,\left\{3\cos^2(\vartheta) - 1\right\} \;=\; \hbar\sqrt{6}\,Y_2^0(\vartheta,\varphi)$$

We find $L_+Y_2^1 = \hbar\sqrt{4}\,Y_2^2$ and $L_-Y_2^1 = \hbar\sqrt{6}\,Y_2^0$.

g: $\boldsymbol{Y_2^{-1}(\vartheta,\varphi) = \sqrt{15/32\pi}\,\sin(2\vartheta)\,e^{-i\varphi}}$.

$$L_+Y_2^{-1}(\vartheta,\varphi) \;=\; \hbar\,e^{i\varphi}\left\{\frac{\partial}{\partial\vartheta} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{\partial}{\partial\varphi}\right\}\left[\sqrt{15/32\pi}\,\sin(2\vartheta)\,e^{-i\varphi}\right] \;=$$

$$=\; \hbar\sqrt{15/32\pi}\,e^{i\varphi}\left\{2\left[2\cos^2(\vartheta) - 1\right]\,e^{-i\varphi} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\sin(2\vartheta)\,\left[-i\,e^{-i\varphi}\right]\right\}$$

$$= \hbar\sqrt{15/32\pi}\, e^{i\varphi} \left\{ 2\left[2\cos^2(\vartheta) - 1\right] e^{-i\varphi} \;+\; \frac{\cos(\vartheta)}{\sin(\vartheta)}\, 2\sin(\vartheta)\cos(\vartheta)\, e^{-i\varphi} \right\}$$

$$= 2\hbar\sqrt{15/32\pi}\, \left\{3\cos^2(\vartheta) - 1\right\} \;=\; \hbar\sqrt{6}\, Y_2^0(\vartheta,\varphi)$$

and

$$L_- Y_2^{-1}(\vartheta,\varphi) \;=\; \hbar\, e^{-i\varphi} \left\{ -\frac{\partial}{\partial\vartheta} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{\partial}{\partial\varphi} \right\} \left[\sqrt{15/32\pi}\,\sin(2\vartheta)\, e^{-i\varphi}\right] \;=$$

$$= \hbar\sqrt{15/32\pi}\, e^{-i\varphi} \left\{ -2\left[\cos^2(\vartheta) - \sin^2(\vartheta)\right] e^{-i\varphi} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\sin(2\vartheta)\, \left[-i\, e^{-i\varphi}\right] \right\}$$

$$= \hbar\sqrt{15/32\pi}\, e^{-i\varphi} \left\{ -2\left[\cos^2(\vartheta) - \sin^2(\vartheta)\right] e^{-i\varphi} \;+\; \frac{\cos(\vartheta)}{\sin(\vartheta)}\, 2\sin(\vartheta)\cos(\vartheta)\, e^{-i\varphi} \right\}$$

$$= 2\hbar\sqrt{15/32\pi}\,\sin^2(\vartheta)\, e^{-2i\varphi} \;=\; 2\hbar\, Y_2^{-2}(\vartheta,\varphi)$$

We find $L_+ Y_2^{-1} = \hbar\sqrt{6}\, Y_2^0$ and $L_- Y_2^{-1} = \hbar\sqrt{4}\, Y_2^{-2}$.

h: $\boldsymbol{Y_2^2(\vartheta,\varphi) = \sqrt{15/32\pi}\,\sin^2(\vartheta)\, e^{2i\varphi}}$.

Let us first determine the derivative of $\sin^2(\vartheta)$.

$$\frac{\partial}{\partial\vartheta}\,\sin^2(\vartheta) \;=\; 2\sin(\vartheta)\cos(\vartheta) \;=\; \sin(2\vartheta) \quad .$$

Using either of the final results, we obtain

$$L_+ Y_2^2(\vartheta,\varphi) \;=\; \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{\partial}{\partial\varphi} \right\} \left[\sqrt{15/32\pi}\,\sin^2(\vartheta)\, e^{2i\varphi}\right] \;=$$

$$= \hbar\sqrt{15/32\pi}\, e^{i\varphi} \left\{ 2\sin(\vartheta)\cos(\vartheta)\, e^{2i\varphi} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\sin^2(\vartheta)\, \left[2i\, e^{2i\varphi}\right] \right\} \;=\; 0$$

and

$$L_- Y_2^2(\vartheta,\varphi) \;=\; \hbar\, e^{-i\varphi} \left\{ -\frac{\partial}{\partial\vartheta} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{\partial}{\partial\varphi} \right\} \left[\sqrt{15/32\pi}\,\sin^2(\vartheta)\, e^{2i\varphi}\right] \;=$$

$$= \hbar\sqrt{15/32\pi}\, e^{-i\varphi} \left\{ -\sin(2\vartheta)\, e^{2i\varphi} \;+\; i\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\sin^2(\vartheta)\, \left[2i\, e^{2i\varphi}\right] \right\}$$

$$= \hbar\sqrt{15/32\pi}\, e^{-i\varphi} \left\{ -\sin(2\vartheta)\, e^{2i\varphi} \;-\; \sin(2\vartheta)\, e^{2i\varphi} \right\}$$

$$= -2\hbar\sqrt{15/32\pi}\,\sin(2\vartheta)\, e^{i\varphi} \;=\; 2\hbar\, Y_2^1(\vartheta,\varphi).$$

