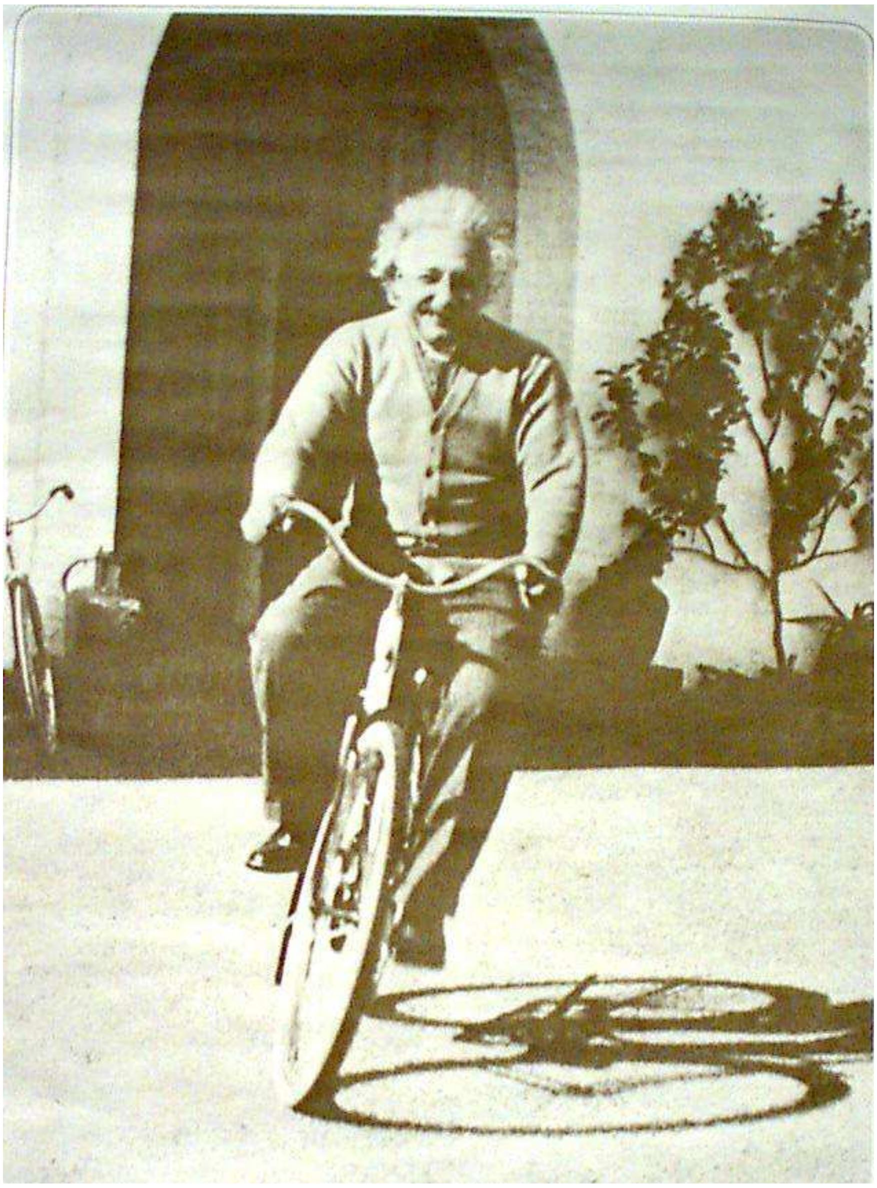

Some topics in Geometry and Gravitation

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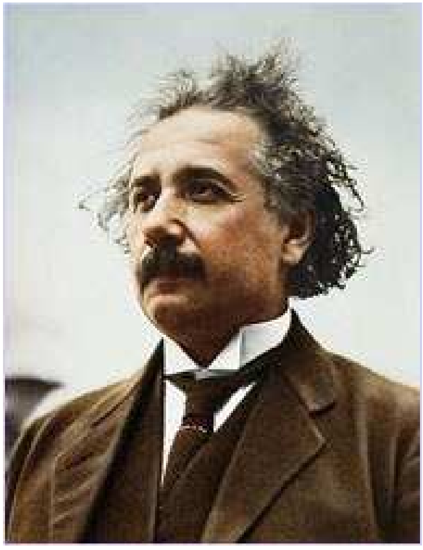


These notes are born as a result of a series of discussions with my colleague Humberto Pascoal from the Centre of Theoretical Physics at the University of Coimbra, Portugal, on the principles of the description of a curved two-dimensional surface embedded in three dimensions. The ideas contained in these notes are not new. On the contrary, the simple and elegant methods which lead Gauss to his Egregium Theorem, lead us to study in more detail a subject which almost two centuries ago has been studied by Carl Friedrich Gauss (1777-1855), Janos Bólyai (1802-1860) and Nikolay Ivanovich Lobachevski (1793-1856).

The notation which I have used in this notes, is the one which is most popular amongst physicists, in order not to discourage first and second year physics students.

Moreover, I do not intend to be rigorous, neither complete.

Below you find the literature which I consulted.

December, 1988.

Eef van Beveren

References

[1] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Annalen der Physik, (1905).

[2] A. Duschek, *Differentialgeometrie*, Handbuch der Physik, Band III, edited by H. Thirring, Verlag von Julius Springer, Berlin, 1928, page 153.

[3] Horst Tietz, *Elementare Differentialgeometrie*, Handbuch der Physik, Band II, edited by S. Flügge, Springer-Verlag, Berlin, 1955, page 146.

[4] H.S.M. Coxeter, F.R.S., *Introduction to Geometry*, John Wiley and Sons, Inc, New York, 1961.

[5] Steven Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, Inc, New York, 1972.

Part I
Introduction

Einstein's discovery that the Lorentz transformations follow from a simple principle, namely the constancy of the light velocity in all inertial reference frames, was an important breakthrough for the development of a full theory of Gravity. But, many *old* concepts had to be replaced as well. In this notes we will pay some attention to important contributions from differential geometry, in particular, to the notion of a freely moving particle in a *curved* manifold.

First we pass through the definition of coordinates and the description of moving objects by a coordinate system. Then we pay some attention to Galilei, Lorentz and Poincaré transformations. In part II we come to introduce the main dish.

1 The aftermath of a physics exam

Praia Perpêtua has a very long and perfectly straight road for bicycles alongside the beach. On a beautiful sunny day just a few weeks before the beginning of the beach season, a young man, named Alex, is seated at the veranda of his beach apartment just next to the road. From there he watches the movements on the beach and enjoys the never ending sound of the waves breaking into foam at the sand of the beach. It is a lovely day, but only a very few people are walking in the sand or taking a bath in the ocean, probably since it is only nine o'clock in the morning. He observes that this morning the ice-cream vendor installed his car next to the road at only 43 meters to his right. At his left at 39 meters distance there is a small restaurant where you can eat fresh fish every day of the year.

Since Alex considers himself the center of the Universe, wherever he is seated, he always represents the origin of his coordinate system. Moreover, since he is nevertheless a reasonably well educated person, he measures distances in meters. As far as the bike road is concerned we will consider only one dimension and indicate the coordinate system of Alex by $x^{(A)}$. For the ice-cream car we find then

$$x^{(A)}_{\text{ice-cream car}} = +43 \quad , \tag{1}$$

whereas, for the fish restaurant we obtain

$$x^{(A)}_{\text{fish restaurant}} = -39 \quad . \tag{2}$$

The plus and minus signs have nothing to do with a classification of Alex towards eating ice cream or fish, he actually loves both ice cream and fresh fish, but indicate that we have chosen the positive sense of the coordinates in the direction of the ice-cream car. The opposite direction is then automatically in the negative sense. On purpose, we have not indicated units in formulas (1) and (2), since we have stated the unit system of Alex before.

A young couple on a bicycle passes by. They are Bruno and Clara who both greet Alex. Bruno does the pedaling, whereas seated on the back the bike, his arms around the waist of Clara who is seated on the saddle and does the steering. They are heading towards the ice-cream car. However, just some meters after passing Alex, Bruno stops the bike and sits with Clara in the sand next to the road, Clara at Bruno's lap. They are talking about the physics exam of yesterday and, in particular, discussing a problem about relative motion, while, in the mean time, hugging and kissing each other. After some half an hour they decide to buy an ice cream, but not without measuring the distance from where they are seated till the car. Bruno measures distances in large steps, which he knows are just equal to one meter. So, he sets out towards the car and arrives after 26 steps. There, he patiently waits for Clara.

Clara measures the world, which consists out of everything which is close to her, in the well-shaped palms of her slim hands. Consequently, she prefers to measure distances with the size of her hands. Bruno enjoys observing her, busy on her knees, slowly approaching him, by putting one hand next to the other in the sand while the sun is shining on her back. Threehundred twenty five she counts when she ends at Bruno's feet. Well done, Bruno agrees, I know that the palms of those tender hands of yours measure precisely eight centimeters. He gives her a big kiss and orders their ice creams. One girlish, with strawberries, for his Clara, and one more serious, with chocolate, for him. They walk back to their bicycle while eating their ice creams.

In the coordinate system of Bruno, which we indicate by $x^{(B)}$, we find for the coordinates of the ice-cream car and the restaurant respectively

$$x^{(B)}_{\text{ice-cream car}} = +26 \quad , \tag{3}$$

and

$$x^{(B)}_{\text{fish restaurant}} = -56 \quad . \tag{4}$$

Furthermore, in the coordinate system of Clara, which we indicate by $x^{(C)}$, we find for the coordinates of the same respectively

$$x^{(C)}_{\text{ice-cream car}} = +325 \quad , \tag{5}$$

and

$$x^{(C)}_{\text{fish restaurant}} = -700 \quad . \tag{6}$$

Notice that the origins of the coordinate systems of Bruno and Clara are at the position where they stopped their bicycle.

While still walking, Bruno and Clara see far behind the restaurant how blond beauty Diane with her elegant legs is pedaling her bicycle towards them. Clara is not very fond of meetings with Bruno's former girlfriend. In particular not, when she is driving her bicycle dressed in a scandalous mini bikini. But then, Bruno puts his arm around Clara's shoulders and gives a long kiss with a little bit of chocolate ice cream in her curly dark hairs full of sand. Both start calculating how much time will elapse before Diane will pass by the place where they were seated.

Bruno starts his stopwatch when Diane is passing by the fishermen monument at 56 meters beyond the fish restaurant. He finds that it takes her exactly 20 seconds from there to the restaurant. So, he concludes that it will take yet another 20 seconds before she will reach them.

Enough time to sit down with Clara, facing the ocean, their backs turned towards the road. Clara counts time with the beats of her heart which, actually, is beating as bit faster than normal right now. Measuring exactly the same distance as Bruno had selected, she comes at 28 beats in total. They sit down in the sand, enjoying their ice creams and pretending not to notice who is passing by on her bike.

In the mean time, Alex also had come aware of who was approaching him at her bicycle. Ever since Diane had broken up with Bruno, he assumed that his chances with her were written in the stars. Therefor, he also measured her speed at his stopwatch to come to the same conclusion as Bruno, and prepared himself for a good impression on his, as yet secret, love. In his coordinate system he found that her position coordinate at time $t^{(A)}$ is given by[1]

$$x_{\text{Diane}}^{(A)}\left(t^{(A)}\right) \;=\; -\,95 \,+\, 2.8 \times t^{(A)} \quad . \tag{7}$$

But, although his formula correctly corresponds to Diane's motion, unfortunately for Alex, Diane hardly responded to his "how are you Diane" when she passed by.

Disappointed, he is still watching her, when he assists how Clara cannot resist to give Diane a provocative look over Bruno's shoulder. It is responded by a slow and sensual hello of Diane. On hearing her attractive voice, Bruno turns his head, utters a soundless hello and, like hypnotised, keeps watching the most pleasant scene of the sun-tanned body of the beautyful girl, pedaling on her bicycle, her blond hair floating in the air like the tail of a comet when it passes close to the sun. However, Clara, disturbed by Bruno's confusion, reacts by putting what is left of her ice cream on his forehead. That gesture rapidly brings him back to reality. While joking about her attack of jealousy, he holds Clara firmly in his arms and kisses her delicately. Both start laughing and soon the incident seems to belong to the past.

While refering to the motion of her bicycle, Bruno affirms that Diane's motion is well described by the following formula[2] in his coordinate system.

$$x_{\text{Diane}}^{(B)}\left(t^{(B)}\right) \;=\; -\,112 \,+\, 2.8 \times t^{(B)} \quad . \tag{8}$$

Clara, still not completely secure of her spell of charm on Bruno, is not exactly eager to share her result on this particular event with Bruno. But, on the other hand, she also does not want him to notice her insecurity. So, she tells Bruno that she found[3]

$$x_{\text{Diane}}^{(C)}\left(t^{(C)}\right) \;=\; -\,1400 \,+\, 25 \times t^{(C)} \quad . \tag{9}$$

Bruno, still absent minded, says that he is not completely sure whether this agrees with his result (8). It is to be admired that Clara manages not to respond that he apparently is not sure about several issues, but, instead, proposes to visit Alex. They decide to go to Alex and check with his findings.

Alex, glad to have some company to distract him from his disappointment, happily shares the findings on the motion of his secret love, with Clara and Bruno. Soon the three start talking about Diane, each with different feelings with regard to the fascinating girl. Bruno with some sadness about the way he lost her, Alex regretting his recent failure to attract her attention and Clara with some latent boiling fury.

[1] 95 is in meters; 2.8 in meters per second (corresponding to about 10 km/h); $t^{(A)}$ in seconds.

[2] 112 is in meters; 2.8 and $t^{(B)}$ as for Alex

[3] 1400 is in widths of Clara's hands; 25 in widths of Clara's hands per heart beat (corresponding to about 10 km/h); $t^{(C)}$ in heart beats.

Based in equations (1) to (6), they had come to the conclusion

$$x^{(B)} = x^{(A)} - 17 \quad , \quad x^{(C)} = 12.5 \times x^{(B)} \quad \text{and} \quad x^{(C)} = 12.5 \times x^{(A)} - 212.5 \quad . \qquad (10)$$

Notice that all relations are linear in the coordinates, whereby the first just stems from the *translation* of the origin of Bruno and Clara with respect to the origin of Alex, the second from the different units of Bruno and Clara, whereas the third relation comes from a combination of the difference in units and the translation of origins.

With respect to instants of time they found

$$t^{(B)} = t^{(A)} \quad , \quad t^{(C)} = 1.4 \times t^{(A)} \quad \text{and} \quad t^{(C)} = 1.4 \times t^{(B)} \quad . \qquad (11)$$

Clara's heart beat was a bit faster than one per second. Hence, her unit of time is a bit smaller than the unit of time for Bruno and Alex. The latter unit is the second.

2 Moving reference system

Alex, Clara and Bruno are still talking about Diane, when they suddenly observe her returning from her bicycle trip. But, from the language of her well-shaped body, Alex deduces that she is not as happy as when she passed by his veranda earlier. He decides to call her.

This time he is succesful, Diane stops her bicycle and accepts his invitation to join them. They greet each other with hugs and kisses. But just after this warm reception ceremony, Clara suddenly remembers that she and Bruno had something urgent to be resolved. Bruno still has the nerve to ask what it was again that is now suddenly so urgent. But, fortunately, from the expression at Clara's face he understands that it is better not to insist on having his ignorance clarified. Alex is delighted with Clara's attitude and impatiently awaits the departure of the couple. As by a miracle he has his blond beauty Diane seated with him at his sofa on his veranda. Much more than he had bargained for earlier this morning.

Without paying much attention to it, Alex had noticed that Diane was not driving her own bicycle when she passed by his veranda for the first time. Actually, it is her father's bicycle she is using. The reason for that, as we will see, is a subject which she by no means is prepared to confess to Alex.

Diane lives in a small house near the fishermen monument. In the morning, when Diane awakes, she usually opens the window of her bedroom completely and stays there for a while in her nightdress, leaning on her elbows, watching the sea, the beach, the seagulls and the people passing by at the road. This morning, when she saw Bruno and Clara passing by, Diane started to slip into her dreamworld, thinking on how much she had enjoyed such trips with Bruno on his bicycle. In particular, when he, like today with Clara, acted as the motor of the bicycle, whereas she, with his arms around her, could just go to any direction which came to her mind. Very often she had steered both of them to a quiet sunny valley in the dunes, where they had once discovered a perfect site for lovers, hidden by pleasant vegetation of red and white flourishing oleanders and yellow broom with their intense perfume. She started feeling like returning to her bed. But, then she saw a glimpse of Eric who comes regularly to the beach for a weekend or for a few days, and who lives in an apartment just two blocks down the road.

Eric, who is probably somewhere in his early twenties, has a strong muscular body and a pleasant face with blond curly hair. He uses to practice jogging when he is at the beach. When Diana saw him moving towards the fishermen monument, she calculated from his speed that if she could manage, within five minutes, to be on her bicycle in front of the fishermen monument, she could just catch up with him at the place where the bike road turns away from the beach, towards the inland, and where he probably would rest a little before returning to his apartment. The distance to that place is 2.1 kilometers measured from the fishermen monument.

Diane already started imagining Eric on the back of her bike with his muscular arms around her body. But, then she remembered that her bike, unfortunately, cannot carry persons in the back. Hence, she decided to take her father's bike instead.

When she started out at the fishermen monument, she knew from Eric's speed that he was some 600 meters far. From her velocity she deduced that in her reference frame, with her father's bicycle at the origin, the distance to Eric was decreasing at 0.8 meters per second, hence, that Eric was approaching her at that speed. It would thus take 12.5 minutes before she would meet Eric right at the place which she had calculated. In her coordinate system Diane describes Eric's position by

$$x^{(D)}_{\text{Eric}}\left(t^{(D)}\right) \; = \; 600 \; - \; 0.8 \times t^{(D)} \quad . \tag{12}$$

Alex also had seen Eric passing by this morning. He had met him a few times in the discothèque. So, they greeted each other. Alex, curious to know at which place Eric would turn around to return home, measured his speed at 2.0 meters per second. In the coordinates of Alex, Eric's position is given by

$$x^{(A)}_{\text{Eric}}\left(t^{(A)}\right) \; = \; 505 \; + \; 2.0 \times t^{(A)} \quad , \tag{13}$$

as long as he continues to move in the positive sense with the same velocity. The latter formula is explained in more detail below.

Alex restarted his stopwatch when he saw Diane passing by the fishermen monument. Consequently, he thereby defines for himself a new beginning of time $t^{(A)} = 0$, at the same time that, coincidently, Diane started her stopwatch. Hence,

$$t^{(A)} \; = \; t^{(D)} \quad . \tag{14}$$

But Alex' choice of a new instant of time $t^{(A)} = 0$, occurred 4 minutes and 12.5 seconds after Eric had passed by his veranda. In that interval of time Eric moved 505 meters. Consequently, at $t^{(A)} = 0$ Eric is in position

$$x^{(A)}_{\text{Eric}}\left(t^{(A)} = 0\right) = 505$$

in the coordinates of Alex. This explains the constant in formula (13) and, moreover, agrees with the observation of Diane,

$$x^{(D)}_{\text{Eric}}\left(t^{(D)} = 0\right) = 600 \quad ,$$

since Diane and Alex are 95 meters apart at $t^{(A)} = t^{(D)} = 0$.

Alex, who is a very curious person, asks Diane why she is driving her father's bicycle. But Diane, who has of course no intention to share her story with Alex, tells him that her bike is broken and has flat tires. Alex immediatly offers himself to repair her bike. However, just when Diane starts explaining that her father is already taking care of the issue, she sees Eric passing by, heading for his apartment. Eric greets Alex and, furthermore, gives a big smile to Diane. Alex notices her blushing and feels his heart broken for the second time this morning.

But, he ignores this feelings. Instead, he tells Diane about his formula (13) on Eric's displacement in function of time, which he had determined earlier this morning when Eric was jogging towards the other end of the road. Diane decides to also share her formula (12) with Alex. For a while they are both puzzled about the differences.

But then Alex finds the solution. He starts explaining it to Diane, by telling her about his formula (7) on her motion, when she was cycling towards the road exit. Diane reacts surprised and first looks a while at Alex before she asks him which espionage service makes him calculating everybody's movements on the bicycle road. Now, Alex starts blushing and confesses that in her case, the calculus had been done for him to know how much time he had to prepare himself for making a good impression on her. He also tells her that he had been very disappointed when she hardly took notice of him. Diane's mouth stays half open while she continues staring at Alex, inspecting the noble regular face of the tall slim boy, outstanding in physics, mathematics and chemistry. Then, suddenly she moves slowly as close to him as she can manage and asks him to please hold her very tight.

At this point, it seems that it might take a while before Alex will explain his solution to Diane, if ever. So, we better do it ourselves.

While Diane was driving her bicycle, her reference frame of $x^{(D)}$ coordinates was moving with her. She considered herself the origin of that reference frame and was moreover moving in the positive sense. In formula (7) Alex had determined her speed at 2.8 (m/s) with respect to his coordinate system which is the reference frame attached to the Earth.

Alex has determined the speed of Eric at 2.0 (m/s) in the positive sense in formula (13). Diane found in formula (12) that Eric moved in the negative sense, towards her, with a speed of 0.8 (m/s). Hence, when we denote speed by v then a relation for the various velocities involved, is given by

$$v^{(D)}_{\text{Eric}} = v^{(A)}_{\text{Eric}} - v^{(A)}_{\text{Diane}} \quad , \tag{15}$$

or in words: Although running in the opposite direction, Eric is approaching Diana since their distance is decreasing with time. Consequently, the velocity of Eric with respect to Diane is in the negative sense in Diane's coordinate system and equals the difference of the velocity of Eric with respect to Alex, which is in the positive sense, and the velocity of Diane with respect to Alex, which is also in the positive sense.

Now, how can the relation (15) be fully recovered from the relations (13) and (12)?

From formula (7) we obtain in the reference frame of Alex the position of the origin of the reference frame of Diane (which is her father's bicycle) in function of the instant of time $t^{(A)}$ measured at the watch of Alex. Hence, we can determine the position of Eric, $x^{(A)}$, at a certain instant of time $t^{(A)}$ in the coordinate system of Alex, starting from his postion in the coordinate system of Diane and the position of her origin with respect to the origin of Alex:

$$x^{(A)}_{\text{Eric}}\left(t^{(A)}\right) = x^{(A)}_{\text{Diane}}\left(t^{(A)}\right) + x^{(D)}_{\text{Eric}}\left(t^{(A)}\right) \quad . \tag{16}$$

Here, we substitute equation (7), to find

$$x^{(A)}_{\text{Eric}}\left(t^{(A)}\right) = -95 + 2.8 \times t^{(A)} + x^{(D)}_{\text{Eric}}\left(t^{(A)}\right) \quad . \tag{17}$$

Next, when we substitute formula (12), also using the relation (14), then we recover relation (13).

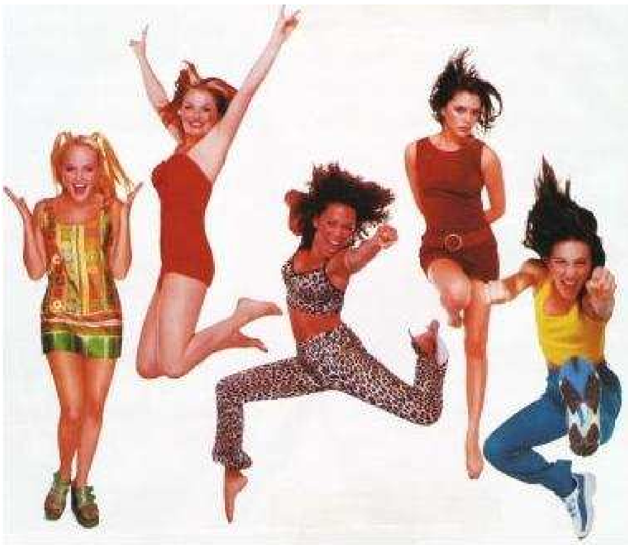
Now, from formula (16) we may deduce

$$\frac{d\,x^{(A)}_{\text{Eric}}\left(t^{(A)}\right)}{d\,t^{(A)}} = \frac{d\,x^{(A)}_{\text{Diane}}\left(t^{(A)}\right)}{d\,t^{(A)}} + \frac{d\,x^{(D)}_{\text{Eric}}\left(t^{(A)}\right)}{d\,t^{(A)}} \quad . \tag{18}$$

In the second term on the righthand side of formula (18), we may substitute relation (14). We obtain then indeed relation (15) in the form

$$v_{\text{Eric}}^{(A)} = v_{\text{Diane}}^{(A)} + v_{\text{Eric}}^{(D)} \quad . \tag{19}$$

The linear transformations (14) and (17) preserve the relation (19), or equivalently relation (15). Transformations which preserve those relations are called Galilei transformations.



3 Transformations

From the previous section we have understood that it is very important to distinguish between a coordinate system and the coordinates of a moving object. Furthermore, we should define well what we *really, really want* (Spice girls, 1994) when we construct coordinate transformations.

1. **Coordinate system (reference frame)**

   Here, we define a coordinate system as a continuous set of points which represent the positions of pointlike objects. We assume that such points fill up the whole space. To each point we may associate a set of real numbers, called *vectors*. In one dimension each point is characterized by one real number, in two dimensions by two real numbers, etc. We assume furthermore that neighbouring points are characterized by neighbouring sets of real numbers. Usually we erect a set of coordinate axes which intersect all in one point, the origin of the coordinate system. The origin is represented by the set (0, 0, ..., 0). All other points are represented by their *projections* on those axes. In that case the whole space is defined once the unit vectors on each axis are defined.

2. **The coordinates of a moving object**

   A moving object is described by a set of coordinates which varies in function of a *time* parameter: $(x_1(t), x_2(t), ..., x_n(t))$ in an n-dimensional space. The resulting space is a one-dimensional subspace of the full coordinate space. In one dimension one does not notice well the difference between the space and the subspace which describes a moving object. However, in higher dimensions it is obvious, since the space described by a moving point particle is just a line.

3. **What we really, really want**

   We want to study coordinate transformations which describe reference frames which are in relative motion with constant velocity. In particular, we want to study what is preserved under such transformations.

In the following we will consider two one-dimensional reference systems A and B and indicate the coordinates and time parameters by respectively $x^{(A)}$ and $t^{(A)}$ in reference frame A and by respectively $x^{(B)}$ and $t^{(B)}$ in reference frame B.

4 Galilei transformations

Besides possible *scale transformations* by choosing different unit systems and which we will not further discuss here, the Galilei transformations consist out of

1. **translations**

   We have *space translations*, given by

   $$x^{(B)} = x^{(A)} + \text{constant} \quad , \tag{20}$$

   which stem from choosing a different origin of space, and *time translations*, given by

   $$t^{(B)} = t^{(A)} + \text{constant} \quad , \tag{21}$$

   which stem from choosing a different beginning of time counting.

2. **rotations**

   In one dimension there are no relevant rotations.

3. **inversion**

   We have *space inversion*, given by

   $$x^{(B)} = -x^{(A)} \quad , \tag{22}$$

   which stem from changing positive and negative sense, and *time inversion*, given by

   $$t^{(B)} = -t^{(A)} \quad , \tag{23}$$

   which stem from counting backward in time.

4. **boosts**

   Under a boost we understand a coordinate transformation for relatively moving reference frames. Suppose that B is moving in the positive sense with respect to A with a constant velocity given by V and assume also that at time $t^{(A)} = 0$ the origins coincide and, moreover, $t^{(B)} = t^{(A)}$. Then this boost is given by the coordinate transformation

   $$x^{(B)} = x^{(A)} - Vt^{(A)} = x^{(A)} - Vt^{(B)} \quad . \tag{24}$$

All the above transformations preserve *distance* and relative motion. Take for example the boost transformation (24) and two pointlike objects a and b in motion. Let their motion be described by

$$x_a^{(A)}\left(t^{(A)}\right) \quad \text{and} \quad x_b^{(A)}\left(t^{(A)}\right) \quad , \tag{25}$$

in reference frame A. There are no restrictions on the dependence of their argument t of the functions $x_a^{(A)}(t)$ and $x_b^{(A)}(t)$. One may consider any motion, like

$$x_a^{(A)}\left(t^{(A)}\right) \;=\; 43 \,+\, 24\cos\left(t^{(A)}\right) \quad \text{and} \quad x_b^{(A)}\left(t^{(A)}\right) \;=\; -512 \,+\, 318\, {t^{(A)}}^5 \quad , \tag{26}$$

or whatever other complicated motion.

In system B, which is here considered to be in motion with respect to system A with constant velocity V, using the transformation (24) and $t^{(B)} = t^{(A)}$, we find for the description of the motion of objects a and b

$$x_a^{(B)}\left(t^{(B)}\right) \;=\; x_a^{(A)}\left(t^{(B)}\right) \;-\; Vt^{(B)} \quad \text{and} \quad x_b^{(B)}\left(t^{(B)}\right) \;=\; x_b^{(A)}\left(t^{(B)}\right) \;-\; Vt^{(B)} \quad . \tag{27}$$

For their relative distance in reference frame B we obtain

$$\left|x_b^{(B)}\left(t^{(B)}\right) - x_a^{(B)}\left(t^{(B)}\right)\right|^2 \;=\; \left|x_b^{(A)}\left(t^{(B)}\right) - x_a^{(A)}\left(t^{(B)}\right)\right|^2 \;=\; \left|x_b^{(A)}\left(t^{(A)}\right) - x_a^{(A)}\left(t^{(A)}\right)\right|^2 \quad . \tag{28}$$

Hence, the relative distance of a and b is equal in both reference frames A and B for all times.