We find $L_+ Y_2^2 = 0$ and $L_- Y_2^2 = \hbar\sqrt{4}\, Y_2^1$.

i: $\boldsymbol{Y_2^{-2}(\vartheta,\varphi) = \sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{-2i\varphi}}$.

$$L_+ Y_2^{-2}(\vartheta,\varphi) \;=\; \hbar\, e^{i\varphi} \left\{ \frac{\partial}{\partial\vartheta} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\, \frac{\partial}{\partial\varphi} \right\} \left[\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{-2i\varphi} \right] \;=$$

$$=\; \hbar\, \sqrt{15/32\pi}\, e^{i\varphi} \left\{ \sin(2\vartheta)\, e^{-2i\varphi} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\ \sin^2(\vartheta)\ \left[-2i\, e^{-2i\varphi} \right] \right\}$$

$$=\; \hbar\, \sqrt{15/32\pi}\, e^{i\varphi} \left\{ \sin(2\vartheta)\, e^{-2i\varphi} \;+\; \sin(2\vartheta)\, e^{-2i\varphi} \right\}$$

$$=\; 2\hbar\, \sqrt{15/32\pi}\ \sin(2\vartheta)\, e^{-i\varphi} \;=\; 2\hbar\, Y_2^{-1}(\vartheta,\varphi)$$

and

$$L_- Y_2^{-2}(\vartheta,\varphi) \;=\; \hbar\, e^{-i\varphi} \left\{ -\, \frac{\partial}{\partial\vartheta} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\, \frac{\partial}{\partial\varphi} \right\} \left[\sqrt{15/32\pi}\ \sin^2(\vartheta)\, e^{-2i\varphi} \right] \;=$$

$$=\; \hbar\, \sqrt{15/32\pi}\, e^{-i\varphi} \left\{ -2\sin(\vartheta)\cos(\vartheta)\, e^{-2i\varphi} \;+\; i\, \frac{\cos(\vartheta)}{\sin(\vartheta)}\ \sin^2(\vartheta)\ \left[-2i\, e^{-2i\varphi} \right] \right\} \;=\; 0 \quad .$$

We find $L_+ Y_2^{-2} = \hbar\, \sqrt{4}\, Y_2^{-1}$ and $L_- Y_2^{-2} = 0$.

In the following table we collect the results of the preceding calculus.

			raising	lowering
Y_ℓ^m	ℓ	m	L_+	L_-
Y_0^0	0	0	0	0
Y_1^1	1	1	0	$\hbar\sqrt{2}\,Y_1^0$
Y_1^0	1	0	$\hbar\sqrt{2}\,Y_1^1$	$\hbar\sqrt{2}\,Y_1^{-1}$
Y_1^{-1}	1	-1	$\hbar\sqrt{2}\,Y_1^0$	0
Y_2^2	2	2	0	$\hbar\sqrt{4}\,Y_2^1$
Y_2^1	2	1	$\hbar\sqrt{4}\,Y_2^2$	$\hbar\sqrt{6}\,Y_2^0$
Y_2^0	2	0	$\hbar\sqrt{6}\,Y_2^1$	$\hbar\sqrt{6}\,Y_2^{-1}$
Y_2^{-1}	2	-1	$\hbar\sqrt{6}\,Y_2^0$	$\hbar\sqrt{4}\,Y_2^{-2}$
Y_2^{-2}	2	-2	$\hbar\sqrt{4}\,Y_2^{-1}$	0

In general one has

$$L_+\,Y_\ell^m \;=\; \sqrt{\ell(\ell+1)-m(m+1)}\;Y_\ell^{m+1}$$

and

$$L_-\,Y_\ell^m \;=\; \sqrt{\ell(\ell+1)-m(m-1)}\;Y_\ell^{m-1} \quad .$$

The signs + or −, or in general the phases, of the spherical harmonics are chosen according to the Condon-Shortley phase convention.

One may use the raising, L_+, and lowering, L_-, operators to construct all spherical harmonics, once one finds the solutions Y_ℓ^0, which are independent of φ, of the equation

$$\begin{aligned}\hbar^2\,\ell(\ell+1)\,Y_\ell^0(\vartheta) \;&=\; L^2\,Y_\ell^0(\vartheta) \;= \\ &=\; -\hbar^2\left\{\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} \;+\; \frac{1}{\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2}\right\}Y_\ell^0(\vartheta) \\ &=\; -\hbar^2\,\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta}\,Y_\ell^0(\vartheta) \quad .\end{aligned}$$

The latter equation, which is called the Legendre differential equation (1785), is not very difficult to be solved (see e.g. http://mathworld.wolfram.com/LegendreDifferentialEquation.html or http:-//en.wikipedia.org/wiki/Legendre_polynomials).