For their relative velocity we obtain

$$\frac{d\,x_b^{(B)}\left(t^{(B)}\right)}{d\,t^{(B)}} - \frac{d\,x_a^{(B)}\left(t^{(B)}\right)}{d\,t^{(B)}} = \frac{d\,x_b^{(A)}\left(t^{(A)}\right)}{d\,t^{(A)}} - \frac{d\,x_a^{(A)}\left(t^{(A)}\right)}{d\,t^{(A)}} \quad . \tag{29}$$

Hence, we find that relative motion is not affected by the Galilei transformations. Newtonian physics is the same in reference frame A as in reference frame B.

4.1 The kinetic energy

The kinetic energy of a moving point particle is related to its mass, m, and its velocity, v, (or, alternatively, its linear momentum), by

$$E(\text{kinetic}) \;=\; \frac{1}{2}mv^2 \quad . \tag{30}$$

Now, since mass is invariant under Galilei transformations, and since, moreover, velocities add up (see formula 19), it is clear that the kinetic energy of a particle is different in different inertial frames, under Galilei transformations.

5 Pioncaré transformations

Instead of leaving $|x_b - x_a|^2$ invariant, as under Galilei transformations, under Poincaré transformations the *event distance*, for $c = 1$ given by

$$(x_b - x_a)^2 - (t_b - t_a)^2 \quad , \tag{31}$$

is left invariant.

The first question is, of course, to define well what this quantity represents. In order to answer that question, we start by defining the notion of an *event*. An event is the happening of something at a certain place and at a certain time. For example, when Bruno stopped the bicycle at 9h02 at 17 meters distance from Alex, or when Eric passed by the veranda of Alex at 9h36. These are two events. We can determine the quantity (31) for those two events. Under Poincaré transformations this gives the same result for Alex, sitting in his sofa at his veranda, as it gives for Diane driving her father's bicycle.

For space and time translations and inversions it is obvious that the quantity (31) is invariant, as well as for space rotations, because space rotations do not touch the time and do leave invariant distances in space. So, we only have to consider Poincaré transformations for reference frames which are moving with relative constant velocity $\beta = V/c$, *i.e. boosts*. The latter transformations are also called *Lorentz* transformations.

Hence, the task is to find a transformation for moving frames which leaves the quantity (31) invariant. This has been done by Albert Einstein (1905). He obtained

$$x^{(B)} = \gamma\left(x^{(A)} - \beta\, t^{(A)}\right) \quad \text{and} \quad t^{(B)} = \gamma\left(t^{(A)} - \beta\, x^{(A)}\right) \quad , \tag{32}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad . \tag{33}$$

It is easy to demonstrate that the coordinate transformation (32) satisfies the condition that the quantity (31) is invariant. Below we express in terms of the coordinates of reference frame A, the event distance (31) for the two events a and b as determined in reference frame B. We make thereby use of expressions (32) and (33).

$$\begin{aligned}
&\left\{x_a^{(B)} - x_b^{(B)}\right\}^2 - \left\{t_a^{(B)} - t_b^{(B)}\right\}^2 = \qquad (34)\\
&= \left\{\gamma\left(x_a^{(A)} - \beta t_a^{(A)}\right) - \gamma\left(x_b^{(A)} - \beta t_b^{(A)}\right)\right\}^2 - \left\{\gamma\left(t_a^{(A)} - \beta x_a^{(A)}\right) - \gamma\left(t_b^{(A)} - \beta x_b^{(A)}\right)\right\}^2\\
&= \gamma^2\left\{\left(x_a^{(A)} - x_b^{(A)}\right) - \beta\left(t_a^{(A)} - t_b^{(A)}\right)\right\}^2 - \gamma^2\left\{\left(t_a^{(A)} - t_b^{(A)}\right) - \beta\left(x_a^{(A)} - x_b^{(A)}\right)\right\}^2\\
&= \gamma^2\left\{\left(1-\beta^2\right)\left(x_a^{(A)} - x_b^{(A)}\right)^2 + \left(\beta^2 - 1\right)\left(t_a^{(A)} - t_b^{(A)}\right)^2\right\}\\
&= \left(x_a^{(A)} - x_b^{(A)}\right)^2 - \left(t_a^{(A)} - t_b^{(A)}\right)^2 \quad .
\end{aligned}$$

Once more using relation (32), we deduce below a relation for the addition of velocities. When reference frame B moves with constant velocity $\beta = V/c$ with respect to reference frame A, and an object moves with velocity $dx^{(A)}/dt^{(A)}$ in reference frame A, then we find for its velocity as measured with respect to the coordinates used in reference frame B the following.

$$\frac{dx^{(B)}}{dt^{(B)}} = \frac{\dfrac{dx^{(B)}}{dt^{(A)}}}{\dfrac{dt^{(B)}}{dt^{(A)}}} = \frac{\gamma\left(\dfrac{dx^{(A)}}{dt^{(A)}} - \beta\right)}{\gamma\left(1 - \beta\,\dfrac{dx^{(A)}}{dt^{(A)}}\right)} = \frac{\dfrac{dx^{(A)}}{dt^{(A)}} - \beta}{1 - \beta\,\dfrac{dx^{(A)}}{dt^{(A)}}} \quad . \tag{35}$$

In particular for an object which travels with the speed of light ($c = 1$) in frame A, we have in frame B

$$c^{(A)} = \frac{dx^{(A)}_{\text{light}}}{dt^{(A)}} = 1 \quad \Longleftrightarrow \quad c^{(B)} = \frac{dx^{(B)}_{\text{light}}}{dt^{(B)}} = \frac{1-\beta}{1-\beta} = 1 \quad . \tag{36}$$

From this result we may conclude that objects which move with the speed of light with respect to the coordinates of one reference frame, also move with the speed of light as observed by using the coordinates of a reference frame which is in motion with a constant velocity β with respect to the first reference system. Or in other words, light moves with the same velocity with respect to observers in different inertial systems.

5.1 Velocities never exceed the velocity of light

In inertial frame A we study an object with velocity v (in units c) with respect to the coordinates of reference frame A. We assume that v is smaller than the velocity of light, *i.e.*

$$|v| < 1 \quad \Longleftrightarrow \quad 1 + v > 0 \quad \text{and} \quad 1 - v > 0 \quad . \tag{37}$$

Let us furthermore consider a reference frame B which moves with constant velocity β, also smaller than the velocity of light, with respect to reference frame A. Now, since both v and β are smaller than the velocity of light, we have, by the use of relations (37), the following inequalities.

$$\begin{array}{rcl} & & 1 - \beta v > 0 \\ -(1+v)(1-\beta) < 0 & \Longleftrightarrow & -(1-\beta v) < v - \beta \\ 0 < (1-v)(1+\beta) & \Longleftrightarrow & v - \beta < 1 - \beta v \quad . \end{array}$$

Putting things together, we find

$$-1 \; < \; \frac{v-\beta}{1-\beta v} = v^{(B)} \; < \; +1 \quad . \tag{38}$$

This proofs that $v^{(B)}$, which is the velocity of the object under study, as measured in the coordinate system B, is always smaller than the velocity of light $c = 1$, for the case that velocities in A do not exceed the velocity of light.

Velocities larger than the velocity of light, while not (yet?) observed, do not seem to make part of our physical world, hence, do not have to be considered (yet?). This does not mean that it is forbidden to study the properties of tachyons, but, just means that we do not yet need to study them.

5.2 Small velocities

Here, we will concentrate on all-day velocities, like jogging Eric and bicycling Diane. With respect to Alex the velocity of jogging Eric is given by (see formula 13)

$$\frac{dx^{(A)}_{\text{Eric}}}{dt^{(A)}} = \frac{2.0 \text{ m/s}}{3.0 \times 10^8 \text{ m/s}} = \frac{2}{3} \times 10^{-8} \quad . \tag{39}$$

Furthermore, the reference frame of Diane moves with a constant velocity (see formula 7) given by

$$\frac{dx^{(A)}_{\text{Diane}}}{dt^{(A)}} = \frac{2.8 \text{ m/s}}{3.0 \times 10^{8} \text{ m/s}} = \frac{2.8}{3} \times 10^{-8} \quad . \tag{40}$$

Hence, by the use of formula (35), we find for the velocity of Eric in the reference frame of Diane the following.

$$\frac{\frac{2}{3} \times 10^{-8} - \frac{2.8}{3} \times 10^{-8}}{1 - \frac{2}{3} \times \frac{2.8}{3} \times 10^{-16}} \approx \frac{-0.8 \text{ m/s}}{3 \times 10^{8} \text{ m/s}} \left(1 + \frac{5.6}{9} \times 10^{-16}\right) \quad . \tag{41}$$

From formula (41) we obtain the result that only in the 16-th decimal we may notice a difference from the addition rule (15). In practice it is impossible to verify this result, since already the errors in the measurements of the velocities of Eric and Diane are orders of magnitude larger.

Consequently, for all-day velocities we will not notice any difference between Galilei and Poincaré transformations. Even with the velocities of supersonic airplanes, which are more than two orders of magnitude larger than running or driving a bicycle, the effects are only in somewhere the 11-th decimal. However, at cosmic scales or in particle accellerators where objects reach velocities close to the light velocity, the differences between the two types of transformations are very well observable. Millions of experiments every single day confirm that Einstein's basic assumption (1905) on the constancy of the light velocity was very clever.

5.3 Energy and momentum

Let us define for a point particle with velocity v

$$\varepsilon = \frac{1}{\sqrt{1 - v^2}} \quad \text{and} \quad \vec{u} = \frac{\vec{v}}{\sqrt{1 - v^2}} \quad , \tag{42}$$

which in one dimension reduces to

$$\varepsilon = \frac{1}{\sqrt{1 - v^2}} \quad \text{and} \quad u = \frac{v}{\sqrt{1 - v^2}} \quad . \tag{43}$$

First, by the use of relations (33) and (35), we determine

$$\varepsilon^{(B)} = \frac{1}{\sqrt{1 - {v^{(B)}}^2}} = \gamma \frac{1 - \beta\, v^{(A)}}{\sqrt{1 - {v^{(A)}}^2}} \quad \text{and} \quad u^{(B)} = \frac{v^{(B)}}{\sqrt{1 - {v^{(B)}}^2}} = \gamma \frac{v^{(A)} - \beta}{\sqrt{1 - {v^{(A)}}^2}} \quad , \tag{44}$$

which can be written in the form

$$\epsilon^{(B)} = \gamma \left(\epsilon^{(A)} - \beta\, u^{(A)}\right) \quad \text{and} \quad u^{(B)} = \gamma \left(u^{(A)} - \beta\, \epsilon^{(A)}\right) \quad . \tag{45}$$

On comparison of formulas (45) with formulas (32), we observe that the pair $(\epsilon\, ,\ u)$ transforms the same way as the pair $(t\, ,\ x)$. Consequently, the quantity

$$\epsilon^2 - u^2 \quad , \tag{46}$$

is an invariant under Lorentz transformations.

Moreover, from

$$\frac{1}{\sqrt{1-\epsilon^2}} \approx \frac{1}{1-\frac{1}{2}\epsilon^2} \approx 1 + \frac{1}{2}\epsilon^2 \quad , \tag{47}$$

we find that to lowest order

$$m\varepsilon \approx m + \frac{1}{2}mv^2 \quad \text{and} \quad mu \approx mv \quad . \tag{48}$$

The first expression in formula (48) corresponds to a constant, m, which represents the mass of a particle, added to the kinetic energy of the particle. In Newtonian mechanics the zero of energy is anyhow not well defined. Hence, $m\varepsilon$ represents for low velocities the kinetic energy of the particle, since the constant term is of no importance. Furthermore, the second term in formula (48) represents the linear momentum of the particle for low velocities.

Those concepts can be generalized. Here, we define for the *total energy*, E, and the *linear momentum*, p of a particle which has a mass m and which moves with velocity v in a certain reference frame

$$E = \frac{m}{\sqrt{1-v^2}} \quad \text{and} \quad p = \frac{mv}{\sqrt{1-v^2}} \quad . \tag{49}$$

Under Lorentz transformations E and p transform the same way as ε and u. From formula (49) we observe that when v approaches the light velocity $c = 1$, then E tends to infinity. This corresponds very well to experimental observation in particle accellerators.

When a particle is at rest, its energy equals m. This is exactly the most famous formula of physics: $E = mc^2$. The rest mass m of a particle is invariant under Lorentz transformations, *i.e.*

$$E^2 - p^2 = m^2 \quad . \tag{50}$$

5.4 Total invariant mass

For a system of two particles a and b the total energy is given by

$$E = E_a + E_b = \sqrt{m_a^2 + p_a^2} + \sqrt{m_b^2 + p_b^2} \quad , \tag{51}$$

and the total linear momentum, obviously, by

$$p = p_a + p_b \quad . \tag{52}$$

It is not difficult to demonstrate that the *total invariant mass*, $\sqrt{s}$, squared, defined by

$$s = \{E_a + E_b\}^2 - \{p_a + p_b\}^2 \quad , \tag{53}$$

is invariant under Lorentz transformations.

6 Relativistic kinematics

Most of the concepts which we studied in the foregoing, can straightforwardly be extended to three dimensions. In the following, we study scattering processes in three dimensions. For scattering processes one can avoid to carry out explicitly the Lorentz transformations by the use of the so-called *Mandelstam* variables. The latter are Lorentz invariant, hence the same in any inertial frame.

The total energy, E_{total}, for a system of two non-interacting on-mass-shell particles of masses m_1 and m_2, which are freely moving with linear momenta respectively $\vec{p}_1$ and $\vec{p}_2$, is given by the sum of the individual energies, $E(\vec{p}_1)$ and $E(\vec{p}_2)$ respectively, according to

$$E_{\text{total}} = E(\vec{p}_1) + E(\vec{p}_2) = \sqrt{\vec{p}_1^{\,2} + m_1^{\,2}} + \sqrt{\vec{p}_2^{\,2} + m_2^{\,2}} \quad . \tag{54}$$

In the center-of-mass frame, where $\vec{p}_1 = -\vec{p}_2 = \vec{p}$, one has the following relations:

$$\begin{aligned} s = (E_{\text{CM, total}})^2 &= 2\vec{p}^{\,2} + m_1^{\,2} + m_2^{\,2} + 2\sqrt{\vec{p}^{\,2} + m_1^{\,2}}\sqrt{-\vec{p}^{\,2} + m_2^{\,2}} \quad , \\ \left(s - 2\vec{p}^{\,2} - m_1^{\,2} - m_2^{\,2}\right)^2 &= 4\left(\vec{p}^{\,2} + m_1^{\,2}\right)\left(-\vec{p}^{\,2} + m_2^{\,2}\right) \quad , \\ 4s\vec{p}^{\,2} &= s^2 - 2s\left(m_1^{\,2} + m_2^{\,2}\right) + m_1^{\,4} + m_2^{\,4} - 2m_1^{\,2}m_2^{\,2} \quad , \\ \text{and} \quad \vec{p}^{\,2} &= \frac{1}{4s}\left\{\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]\right\} \quad . \end{aligned} \tag{55}$$

The Mandelstam variables, s, t and u, for the process

$$1 + 2 \longrightarrow 3 + 4 \tag{56}$$

are defined by

$$s = (p_1 + p_2)^2 \quad , \quad t = (p_1 - p_3)^2 \quad , \quad u = (p_1 - p_4)^2 \quad , \tag{57}$$

or, alternatively, by using total four-momentum conservation which is given by

$$p_1 + p_2 = p_3 + p_4 \quad , \tag{58}$$

one also has

$$s = (p_3 + p_4)^2 \quad , \quad t = (p_2 - p_4)^2 \quad , \quad u = (p_2 - p_3)^2 \quad , \tag{59}$$

Notice that we use here the metric $(+, -, -, -)$, which for s gives

$$s = (p_1 + p_2)^2 = (E(\vec{p}_1) + E(\vec{p}_2))^2 - (\vec{p}_1 + \vec{p}_2)^2 \quad . \tag{60}$$

In the the center-of-mass frame, where $\vec{p}_1 = -\vec{p}_2$, one obtains, moreover

$$s = (E(\vec{p}_1) + E(\vec{p}_2))^2 = (E_{\text{CM}})^2 \quad , \tag{61}$$

which equals the total invariant mass, as already anticipated in formula (55).

Furthermore, from their definition one observes that the Mandelstam variables (57) are Lorentz invariant and hence invariants with respect to any Lorentz transformation.

By total momentum conservation (58), one deduces

$$\begin{aligned} s+t+u &= 3{p_1}^2+{p_2}^2+{p_3}^2+{p_4}^2+2p_1\cdot(p_2-p_3-p_4) \\ &= 3{p_1}^2+{p_2}^2+{p_3}^2+{p_4}^2-2{p_1}^2 \\ &= {p_1}^2+{p_2}^2+{p_3}^2+{p_4}^2 \\ &= {m_1}^2+{m_2}^2+{m_3}^2+{m_4}^2 \quad . \end{aligned} \tag{62}$$

Consequently, for on-mass-shell processes s, t and u are not independent.

In Fig. 1 we visualize things for the center-of-mass frame.

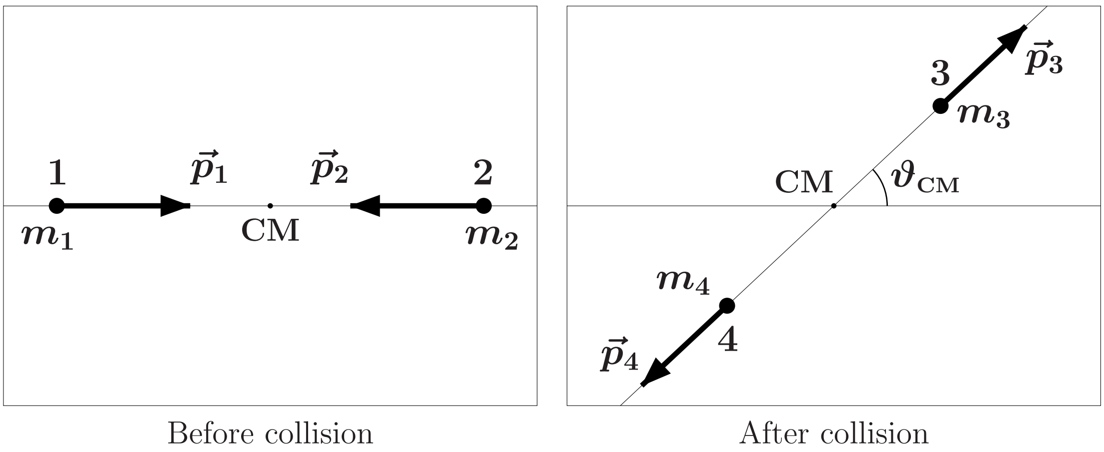


Figure 1: Collision in the center-of-mass system. Before collision particle 1 and particle 2 move towards their center of mass with equal and opposite three-momenta, $\vec{p}_1$ and $\vec{p}_2$ respectively. After collision particle 3 and particle 4 move away from their center of mass with equal and opposite three-momenta, $\vec{p}_3$ and $\vec{p}_4$ respectively.

The angle between the direction of motion of the outgoing particle 3 and the direction of motion of the incoming particle 1 is defined as the angle $\vartheta_{\rm CM}$ of the scattering process of formula (56) in the center-of-mass system. It has the following relation with the Mandelstam variable t.

$$\begin{aligned} t &= (p_1-p_3)^2 \qquad (63) \\ &= (p_1)^2 + (p_3)^2 - 2p_1\cdot p_3 \\ &= (p_1)^2 + (p_3)^2 - 2E(\vec{p}_1)\,E(\vec{p}_3) + 2\vec{p}_1\cdot\vec{p}_3 \\ &= (m_1)^2 + (m_3)^2 - 2\sqrt{(\vec{p}_1)^2+(m_1)^2}\sqrt{(\vec{p}_3)^2+(m_3)^2} + 2\,|\vec{p}_1|\,|\vec{p}_3|\cos(\vartheta_{\rm CM}) \quad . \end{aligned}$$

6.1 $\pi^+\pi^- \to K^+K^{*-}$

Consider a π^+ meson which annihilates with a π^- meson, resulting in two outgoing Kaon mesons, a K^+ meson and a K^{*-} meson. The π^- meson is at rest in the laboratory, whereas the π^+ meson has a total energy of 9.0 GeV. The Kaon meson comes out with an angle of 60° with respect to the direction of the incoming pion, in the center-of-mass system.

Given this information, we may determine the other kinematical quantities. For the masses of the particles we take

$$m_\pi = 0.14 \text{ GeV} \quad , \quad m_K = 0.50 \text{ GeV} \quad , \quad m_{K^*} = 0.89 \text{ GeV} \quad .$$

Let us first determine the total invariant mass of the system.

$$\sqrt{s} \;=\; \sqrt{2m_\pi^2 + 2E_{\pi^+}m_\pi} \;=\; 1.60 \text{ GeV} \quad .$$

With that result, also using formula (55), we may determine $\vec{p}_\pi^{\;2}$ in the center-of-mass frame.

$$(\vec{p}_\pi)^2 \;=\; \frac{1}{4}\left[s - 4m_\pi^2\right] \;=\; 0.62 \text{ (GeV)} \quad .$$

Next, we can determine $(\vec{p}_K)^2$ in the center-of-mass frame, as follows

$$(\vec{p}_K)^2 \;=\; \frac{1}{4s}\left\{\left[s - (m_K + m_{K^*})^2\right]\left[s - (m_K - m_{K^*})^2\right]\right\} \;=\; 0.148 \text{ (GeV)} \quad .$$

The linear momentum of K^* is in the center-of-mass frame opposite to $\vec{p}_K$, of course. Consequently, we can check our calculations by determining the total invariant mass $\sqrt{s}$ after collision. This gives

$$\sqrt{s} \;=\; \sqrt{\vec{p}_K^{\;2} + m_K^2} + \sqrt{\vec{p}_{K^*}^{\;2} + m_{K^*}^2} \;=\; 0.63 \text{ (GeV)} \;+\; 0.97 \text{ (GeV)} \;=\; 1.60 \text{ (GeV)} \quad ,$$

which is indeed what we obtained for the situation before collision. We obtain here thus that the total energy before and after collision is the same, hence, total energy is conserved.

Then we may determine t and u

$$t \;=\; (m_\pi)^2 \;+\; (m_K)^2 \;-\; 2E_\pi E_K \;+\; 2\left|\vec{p}_\pi\right|\left|\vec{p}_K\right|\cos\left(\vartheta_{\text{CM}}\right) \;=\; -0.437 \text{ GeV}^2 \quad .$$

and

$$u \;=\; (m_\pi)^2 \;+\; (m_{K^*})^2 \;-\; 2E_\pi E_{K^*} \;+\; 2\left|\vec{p}_\pi\right|\left|\vec{p}_{K^*}\right|\cos\left(\pi - \vartheta_{\text{CM}}\right) \;=\; -1.04 \text{ GeV}^2 \quad .$$

We will denote the total momentum of a particle in the laboratory by q and its linear momentum by $\vec{q}$. The π^- meson is at rest, hence $\vec{q}_{\pi^-} = 0$. For the π^+ we have

$$E\left(\vec{q}_{\pi^+}\right) \;=\; \sqrt{\left(\vec{q}_{\pi^+}\right)^2 + m_{\pi^+}^2} \;=\; 9 \text{ (GeV)} \quad \Longleftrightarrow \quad \vec{q}_{\pi^+} \;\approx\; 9 \text{ (GeV)} \quad .$$

Moreover, since $\vec{q}_{\pi^-} = 0$, we have for t the relation

$$t \;=\; \left(q_{\pi^-} - q_{K^*}\right)^2 \;=\; m_{\pi^-}^2 + m_{K^*}^2 - 2m_{\pi^-}\sqrt{\vec{q}_{K^*}^{\;2} + m_{K^*}^2} \quad ,$$

hence

$$\vec{q}_{K^*}^{\;2} \;=\; \left(\frac{m_{\pi^-}^2 + m_{K^*}^2 - t}{2m_{\pi^-}}\right)^2 \;-\; m_{K^*}^2 \;=\; 4.37 \text{ GeV}^2 \quad .$$

Similarly

$$u = (q_{\pi^-} - q_K)^2 = m_{\pi^-}^2 + m_K^2 - 2m_{\pi^-}\sqrt{\vec{q}_K^{\;2} + m_K^2} \quad ,$$

hence

$$\vec{q}_K^{\;2} = \left(\frac{m_{\pi^-}^2 + m_K^2 - u}{2m_{\pi^-}}\right)^2 - m_K^2 = 4.66 \text{ GeV}^2 \quad .$$

Also

$$E(\vec{q}_K) = \sqrt{\vec{q}_K^{\;2} + m_K^2} = 4.68 \text{ (GeV)} \quad ,$$

and

$$E(\vec{q}_{K^*}) = \sqrt{\vec{q}_{K^*}^{\;2} + m_{K^*}^2} = 4.46 \text{ (GeV)} \quad .$$

Finally, we determine the angle in the frame of the laboratory, of the direction of the outgoing K meson with respect to the direction of the incoming π^+ meson.

$$\cos(\theta_{K,\text{lab}}) = \frac{t - m_\pi^2 - m_K^2 + 2E(\vec{q}_{\pi^+})\,E(\vec{q}_K)}{2\,|\vec{q}_{\pi^+}|\,|\vec{q}_K|} = 0.9974 \quad ,$$

corresponding to an angle of 4.1 degrees.

Although hidden, by the use of Mandelstam variables, the above calculus is based in Lorentz transformations and the constancy of the light velocity. It is, furthermore based in the definitions (49) for the energy and linear momentum of a moving particle. The results are verified by experiment, the ultimate judge on our guesses. Every single day, in various particle accellerators, millions of such scattering processes, involving two, or many more particles, are performed. Up till today, nothing has been found which conflicts with Einstein's assumptions.

6.2 Elastic Scattering in the center-of-mass system

In elastic scattering the outgoing particles are identical to the incoming particles, which at this level amounts to $m_3 = m_1$ and $m_4 = m_2$. When we define the three-momenta before and after collision by respectively $\vec{p}$ and $\vec{p}\,'$, then we have

$$\vec{p}_1 = -\vec{p}_2 = \vec{p} \quad \text{and} \quad \vec{p}_3 = -\vec{p}_4 = \vec{p}\,' \quad , \tag{64}$$

hence, by using formula (55),

$$\vec{p}^{\,2} = \frac{1}{4s}\left\{\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]\right\}$$

$$\text{and} \quad \vec{p}\,'^2 = \frac{1}{4s}\left\{\left[s - (m_3 + m_4)^2\right]\left[s - (m_3 - m_4)^2\right]\right\} \quad . \tag{65}$$

Since, moreover, $m_3 = m_1$ and $m_4 = m_2$ for elastic scattering, we have for the center-of-mass three-momenta in that case

$$\vec{p}^{\,2} = \vec{p}\,'^2 \quad . \tag{66}$$

Substitution of the result (66) in expressions (61) and (63) gives the results

$$\begin{aligned} s &= \left(\sqrt{(\vec{p})^2 + (m_1)^2} + \sqrt{(\vec{p})^2 + (m_2)^2}\right)^2 \\ &= (\vec{p})^2 + (m_1)^2 + (\vec{p})^2 + (m_2)^2 + 2\sqrt{(\vec{p})^2 + (m_1)^2}\sqrt{(\vec{p})^2 + (m_2)^2} \\ &= 2(\vec{p})^2 + (m_1)^2 + (m_2)^2 + 2\sqrt{(\vec{p})^2 + (m_1)^2}\sqrt{(\vec{p})^2 + (m_2)^2} \quad , \end{aligned} \tag{67}$$

and

$$t = (m_1)^2 + (m_3)^2 - 2\sqrt{\vec{p}^{\,2} + (m_1)^2}\sqrt{\vec{p}^{\,\prime 2} + (m_3)^2} + 2\,|\vec{p}|\,\left|\vec{p}^{\,\prime}\right|\cos(\vartheta_{\text{CM}})$$
$$= 2\,(m_1)^2 - 2\left(\vec{p}^{\,2} + (m_1)^2\right) + 2\vec{p}^{\,2}\cos(\vartheta_{\text{CM}})$$
$$= 2\vec{p}^{\,2}\left\{-1 + \cos(\vartheta_{\text{CM}})\right\} \quad . \tag{68}$$

Furthermore

$$u = (m_1)^2 + (m_4)^2 - 2\sqrt{\vec{p}^{\,2} + (m_1)^2}\sqrt{\vec{p}^{\,\prime 2} + (m_4)^2} + 2\vec{p}_1 \cdot \vec{p}_4$$
$$= (m_1)^2 + (m_2)^2 - 2\sqrt{(\vec{p})^2 + (m_1)^2}\sqrt{(\vec{p})^2 + (m_2)^2} - 2\vec{p}^{\,2}\cos(\vartheta_{\text{CM}}) \ . \tag{69}$$

As is obvious from the definitions of ϑ_{CM} in Fig. 1 and $\vec{p}$ in formula (64), one has

$$\vec{p}^{\,2} \geq 0 \quad \text{and} \quad -1 \leq \cos(\vartheta_{\text{CM}}) \leq +1 \quad , \tag{70}$$

hence for the Mandelstam variables s (formula 67) and t (formula 68), we find

$$s \geq (m_1 + m_2)^2 \quad \text{and} \quad t \leq 0 \quad . \tag{71}$$

6.3 Elastic Scattering in the lab system

In the laboratory system particle 2 is assumed to be at rest. This is visualized in Fig. 2. We define the laboratory four-momenta by q_1, q_2, q_3 and q_4, in order to distinguish from the center-of-mass four-momenta. We study here again the case of elastic scattering, which implies $m_3 = m_1$ and $m_4 = m_2$.

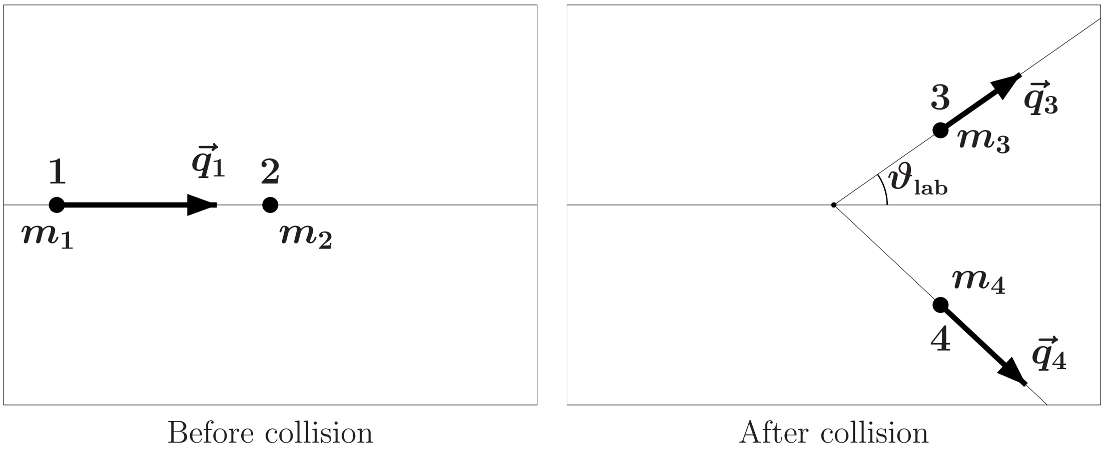


Figure 2: Collision in the laboratory system. Before collision particle 1 moves with three-momentum $\vec{q}_1$ towards particle 2 at rest ($\vec{q}_2 = 0$) in the center of coordinates. After collision particle 3 and particle 4 move away from the center of coordinates with three-momenta $\vec{q}_3$ and $\vec{q}_4$ respectively.

Here, we obtain for the Mandelstam variables (57), which, as mentioned before, are invariant under Lorentz transformations, hence the same for the laboratory system and the center-of-mass system,

$$\begin{aligned} s &= (q_1 + q_2)^2 = (m_1)^2 + (m_2)^2 + 2E(\vec{q}_1)E(\vec{q}_2) - 2\vec{q}_1 \cdot \vec{q}_2 \\ &= (m_1)^2 + (m_2)^2 + 2E(\vec{q}_1)m_2 \\ t &= (q_1 - q_3)^2 = 2(m_1)^2 - 2E(\vec{q}_1)E(\vec{q}_3) + 2\vec{q}_1 \cdot \vec{q}_3 \\ &= 2(m_1)^2 - 2E(\vec{q}_1)E(\vec{q}_3) + 2|\vec{q}_1||\vec{q}_3|\cos(\vartheta_{\rm lab}) \\ u &= (q_2 - q_3)^2 = (m_1)^2 + (m_2)^2 - 2m_2E(\vec{q}_3) \quad . \end{aligned} \tag{72}$$

Also using formula (62), we find for t

$$t = 2m_1 + 2m_2 - s - u = 2m_2(E(\vec{q}_3) - E(\vec{q}_1)) \quad . \tag{73}$$

One defines the kinetic energy of the incoming particle by

$$T_1 = E(\vec{q}_1) - m_1 \quad . \tag{74}$$

At threshold, where $\vec{q}_1 = 0$, we obtain $T_1 = 0$.

Part II
Generalities

7 Covariant and contravariant components

In a vector space of N dimensions we define an arbitrary set of basis vectors,

$$\boldsymbol{e}_i \quad , \quad i = 1, \ldots, N \quad , \tag{75}$$

and their innerproducts, given by

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j \; = \; g_{ij} \quad , \quad i, j = 1, \ldots, N \quad . \tag{76}$$

An arbitrary vector $\boldsymbol{v}$ in this N-dimensional vector space may be characterized by its components, v^i, on the basis (75), as follows

$$\boldsymbol{v} \; = \; v^i \boldsymbol{e}_i \quad , \tag{77}$$

where, as usually, repeated indices imply summation.

Using expression (76), we obtain for the innerproduct of two vectors the result

$$\boldsymbol{v} \cdot \boldsymbol{w} \; = \; v^i w^j \boldsymbol{e}_i \cdot \boldsymbol{e}_j \; = \; g_{ij} v^i w^j \quad . \tag{78}$$

The vector $\boldsymbol{v}$ of formula (77) might equally well be characterized by its innerproducts with the basis vectors (75). For this purpose we define

$$v_i \; = \; \boldsymbol{v} \cdot \boldsymbol{e}_i \quad . \tag{79}$$

There exits of course a relation between the two quantities v^i and v_i, defined in formulas (77) and (79) respectively. Also using definition (76), we obtain for that relation the following result

$$v_i \; = \; \boldsymbol{v} \cdot \boldsymbol{e}_i \; = \; v^j \boldsymbol{e}_j \cdot \boldsymbol{e}_i \; = \; g_{ij} v^j \quad . \tag{80}$$

Since we are free to choose any basis in the N-dimensional vector space under consideration, let us select a basis $\{\boldsymbol{a}\}$, which is related to the basis (75) by a nonsingular transformation A given by

$$\boldsymbol{a}_j \; = \; A_j^{\;i} \, \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{e}_i \; = \; \left(A^{-1}\right)_i^{\;j} \boldsymbol{a}_j \quad . \tag{81}$$

The vector $\boldsymbol{v}$ defined in formula (77) has new components, say v'^j, at the new basis $\{\boldsymbol{a}\}$. A relation between the two sets of components is, by the use of the transformations (81), readily found, trough the relation

$$v'^j \boldsymbol{a}_j \; = \; \boldsymbol{v} \; = \; v^i \boldsymbol{e}_i \; = \; v^i \left(A^{-1}\right)_i^{\;j} \boldsymbol{a}_j \quad ,$$

to yield

$$v'^j = v^i \left(A^{-1}\right)_i^{\ j} \quad . \tag{82}$$

We find thus that the components of a vector transform with the inverse of the transformation of the basis elements, for which reason those components are said to be *contra-variant.*

The components of $\boldsymbol{v}$ which are defined in formula (79) transform under the basis transformation (81) as follows

$$v'_j = \boldsymbol{v} \cdot \boldsymbol{a}_j = \boldsymbol{v} \cdot A_j^{\ i}\, \boldsymbol{e}_i = A_j^{\ i}\, v_i \quad . \tag{83}$$

Aparantly, the quantities (79) transform in the same way as the basis vectors, for which reason they are said to be *covariant.*

The inverse transformation is for the contravariant components given by

$$v^i = v'^j A_j^{\ i} \quad , \tag{84}$$

as, by the use of the transformation property (82) for contravariant components and the usual rules for the components of products of transition matrices, can easily be seen from

$$v'^j A_j^{\ i} = v^k \left(A^{-1}\right)_k^{\ j} A_j^{\ i} = v^k \left(A^{-1}A\right)_k^{\ i} = v^k \delta_k^i = v^i \quad .$$

An arbitrary point P in the N-dimensional vector space under consideration can be characterized by the components $\{x\}$ of its position vector $\boldsymbol{x}(P)$, with respect to the basis $\{\boldsymbol{e}\}$ which is defined in formula (75), but equally well by the components $\{x'\}$ with respect to the basis $\{\boldsymbol{a}\}$ defined in formula (81), according to

$$\boldsymbol{x}(P) = x^i \boldsymbol{e}_i = x'^j \boldsymbol{a}_j \quad . \tag{85}$$

Relations between both sets of coordinates $\{x\}$ and $\{x'\}$ are, according to formulas (82) and (84), given by

$$x'^j = x^i \left(A^{-1}\right)_i^{\ j} \quad \text{and} \quad x^i = x'^j A_j^{\ i} \quad . \tag{86}$$

Such relations might also be seen as the definitions of the basis transformations (81). Moreover, the relations (86) are linear relations between the two sets of coordinates and hence it follows

$$\frac{\partial x'^j}{\partial x^i} = \left(A^{-1}\right)_i^{\ j} \quad \text{and} \quad \frac{\partial x^i}{\partial x'^j} = A_j^{\ i} \quad , \tag{87}$$

such that we also obtain

$$x'^j = x^i \frac{\partial x'^j}{\partial x^i} \quad \text{and} \quad x^i = x'^j \frac{\partial x^i}{\partial x'^j} \quad . \tag{88}$$

8 The metrical tensor

The object g which is defined in formula (76), will be given the name *metrical tensor*. We study in this section its transformation rules under a coordinate transformation of the form (88).

From relations (76) and (81) one obtains for the metric g' at the $\{\boldsymbol{a}\}$ basis, the transformation rule

$$g'_{k\ell} = \boldsymbol{a}_k \cdot \boldsymbol{a}_\ell = A_k{}^i A_\ell{}^j \boldsymbol{e}_i \cdot \boldsymbol{e}_j = A_k{}^i A_\ell{}^j g_{ij} \quad ,$$

which upon substitution of relation (87) takes the form

$$g'_{k\ell} = \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^\ell} g_{ij} \quad . \tag{89}$$

Relation (89) is one of the basic relations in *differential geometry.*

The components of the inverse of the metrical tensor are denoted with upper indices, according to

$$\left(g^{-1}g\right)^i{}_j = g^{ik} g_{kj} = \delta^i_j \quad . \tag{90}$$

Those components transform under the coordinate transformation (88) as follows

$$g'^{k\ell} = \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^\ell}{\partial x^j} g^{ij} \quad . \tag{91}$$

Notice, that by the definition (76) both the metric tensor and its inverse are symmetric in their indices.

9 Local basis

In this section we consider an orthonormal basis $\{\boldsymbol{e}\}$ in an N-dimensional Euclidean space and associated with it a set of coordinates $\{x\}$. The inner product of the basis vectors (or the metric of the space) is then given by

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \quad , \tag{92}$$

and any point P in space by its components, according to

$$\boldsymbol{x}(P) = x^i \boldsymbol{e}_i \quad . \tag{93}$$

As an example consider ordinary three-dimensional space. For the three basis vectors we select for our example $e^1 = \hat{x}$, $e^2 = \hat{y}$ *and* $e^3 = \hat{z}$, *whereas for the coordinates we take* x, y *and* z.

Now, in this space we select a different set of coordinates $\{x'\}$, such that each point in the vector space can uniquely be described by those coordinates. This means in general that there exists relations between the two sets of coordinates $\{x\}$ and $\{x'\}$ which are sufficiently well behaved, such that we may take as many derivatives as we need in the following.

Take as an example the set of spherical coordinates r, ϑ *and* φ *which are related to the ordinary* $3D$ *coordinates,* x, y *and* z, *by*
$x = x(r, \vartheta, \varphi) = r\sin(\vartheta)\cos(\varphi)$, $y = y(r, \vartheta, \varphi) = r\sin(\vartheta)\sin(\varphi)$ *and*
$z = z(r, \vartheta, \varphi) = r\cos(\vartheta)$.

Associated with the set of coordinates $\{x'\}$ we choose at each point of our space a new local basis $\{\boldsymbol{u}(x'(x))\}$, which basis serves for measurements in the direct vicinity of the point under consideration, but has no meaning at a global level. The transformations which relate the global basis $\{\boldsymbol{e}\}$ and the local basis $\{\boldsymbol{u}(x'(x))\}$, are given by

$$\boldsymbol{u}_j(x') = \frac{\partial x^i}{\partial x'^j}\, \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{e}_i = \frac{\partial x'^j}{\partial x^i}\, \boldsymbol{u}_j(x') \quad . \tag{94}$$

For the 3D spherical coordinates, we obtain for a local basis the $\hat{r}$, $\hat{\vartheta}$ *and* $\hat{\varphi}$ *unit vectors:*
$\hat{r}(r, \vartheta, \varphi) = \{\hat{x}\cos(\varphi) + \hat{y}\sin(\varphi)\}\sin(\vartheta) + \hat{z}\cos(\vartheta)$,
$\hat{\vartheta}(r, \vartheta, \varphi) = r\left[\{\hat{x}\cos(\varphi) + \hat{y}\sin(\varphi)\}\cos(\vartheta) - \hat{z}\sin(\vartheta)\right]$ *and*
$\hat{\varphi}(r, \vartheta, \varphi) = r\left[-\hat{x}\sin(\varphi) + \hat{y}\cos(\varphi)\right]\sin(\vartheta)$.

10 The metric of the local coordinates

The metric g of the new coordinates is, by analogy of formula (76), determined by the inner products of the local basis vectors. Using formulas (92) and (94), one finds

$$g_{k\ell}(x') \;=\; \boldsymbol{u}_k(x') \cdot \boldsymbol{u}_\ell(x') \;=\; \frac{\partial x^i}{\partial {x'}^k} \frac{\partial x^j}{\partial {x'}^\ell} \, \delta_{ij} \quad , \tag{95}$$

whereas, as in formula (91), for the inverse metric follows

$$g^{k\ell}(x') \;=\; \frac{\partial {x'}^k}{\partial x^i} \frac{\partial {x'}^\ell}{\partial x^j} \, \delta^{ij} \quad . \tag{96}$$

For the 3D spherical coordinates, we find for the metric

$$g(r,\vartheta,\varphi) \;=\; \begin{pmatrix} g_{rr} & g_{r\vartheta} & g_{r\varphi} \\ g_{\vartheta r} & g_{\vartheta\vartheta} & g_{\vartheta\varphi} \\ g_{\varphi r} & g_{\varphi\vartheta} & g_{\varphi\varphi} \end{pmatrix} \;=\; \begin{pmatrix} \hat{r}\cdot\hat{r} & \hat{r}\cdot\hat{\vartheta} & \hat{r}\cdot\hat{\varphi} \\ \hat{\vartheta}\cdot\hat{r} & \hat{\vartheta}\cdot\hat{\vartheta} & \hat{\vartheta}\cdot\hat{\varphi} \\ \hat{\varphi}\cdot\hat{r} & \hat{\varphi}\cdot\hat{\vartheta} & \hat{\varphi}\cdot\hat{\varphi} \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\vartheta) \end{pmatrix} \tag{97}$$

11 Differentiation with respect to the local basis

For differentiation with respect to the local coordinates we introduce a new notation, in order to simplify the formulas to come. We write:

$$F(x')_{,\,i} \;=\; \frac{\partial F(x')}{\partial {x'}^i} \quad . \tag{98}$$

In this new notation we may cast, for example, expression (95) for the local metric, in the form

$$g_{k\ell}(x') \;=\; \delta_{ij}\, {x^i}_{,\,k}\, {x^j}_{,\,\ell} \quad . \tag{99}$$

Notice however, that for the transformation (96) the notation remains as it was.

For double derivatives we have two alternatives, one of which is more compact and will be used in this notes, *i.e.*

$$F(x')_{,\,ij} \;=\; F(x')_{,\,i,\,j} \;=\; \frac{\partial^2 F(x')}{\partial {x'}^i \partial {x'}^j} \quad . \tag{100}$$

Since differentation does not depend on the order, one has moreover that

$$F(x')_{,\,ij} \;=\; F(x')_{,\,ji} \quad .$$

12 Christoffel symbols (affine connection)

In this section we study how the local basis vectors change if we move in space from one place to the other. It might be clear that the variations of the local basis vectors in such a process are completely determined once we know the derivatives of these vectors in each point. So, we restrict ourselves to infinitesimal small variations in space. Moreover, we may assume that the derivative of one of the local basis vectors can be expressed as a linear combination of the complete local basis, since locally the latter form a complete basis. The coefficients of such expression are called *affine connections*, symbol Γ. The relation is, by the use of the notation defined in formula (98), given by:

$$\boldsymbol{u}_j(x')_{,\,i} \;=\; \Gamma^k_{ij}(x')\,\boldsymbol{u}_k(x') \quad . \tag{101}$$

In order to express the Γ's in terms of derivatives, let us, also using formulas (94) and (100), consider the following expression

$$\boldsymbol{u}_j(x')_{,\,i} \;=\; \left(x^\ell{}_{,\,j}\,\boldsymbol{e}_\ell\right)_{,\,i} \;=\; x^\ell{}_{,\,ij}\,\boldsymbol{e}_\ell \;=\; x^\ell{}_{,\,ij}\,\frac{\partial x'^k}{\partial x^\ell}\,\boldsymbol{u}_k(x') \quad ,$$

from which we, by comparing with formula (101), deduce the identity

$$\Gamma^k_{ij}(x') \;=\; x^\ell{}_{,\,ij}\,\frac{\partial x'^k}{\partial x^\ell} \quad . \tag{102}$$

Notice that the affine connection is symmetric in the lower indices, since differentation is, *i.e.*

$$\Gamma^k_{ij}(x') \;=\; \Gamma^k_{ji}(x') \quad . \tag{103}$$

The non-zero components of the affine connections for the 3D *spherical coordinates, are the following*

$$\Gamma^r_{\vartheta\vartheta} \;=\; x^\ell{}_{,\,\vartheta\vartheta}\,\frac{\partial r}{\partial x^\ell} \;=\; -r\sin(\vartheta)\cos(\varphi)\frac{x}{r} - r\sin(\vartheta)\sin(\varphi)\frac{y}{r} - r\cos(\vartheta)\frac{z}{r} \;=\; -r \quad ,$$

$$\Gamma^r_{\varphi\varphi} \;=\; x^\ell{}_{,\,\varphi\varphi}\,\frac{\partial r}{\partial x^\ell} \;=\; -r\sin(\vartheta)\cos(\varphi)\frac{x}{r} - r\sin(\vartheta)\sin(\varphi)\frac{y}{r} \;=\; -r\;\sin^2(\vartheta) \quad ,$$

$$\Gamma^\vartheta_{\varphi\varphi} \;=\; x^\ell{}_{,\,\varphi\varphi}\,\frac{\partial \vartheta}{\partial x^\ell} \;=\; -r\sin(\vartheta)\frac{x\cos(\varphi)+y\sin(\varphi)}{r^2\sqrt{x^2+y^2}}z \;=\; -\cos(\vartheta)\;\sin(\vartheta) \quad ,$$

similarly

$$\Gamma^\vartheta_{r\vartheta} \;=\; \frac{1}{r} \;=\; \Gamma^\vartheta_{\vartheta r} \quad , \quad \Gamma^\varphi_{r\varphi} \;=\; \frac{1}{r} \;=\; \Gamma^\varphi_{\varphi r} \quad , \quad \Gamma^\varphi_{\vartheta\varphi} \;=\; \frac{\cos(\vartheta)}{\sin(\vartheta)} \;=\; \Gamma^\varphi_{\varphi\vartheta} \tag{104}$$

13 The relation between the affine connection and the metric

For completeness, we repeat here the derivation of the relation between the metrical tensor (95) and the affine connection (102).

First, we determine the derivative of the components of the metrical tensor with respect to the local coordinates

$$g_{ij}(x')_{,k} = \left(\delta_{\ell m}\, x^{\ell}{}_{,i}\, x^{m}{}_{,j}\right)_{,k} = \delta_{\ell m}\, x^{\ell}{}_{,ik}\, x^{m}{}_{,j} + \delta_{\ell m}\, x^{\ell}{}_{,i}\, x^{m}{}_{,jk} \quad .$$

Next, we notice that if we take a linear combination of the above expression for the derivatives, obtained by permuting the indices (i, j, k), then we can single out one term, according to

$$\Gamma_{kij} = \frac{1}{2}\left\{ g_{ki,j} + g_{kj,i} - g_{ij,k} \right\} = \delta_{\ell m}\, x^{\ell}{}_{,ij}\, x^{m}{}_{,k} \quad . \tag{105}$$

The above quantities are the Christoffel symbols, which can be related to the affine connections (102), also using formula (96), by

$$\begin{aligned} g^{k\ell}\,\Gamma_{\ell ij} &= \frac{\partial x'^{k}}{\partial x^{p}}\frac{\partial x'^{\ell}}{\partial x^{q}}\,\delta^{pq}\,\delta_{mn}\,x^{m}{}_{,ij}\,x^{n}{}_{,\ell} \\ &= \frac{\partial x'^{k}}{\partial x^{p}}\,\delta^{pq}\,\delta^{n}_{q}\,\delta_{mn}\,x^{m}{}_{,ij} = \Gamma^{k}_{ij} \quad . \end{aligned} \tag{106}$$

Using the metrical tensor which is given in formula (97), we obtain for the non-zero components of the affine connections for the 3D spherical coordinates, the following

$$\begin{aligned} \Gamma^{r}_{\vartheta\vartheta} &= g^{rr}\,\Gamma_{r\vartheta\vartheta} = -r \quad , \quad \Gamma^{\vartheta}_{r\vartheta} = \Gamma^{\vartheta}_{\vartheta r} = g^{\vartheta\vartheta}\,\Gamma_{\vartheta r\vartheta} = \frac{1}{r} \quad , \\ \Gamma^{r}_{\varphi\varphi} &= g^{rr}\,\Gamma_{r\varphi\varphi} = -r\,\sin^{2}(\vartheta) \quad , \quad \Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = g^{\varphi\varphi}\,\Gamma_{\varphi r\varphi} = \frac{1}{r} \quad , \\ \Gamma^{\vartheta}_{\varphi\varphi} &= g^{\vartheta\vartheta}\,\Gamma_{\vartheta\varphi\varphi} = -\sin(\vartheta)\,\cos(\vartheta) \quad , \quad \Gamma^{\varphi}_{\vartheta\varphi} = \Gamma^{\varphi}_{\varphi\vartheta} = g^{\varphi\varphi}\,\Gamma_{\varphi\vartheta\varphi} = \frac{\cos(\vartheta)}{\sin(\vartheta)} \quad . \end{aligned} \tag{107}$$

14 The derivatives of a vector field

Suppose that in the N-dimensional Euclidean space (92) is defined an arbitrary vector field $\boldsymbol{v}(x)$. With respect to the global Cartesean basis $\{e\}$ let its components in some point P, which has coordinates x, be given by

$$\boldsymbol{v}(x) \;=\; v^{i}(x)\,\boldsymbol{e}_{i} \quad , \tag{108}$$

and with respect to the local basis $u(x')$ at P, which has coordinates x' in the corresponding coordinate system, by

$$\boldsymbol{v}\,(x(x')) \;=\; v'^{i}(x')\,\boldsymbol{u}_{i}(x') \quad . \tag{109}$$

In the vicinity of the point P one might wish to determine the derivatives of the vector field with respect to the local coordinates. However, since the local basis vectors u differ from place to place, also their derivatives will be involved, *i.e.*

$$\boldsymbol{v}_{,\,i} \;=\; \Big(v'^{k}\,\boldsymbol{u}_{k}\Big)_{,\,i} \;=\; v'^{k}{}_{,\,i}\,\boldsymbol{u}_{k} \;+\; v'^{j}\,\boldsymbol{u}_{j,\,i} \quad ,$$

which, by the use of (101), leads to

$$\boldsymbol{v}_{,\,i} \;=\; \Big\{v'^{k}{}_{,\,i} \;+\; v'^{j}\,\Gamma^{k}_{ij}\Big\}\,\boldsymbol{u}_{k} \quad . \tag{110}$$

Another quantity of interest is the covariant component of the derivative of the vector field. Using the definition (83) of the covariant component of a vector and formula (101), one obtains

$$\begin{aligned}\boldsymbol{v}_{,\,j}\cdot\boldsymbol{u}_{i} \;&=\; \Big(\boldsymbol{v}\cdot\boldsymbol{u}_{i}\Big)_{,\,j} \;-\; \boldsymbol{v}\cdot\boldsymbol{u}_{i,\,j} \\ &=\; v'_{i,\,j} \;-\; \Gamma^{k}_{ij}\boldsymbol{v}\cdot\boldsymbol{u}_{k} \;=\; v'_{i,\,j} \;-\; \Gamma^{k}_{ij}v'_{k} \quad .\end{aligned} \tag{111}$$

15 Covariant derivative

It is common practice to introduce a new notation for the components of the derivative of a vector field with respect to the local basis. Using the result (110), we write for those components

$$\boldsymbol{v}_{,\,i} \;=\; v'^{k}{}_{;\,i}\,\boldsymbol{u}_{k} \quad\text{with}\quad v'^{k}{}_{;\,i} \;=\; v'^{k}{}_{,\,i} \;+\; \Gamma^{k}_{ij}v'^{j} \quad , \tag{112}$$

and which are sometimes called the *covariant derivative of the contravariant components* of a vector field.

For the covariant components of a vector field, using the result (111), we write similarly its covariant derivatives by

$$v'_{i;\,j} \;=\; \boldsymbol{v}_{,\,j}\cdot\boldsymbol{u}_{i} \;=\; v'_{i,\,j} \;-\; \Gamma^{k}_{ij}v'_{k} \quad . \tag{113}$$

Part III

Three dimensions

In the following, we study curves and surfaces in three dimensions.

16 Curves in three dimensions

In the case $N = 3$, the basis vectors $\{\boldsymbol{e}\}$ and the coordinates $\{x\}$ represent the well-known Cartesean coordinates in three dimensions. In that space we parametrize an arbitrary curve by a real parameter t, such that when one moves along the curve, t changes in a continuous way from one real value to the other. Now, in formula (93) we characterized one point $\boldsymbol{x}$ in space by its coordinates $\{x\}$ at the basis $\{\boldsymbol{e}\}$. Similarly, we characterize here a whole curve by letting the coordinates depend continuously on the parameter t, *i.e.*

$$\boldsymbol{x}(t) \;=\; x^i(t)\,\boldsymbol{e}_i \quad . \tag{114}$$

Specific curves can be characterized by the adequate choice for the three functions of t, $x^1(t)$, $x^2(t)$ and $x^3(t)$. We will here only allow for curves for which those three functions are infinitely many times differentiable as functions of t.

The tangent vector in an arbitrary point of the curve is given by the first derivative in t of the curve

$$\dot{\boldsymbol{x}}(t) \;=\; \frac{d\boldsymbol{x}(t)}{dt} \;=\; \frac{dx^i(t)}{dt}\,\boldsymbol{e}_i \;=\; \dot{x}^i(t)\,\boldsymbol{e}_i \quad . \tag{115}$$

Its length is evidently related to the usual expression of the square of the length of a vector (remember that we have an orthonormal basis $\{\boldsymbol{e}\}$ here):

$$|\dot{\boldsymbol{x}}(t)|^2 \;=\; \dot{\boldsymbol{x}}(t)\cdot\dot{\boldsymbol{x}}(t) \;=\; \dot{x}^i(t)\dot{x}^j(t)\,\delta_{ij} \quad . \tag{116}$$

This quantity can also be used to measure a distance $s_2 - s_1$ of a line segment along the curve. Clearly, one must therefor integrate over the interval (t_1, t_2) in the parameter t which corresponds to that segment. We find then

$$s_2 - s_1 \;=\; \int_{t_1}^{t_2} dt\;|\dot{\boldsymbol{x}}(t)| \;=\; \int_{t_1}^{t_2} dt\,\sqrt{\dot{x}^i(t)\dot{x}^j(t)\,\delta_{ij}} \quad . \tag{117}$$

From the above expression one may deduce that an infinitesimal distance ds is determined by

$$ds^2 \;=\; dx^i\,dx^j\,\delta_{ij} \quad . \tag{118}$$

Now, since the parameter s, which is called the *proper length* along the curve, is a more convenient parametrization of the curve, than an arbitrary choice, we will use in the following s as the parameter along the curve. Derivatives with respect to s will be denoted by a prime, instead of a dot (not to be confused with the notation for a new coordinate set as defined in section (9), but that might be clear from the context).

So, expressed in the proper length parameter s, the curve is now characterized by

$$\boldsymbol{x}(s) \;=\; x^i(s)\,\boldsymbol{e}_i \quad , \tag{119}$$

and its tangent vector by

$$\boldsymbol{\tau}(s) \;=\; \boldsymbol{x}'(s) \;=\; \frac{dx^i(s)}{ds}\,\boldsymbol{e}_i \quad . \tag{120}$$

Notice, that, by our choice (118) of parametrization, the tangent vector $\boldsymbol{\tau}(s)$ has unit length in any point of the curve. This can also be seen as follows:

$$|\boldsymbol{\tau}(s)|^2 \;=\; \frac{dx^i(s)}{ds}\,\boldsymbol{e}_i \cdot \frac{dx^j(s)}{ds}\,\boldsymbol{e}_j \;=\; \frac{dx^i(s)}{ds}\,\frac{dx^j(s)}{ds}\,\delta_{ij} \;=\; \frac{dx^i dx^j \delta_{ij}}{ds^2} \;=\; \frac{ds^2}{ds^2} \;=\; 1 \quad . \tag{121}$$

For a first example, let us consider a circle of radius R in the (x, y)-plane, centered at the origin. We parametrize a point, P, of the circle by the angle, t, its position vector, $\boldsymbol{x}_P$, makes with the x-axis. Below we write our parametrization, $\boldsymbol{x}(t)$, of the position vectors of points at the circumference of the circle and their related tangent vectors, $\dot{\boldsymbol{x}}(t)$, respectively:

$$\boldsymbol{x}(t) \;=\; \begin{pmatrix} R\cos(t) \\ R\sin(t) \\ 0 \end{pmatrix} \quad \text{and} \quad \dot{\boldsymbol{x}}(t) \;=\; \begin{pmatrix} -R\sin(t) \\ R\cos(t) \\ 0 \end{pmatrix} \quad .$$

The length of the tangent vector equals R. For the length of a line segment of the curve, we find

$$s_2 - s_1 \;=\; \int_{t_1}^{t_2} dt\, R \;=\; R\,(t_2 - t_1) \quad ,$$

from which we deduce that for the proper length parameter we may select $s = Rt$. In this parametrization we have then

$$\boldsymbol{x}(s) \;=\; \begin{pmatrix} R\cos(s/R) \\ R\sin(s/R) \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\tau}(s) \;=\; \boldsymbol{x}'(s) \;=\; \begin{pmatrix} -\sin(s/R) \\ \cos(s/R) \\ 0 \end{pmatrix} \quad . \tag{122}$$

Notice that $\boldsymbol{x}'(s)$ has unit length.

For a second example, we study a circular helix, or screw, of radius a and constant speed b centered around the z-axis. For the parametrization of the position vectors, $\boldsymbol{x}(t)$*, of points at this curve and their related tangent vectors,* $\dot{\boldsymbol{x}}(t)$*, we choose respectively:*

$$\boldsymbol{x}(t) \; = \; \begin{pmatrix} a\cos(t) \\ a\sin(t) \\ bt \end{pmatrix} \quad \text{and} \quad \dot{\boldsymbol{x}}(t) \; = \; \begin{pmatrix} -a\sin(t) \\ a\cos(t) \\ b \end{pmatrix} \quad .$$

The length of the tangent vector equals $\sqrt{a^2+b^2}$*. For the length of a line segment of the curve, we find*

$$s_2 - s_1 \; = \; \int_{t_1}^{t_2} dt\, \sqrt{a^2+b^2} \; = \; \sqrt{a^2+b^2}\,(t_2 - t_1) \quad ,$$

from which we deduce that for the proper length parameter we may select $s = \sqrt{a^2+b^2}t$*. In this parametrization we have then*

$$\boldsymbol{x}(s) \; = \; \begin{pmatrix} a\cos\left(\dfrac{s}{\sqrt{a^2+b^2}}\right) \\ a\sin\left(\dfrac{s}{\sqrt{a^2+b^2}}\right) \\ \dfrac{bs}{\sqrt{a^2+b^2}} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\tau}(s) \; = \; \begin{pmatrix} \dfrac{-a}{\sqrt{a^2+b^2}}\sin\left(\dfrac{s}{\sqrt{a^2+b^2}}\right) \\ \dfrac{a}{\sqrt{a^2+b^2}}\cos\left(\dfrac{s}{\sqrt{a^2+b^2}}\right) \\ \dfrac{b}{\sqrt{a^2+b^2}} \end{pmatrix} \quad . \tag{123}$$

Notice that $\boldsymbol{x}'(s)$ *has unit length.*

17 The natural local basis of a curve

In this section we will construct a set of three vectors at each point $P(s)$ along the curve, which serves as a local basis for the Euclidean three-dimensional space in the vicinity of $P(s)$.

The first vector of this set is the unit tangent vector $\boldsymbol{\tau}(s)$, defined in formula (120). For the second vector one may select the derivative of the tangent vector, normalized to unity

$$\boldsymbol{h}(s) \; = \; \frac{\dfrac{d\boldsymbol{\tau}(s)}{ds}}{\left|\dfrac{d\boldsymbol{\tau}(s)}{ds}\right|} \quad . \tag{124}$$

The tangent vector $\boldsymbol{\tau}(s)$ and its derivative $\boldsymbol{\tau}'(s)$ are orthogonal in any point of the curve, as can be seen, also using formula (121), from

$$0 \; = \; \frac{d1}{ds} \; = \; \frac{d\left\{\boldsymbol{\tau}(s)\cdot\boldsymbol{\tau}(s)\right\}}{ds} \; = \; 2\,\boldsymbol{\tau}'(s)\cdot\boldsymbol{\tau}(s) \quad . \tag{125}$$

The variation of the tangent vector along the curve in the vicinity of a point $P(s)$ indicates the amount of curvature of the curve at that point. In order to see this more explicitly, let us determine the difference between the tangent vector at the point $P(s)$ and the tangent vector at the point $P(s + \Delta s)$. To first order this difference is given by

$$\boldsymbol{\tau}(s+\Delta s) \; - \; \boldsymbol{\tau}(s) \; \approx \; \boldsymbol{\tau}'(s)\,\Delta s \quad . \tag{126}$$

Now, since the tangent vectors have unit length, the modulus of the above difference yields, also to first order, the angle $\alpha(s)$ between the two vectors at location $P(s)$, *i.e.*

$$|\boldsymbol{\tau}(s+\Delta s) \; - \; \boldsymbol{\tau}(s)| \; \approx \; \alpha(s) \quad . \tag{127}$$

Moreover, the curvature radius of the curve at location $P(s)$ multiplied by the angle $\alpha(s)$ equals the distance along the curve between the two points $P(s)$ and $P(s+\Delta s)$. Whereas, furthermore the length of this line segment equals Δs, since the parametrization (119) corresponds to the length of line segments on the curve. Hence, one finds

$$|\Delta s| \; \approx \; \frac{\alpha(s)}{\kappa(s)} \quad , \tag{128}$$

where $\kappa(s)$ stands for the inverse of the radius of curvature at location $P(s)$. The choice to parametrize curvature by the inverse of the radius, rather than by the radius itself, is quite logical, since then one has vanishing parameter in the absence of curvature.

Now, by joining the three pieces (126), (127) and (128), we obtain the relation

$$\begin{aligned} |\Delta s| \; &\approx \; \frac{\alpha(s)}{\kappa(s)} \; \approx \; \frac{|\boldsymbol{\tau}(s+\Delta s) \; - \; \boldsymbol{\tau}(s)|}{\kappa(s)} \\ &\approx \; \frac{|\boldsymbol{\tau}'(s)|\;|\Delta s|}{\kappa(s)} \quad , \end{aligned}$$

from which we consequently may conclude that the derivative of the tangent vector (120) and the normalized derivative (124), are related via the parameter of local curvature, by

$$\boldsymbol{\tau}'(s) \ = \ \kappa(s)\,\boldsymbol{h}(s) \quad . \tag{129}$$

The third vector $\boldsymbol{b}(s)$ of the natural local basis at $P(s)$ is given by the outer product of $\boldsymbol{\tau}(s)$ and $\boldsymbol{h}(s)$, *i.e.*

$$\boldsymbol{b}(s) \ = \ \boldsymbol{\tau}(s) \ \times \ \boldsymbol{h}(s) \quad . \tag{130}$$

Since both $\boldsymbol{\tau}(s)$ and $\boldsymbol{h}(s)$ are unit vectors and moreover orthogonal, it is clear from the definition (130) that also $\boldsymbol{b}(s)$ must be unity. Hence, the natural local basis vectors $\boldsymbol{\tau}(s)$, $\boldsymbol{h}(s)$ and $\boldsymbol{b}(s)$ form an orthonormal set.

For the circle of example (122) we obtain for the derivative of the tangent vector

$$\boldsymbol{\tau}'(s) \ = \ \begin{pmatrix} -\frac{1}{R}\cos(s/R) \\ -\frac{1}{R}\sin(s/R) \\ 0 \end{pmatrix} \quad \text{and} \quad \kappa(s) \ = \ |\boldsymbol{\tau}'(s)| \ = \ \frac{1}{R} \quad , \tag{131}$$

which leads for $\boldsymbol{h}(s)$ *and* $\boldsymbol{b}(s)$ *to*

$$\boldsymbol{h}(s) \ = \ \begin{pmatrix} -\cos(s/R) \\ -\sin(s/R) \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}(s) \ = \ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ = \ \hat{z} \quad . \tag{132}$$

For the circular helix of example (123) we obtain for the derivative of the tangent vector

$$\boldsymbol{\tau}'(s) \ = \ \frac{a}{a^2+b^2} \begin{pmatrix} -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ 0 \end{pmatrix} \quad \text{and} \quad \kappa(s) \ = \ |\boldsymbol{\tau}'(s)| \ = \ \frac{a}{a^2+b^2} \quad . \tag{133}$$

Notice that in the limit $b = 0$*, we return to the case of the circle for* $R = a$*. In the limit of* $b \to \infty$ *one has a straight line parallel to the* z*-axis with vanishing curvature.*
For $\boldsymbol{h}(s)$ *and* $\boldsymbol{b}(s)$ *one has for the circular helix*

$$\boldsymbol{h}(s) \ = \ \begin{pmatrix} -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}(s) \ = \ \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} b\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ -b\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ a \end{pmatrix} \quad . \tag{134}$$

18 The derivatives of the natural basis

Knowledge of the derivatives of the natural local basis vectors $\boldsymbol{\tau}(s)$, $\boldsymbol{h}(s)$ and $\boldsymbol{b}(s)$ informs us about the development of this basis for displacements along the line. Below, we show that these derivatives are given by the *Frenet* relations:

$$\begin{array}{rclcrcll} \boldsymbol{\tau}'(s) & = & \kappa(s)\,\boldsymbol{h}(s) & , & \kappa(s) & = & |\boldsymbol{\tau}'(s)| & ; \\ \boldsymbol{h}'(s) & = & -\kappa(s)\,\boldsymbol{\tau}(s)\;+\;\sigma(s)\boldsymbol{b}(s) & , & \sigma(s) & = & |\boldsymbol{\tau}(s)\times\boldsymbol{h}'(s)| & ; \\ \boldsymbol{b}'(s) & = & -\sigma(s)\,\boldsymbol{h}(s) & . & & & & \end{array} \tag{135}$$

In the proof of the *Frenet* relations we use the fact that the derivative of a unit vector is always perpendicular to the unit vector itself (see formula 125). Moreover, we know that by their definitions (120), (124) and (130), $\boldsymbol{\tau}(s)$, $\boldsymbol{h}(s)$ and $\boldsymbol{b}(s)$ are unit vectors (see also formula 121). Hence, the derivatives $\boldsymbol{\tau}'(s)$, $\boldsymbol{h}'(s)$ and $\boldsymbol{b}'(s)$ are perpendicular to respectively $\boldsymbol{\tau}(s)$, $\boldsymbol{h}(s)$ and $\boldsymbol{b}(s)$.

18.1 Proof of the Frenet relations

The relation between $\boldsymbol{\tau}'(s)$ and $\boldsymbol{h}(s)$ has been studied in section (17), where also the definition of the curvature parameter $\kappa(s)$ has been given.

For the proof of the third line in formula (135), we remember that both $\boldsymbol{\tau}(s)$ and $\boldsymbol{h}'(s)$ are perpendicular to $\boldsymbol{h}(s)$. Hence, their outer product is parallel to $\boldsymbol{h}(s)$. Using definition (130) and formula (129), we find then

$$\boldsymbol{b}'(s) \;=\; \boldsymbol{\tau}'(s)\;\times\;\boldsymbol{h}(s)\;+\;\boldsymbol{\tau}(s)\;\times\;\boldsymbol{h}'(s)\;=\;\boldsymbol{\tau}(s)\;\times\;\boldsymbol{h}'(s) \quad ,$$

which is a vector parallel and opposite to $\boldsymbol{h}(s)$. Moreover, since $\boldsymbol{h}(s)$ is unity, one obtains

$$-\,|\boldsymbol{b}'(s)| \;=\; -\,|\boldsymbol{\tau}(s)\;\times\;\boldsymbol{h}'(s)| \quad ,$$

for the constant of proportionality $-\sigma(s)$. Which proofs the third of the Frenet relations.

For the second relation, we just study the innerproducts of $\boldsymbol{h}'(s)$ with $\boldsymbol{\tau}$(s) and $\boldsymbol{b}$(s), since the natural basis is orthonormal at each point $P(s)$ of the curve. Using formulas (125) and the first relation of Frenet, we obtain

$$0 \;=\; \frac{d\left\{\boldsymbol{\tau}'(s)\cdot\boldsymbol{\tau}(s)\right\}}{ds} \;=\; \boldsymbol{\tau}''(s)\cdot\boldsymbol{\tau}(s)+\boldsymbol{\tau}'(s)\cdot\boldsymbol{\tau}'(s) \;=\; \boldsymbol{\tau}''(s)\cdot\boldsymbol{\tau}(s)+\kappa^2(s) \quad ,$$

which leads to

$$\boldsymbol{h}'(s)\cdot\boldsymbol{\tau}(s) \;=\; \left(\frac{d}{ds}\,\frac{\boldsymbol{\tau}'(s)}{\kappa(s)}\right)\cdot\boldsymbol{\tau}(s) \;=\; \left\{-\kappa'(s)\frac{\boldsymbol{\tau}'(s)}{\kappa^2(s)}\;+\;\frac{\boldsymbol{\tau}''(s)}{\kappa(s)}\right\}\cdot\boldsymbol{\tau}(s) \;=\; -\kappa(s) \quad , \tag{136}$$

which proofs the first part of the second Frenet relation.

For the second part, we use the fact that $\boldsymbol{h}(s)$ and $\boldsymbol{b}(s)$ are perpendicular and the third Frenet relation, resulting to

$$\boldsymbol{h}'(s)\cdot\boldsymbol{b}(s) \;=\; \frac{d\left\{\boldsymbol{h}(s)\cdot\boldsymbol{b}(s)\right\}}{ds} \;-\; \boldsymbol{h}(s)\cdot\boldsymbol{b}'(s) \;=\; \sigma(s) \quad , \tag{137}$$

which proofs the second part of the second Frenet relation and, hence, completes the proof of the Frenet relations.

18.2 Darboux vector

For completeness, we study here the variation of the natural local basis vectors $\boldsymbol{\tau}(s)$, $\boldsymbol{h}(s)$ and $\boldsymbol{b}(s)$ for displacements along the curve. We will find that those vectors rotate around the so-called *Darboux* vector, defined by

$$\boldsymbol{d}(s) \;=\; \sigma(s)\,\boldsymbol{\tau}(s) \;+\; \kappa(s)\,\boldsymbol{b}(s) \quad . \tag{138}$$

Under a small displacement Δs along the curve one may write, to first order, the transformation of the natural local basis vectors, by

$$\begin{pmatrix} \boldsymbol{\tau}(s+\Delta s) \\ \boldsymbol{h}(s+\Delta s) \\ \boldsymbol{b}(s+\Delta s) \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{\tau}(s) \\ \boldsymbol{h}(s) \\ \boldsymbol{b}(s) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\tau}'(s) \\ \boldsymbol{h}'(s) \\ \boldsymbol{b}'(s) \end{pmatrix} \Delta s \quad ,$$

which, using formula (135), can be casted in the form

$$\begin{pmatrix} \boldsymbol{\tau}(s+\Delta s) \\ \boldsymbol{h}(s+\Delta s) \\ \boldsymbol{b}(s+\Delta s) \end{pmatrix} \approx \begin{pmatrix} \boldsymbol{\tau}(s) \\ \boldsymbol{h}(s) \\ \boldsymbol{b}(s) \end{pmatrix} + \begin{pmatrix} \cdot & \kappa(s) & \cdot \\ -\kappa(s) & \cdot & \sigma(s) \\ \cdot & -\sigma(s) & \cdot \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}(s) \\ \boldsymbol{h}(s) \\ \boldsymbol{b}(s) \end{pmatrix} \Delta s \quad .$$

The matrix in the above expression represents an infinitesimal rotation around the so-called Darboux vector, which is given in formula (138), whereas the angle of the infinitesimal rotation follows from

$$\text{rotation angle} \;\approx\; |\boldsymbol{d}(s)|\,\Delta s \;=\; \sqrt{\sigma^2(s) \;+\; \kappa^2(s)}\,\Delta s \quad . \tag{139}$$

When passing from one position to a nearby position along the curve, the natural basis vectors, while remaining a righthand orthonormal set, changes its orientation following the above described rotation.

19 A two-dimensional surface in three dimensions

In the following, we study a two-dimensional arbitrarily curved surface embedded in a three-dimensional Euclidean space. The three-dimensional space we endow with a set of orthonormal basis vectors $\{\boldsymbol{e}\}$ and coordinates $\{x\}$. We assume that the two-dimensional surface is characterized by a set of two coordinates, $\{u\}$. This means that we assume that each point P in the surface is associated with a unique set of two coordinates u^1 and u^2. We assume moreover that those coordinates vary in a continuous way when one moves from one place to a nearby position. The components of the three-dimensional vectors $\boldsymbol{x}(P)$ which connect the origin of the three-dimensional embedding space to points P on the surface, are this way functions of the two coordinates which characterize the surface, *i.e.*

$$\boldsymbol{x}_P\left(u^1,u^2\right) \;=\; x^i\left(u^1(P),u^2(P)\right)\,\boldsymbol{e}_i \;=\; x^i(u)\,\boldsymbol{e}_i \quad . \tag{140}$$

We assume then that the three functions $x^i(u)$ are infinitely many times differentiable functions of the variables u^1 and u^2.

Now, at each point of the surface we can define a tangent plane by selecting two vectors which uniquely characterize this plane. Such local tangent basis vectors can be constructed from the three-dimensional vectors associated to the points on the surface, since their derivatives with respect to the surface coordinates $\{u\}$ indicate locally the tangent directions, *i.e.*

$$\boldsymbol{a}_\alpha(u) \;=\; \boldsymbol{x}(u)_{,\,\alpha} \;=\; \frac{\partial x^i(u)}{\partial u^\alpha}\,\boldsymbol{e}_i \quad . \tag{141}$$

where $\alpha = 1$ or 2, and where i runs over 1, 2 and 3.

Notice that we indicate here with a comma differentation with respect to the arbitrary surface coordinates $\{u\}$, since we consider those the local coordinates of the two-dimensional surface (compare section 11).

With the difference that here we denote the coordinates by $\{u\}$ and the local basis vectors by $\{\boldsymbol{a}\}$, we can use formula (95) in order to determine the local metric in the vicinity of any point P. We obtain in general

$$g_{\alpha\beta}(u) \;=\; \boldsymbol{a}_\alpha(u)\cdot\boldsymbol{a}_\beta(u) \;=\; \boldsymbol{x}(u)_{,\,\alpha}\cdot\boldsymbol{x}(u)_{,\,\beta} \;=\; x^i{}_{,\,\alpha}\,x^j{}_{,\,\beta}\,\delta_{ij} \quad . \tag{142}$$

Being useful later on, we complete here the set of local basis vectors by a third unit vector, normal to the plane, which thus allows for the description of points outside the two-dimensional surface in the vicinity of a point P on the surface. We define therefor

$$\boldsymbol{n}(u) \;=\; \frac{\boldsymbol{a}_1(u)\times\boldsymbol{a}_2(u)}{|\boldsymbol{a}_1(u)\times\boldsymbol{a}_2(u)|} \quad . \tag{143}$$

By the use of formula (141) we may rewrite the outer product of the above expression as follows:

$$\boldsymbol{a}_1(u)\times\boldsymbol{a}_2(u) \;=\; x^i{}_{,1}\,x^j{}_{,2}\,\boldsymbol{e}_i\times\boldsymbol{e}_j \;=\; x^i{}_{,1}\,x^j{}_{,2}\,\epsilon_{ijk}\boldsymbol{e}_k \quad . \tag{144}$$

Where, ϵ_{ijk} represents the Levi-Civita symbol, which has the following property

$$\epsilon_{ijk}\epsilon_{\ell mk} \;=\; \sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{\ell mk} \;=\; \delta_{i\ell}\delta_{jm} \;-\; \delta_{im}\delta_{j\ell} \quad . \tag{145}$$

Using this equality and the expression (142), we find for the square of the modulus of the outer product (144) the result

$$\begin{aligned}
|\boldsymbol{a}_1(u) \times \boldsymbol{a}_2(u)|^2 &= x^i{}_{,1}\, x^j{}_{,2}\, \epsilon_{ijk}\, \boldsymbol{e}_k \,\cdot\, x^\ell{}_{,1}\, x^m{}_{,2}\, \epsilon_{\ell mn}\, \boldsymbol{e}_n \\
&= |\boldsymbol{x}_{,1}|^2\, |\boldsymbol{x}_{,2}|^2 \,-\, (\boldsymbol{x}_{,1} \cdot \boldsymbol{x}_{,2})^2 \\
&= g_{11}(u)\, g_{22}(u) \,-\, g_{12}(u)\, g_{21}(u) \\
&= \det\,(\text{metrical tensor}) \;=\; g(u) \quad . \qquad (146)
\end{aligned}$$

Using this result, we obtain for the local normal vector (143) the form

$$\boldsymbol{n}(u) \;=\; \frac{\boldsymbol{a}_1(u) \times \boldsymbol{a}_2(u)}{\sqrt{g(u)}} \quad . \qquad (147)$$

Notice, as may be obvious, that

$$\boldsymbol{n}(u) \,\cdot\, \boldsymbol{a}_1(u) \;=\; \boldsymbol{n}(u) \,\cdot\, \boldsymbol{a}_2(u) \;=\; 0 \quad . \qquad (148)$$

Also for later use we determine here the innerproduct of the local normal vector and an arbirary three-vector, *i.e.*

$$\sqrt{g(u)}\boldsymbol{v} \cdot \boldsymbol{n}(u) \;=\; v^i\, x^j{}_{,1}\, x^k{}_{,2}\, \epsilon_{jk\ell}\, \boldsymbol{e}_\ell \cdot \boldsymbol{e}_i \;=\; \boldsymbol{v} \cdot (\boldsymbol{x}_{,1} \times \boldsymbol{x}_{,2}) \quad . \qquad (149)$$

20 The derivatives of the local basis

In this section we study the derivatives of the local basis on the two-dimensional surface with respect to the local coordinates $\{u\}$. It is worthwhile to notice from the start that, because of the way the tangent vecors $\{\boldsymbol{a}\}$ are defined in formula (141), one has that

$$\boldsymbol{a}_{\alpha,\,\beta}(u) \;=\; \boldsymbol{x}(u)_{,\,\alpha\beta} \quad , \tag{150}$$

is *symmetric* in the indices α and β.

From formula (142) we deduce that

$$\begin{aligned} g_{\alpha\beta,\,\mu}(u) \;&=\; \boldsymbol{a}_{\alpha,\,\mu}(u)\cdot\boldsymbol{a}_{\beta}(u) \;+\; \boldsymbol{a}_{\alpha}(u)\cdot\boldsymbol{a}_{\beta,\,\mu}(u) \\ &=\; \boldsymbol{x}(u)_{,\,\alpha\mu}\cdot\boldsymbol{x}(u)_{,\,\beta} \;+\; \boldsymbol{x}(u)_{,\,\alpha}\cdot\boldsymbol{x}(u)_{,\,\beta\mu} \quad , \end{aligned}$$

for which, moreover using expression (105), we obtain for the Christoffel symbols of the two-dimensional surface the expression

$$\begin{aligned} \Gamma_{\mu\alpha\beta}(u) \;&=\; \frac{1}{2}\left\{ g_{\mu\alpha,\,\beta} \;+\; g_{\mu\beta,\,\alpha} \;-\; g_{\alpha\beta,\,\mu}\right\} \\ &=\; \boldsymbol{x}_{,\,\alpha\beta}(u)\cdot\boldsymbol{x}_{,\,\mu}(u) \;=\; \boldsymbol{a}_{\alpha,\,\beta}(u)\cdot\boldsymbol{a}_{\mu}(u) \quad . \end{aligned} \tag{151}$$

In order to determine the affine connections from formula (102), we need moreove the inverse of the local metric in the two-dimensional surface, the components of which we denote here with upper indices as before (see formula 90). We write then

$$\Gamma^{\mu}_{\alpha\beta}(u) \;=\; g^{\mu\sigma}(u)\,\Gamma_{\sigma\alpha\beta}(u) \quad . \tag{152}$$

Now, let us write for the derivative of a local tangent vector (141), the following decomposition in terms of the three local basis vectors

$$\boldsymbol{a}_{\alpha,\,\beta} \;=\; A^{\mu}_{\alpha\beta}\,\boldsymbol{a}_{\mu} \;+\; B_{\alpha\beta}\,\boldsymbol{n} \quad .$$

The coefficients $A^{\mu}_{\alpha\beta}$ are readily determined by the use of formulas (79), (80), (90), (151), (152) and (148). One obtains

$$\begin{aligned} A^{\mu}_{\alpha\beta} \;&=\; \delta^{\mu}_{\nu}\,A^{\nu}_{\alpha\beta} \;=\; g^{\mu\sigma}\,g_{\sigma\nu}\,A^{\nu}_{\alpha\beta} \;=\; g^{\mu\sigma}\,\boldsymbol{a}_{\sigma}\cdot\boldsymbol{a}_{\nu}\,A^{\nu}_{\alpha\beta} \\ &=\; g^{\mu\sigma}\,\boldsymbol{a}_{\sigma}\cdot\boldsymbol{a}_{\alpha,\,\beta} \;=\; g^{\mu\sigma}\,\Gamma_{\sigma\alpha\beta} \;=\; \Gamma^{\mu}_{\alpha\beta} \quad . \end{aligned}$$

Remembering that the local unit normal vector $\boldsymbol{n}$ is normal to both local tangent vectors, we define here the *torsion tensor* L by

$$L_{\alpha\beta}(u) \;=\; \boldsymbol{a}_{\alpha,\,\beta}(u)\cdot\boldsymbol{n}(u) \;=\; B_{\alpha\beta} \quad . \tag{153}$$

Notice, that because of property (150) the torsion tensor is symmetric in its indices. Furthermore, by the use of formulas (141) and (149), we may cast the expression (153) in the following form

$$L_{\alpha\beta}(u) \;=\; \boldsymbol{x}_{,\,\alpha\beta}(u)\cdot\boldsymbol{n}(u) \;=\; \boldsymbol{x}_{,\,\alpha\beta}(u)\;\cdot\;\frac{\boldsymbol{x}_{,1}(u)\times\boldsymbol{x}_{,2}(u)}{\sqrt{g(u)}} \quad . \tag{154}$$

We obtain then for the decompostion of the derivative of the local tangent vectors, the expression

$$\boldsymbol{a}_{\alpha,\,\beta}(u) \;=\; \Gamma^{\mu}_{\alpha\beta}(u)\,\boldsymbol{a}_{\mu}(u) \;+\; L_{\alpha\beta}(u)\,\boldsymbol{n}(u) \quad . \tag{155}$$

For the derivatives of the local normal we first remember that $\boldsymbol{n}$ is a unit vector, which according to formula (125) implies that $\boldsymbol{n}$ is perpendicular to its derivative and hence only has components in the tangent plane. Let

$$\boldsymbol{n}_{,\,\alpha} \;=\; N^{\beta}_{\;\alpha}\,\boldsymbol{a}_{\beta} \quad ,$$

then, using once more formulas (79) and (80), and moreover definition (153), one deduces

$$\begin{aligned} N^{\beta}_{\;\alpha} \;&=\; g^{\beta\nu}\,N_{\nu\alpha} \;=\; g^{\beta\nu}\,\boldsymbol{n}_{,\,\alpha}\cdot\boldsymbol{a}_{\nu} \;=\; g^{\beta\nu}\,\left\{(\boldsymbol{n}\cdot\boldsymbol{a}_{\nu})_{,\,\alpha} \;-\; \boldsymbol{n}\cdot\boldsymbol{a}_{\nu,\,\alpha}\right\} \\ &=\; -g^{\beta\nu}\,\boldsymbol{n}\cdot\boldsymbol{a}_{\nu,\,\alpha} \;=\; -g^{\beta\nu}\,L_{\nu\alpha} \quad . \end{aligned}$$

Hence,

$$\boldsymbol{n}_{,\,\alpha}(u) \;=\; -g^{\beta\nu}\,L_{\nu\alpha}\,\boldsymbol{a}_{\beta}(u) \quad . \tag{156}$$

21 Curves on the surface

As we have seen before, in section (16), we give the name *curve* to a one-dimensional subspace, which is continuous and thus can be parametrized by real numbers and which moreover is differentiable, as many times as we desire. What we have in mind here is something like a smooth line drawn with an infinitesimally sharp pencil on a piece of paper. Furthermore, what we actually only needed from those curves is a small segment in the neighbourhood of a well defined point of space, as explored in section (17).

A curve embedded in the two-dimensional surface defined in section (19), can obviously be parametrized by a parameter, t, such that the surface coordinates, u, which indicate points of the curve, become functions, $u(t)$, of t. We consider here curves in the surface which are infinitely many times differentiable.

A curve at the two-dimensional surface is thus charaterized by

$$\boldsymbol{x}(t) \;=\; \boldsymbol{x}(u(t)) \;=\; x^{i}(u(t))\boldsymbol{e}_{i} \quad . \tag{157}$$

The tangent to the curve at the point $P(t)$ is, by the use of equation (141), given by

$$\dot{\boldsymbol{x}}(t) \;=\; \frac{du^{\alpha}}{dt} x^{i}{}_{,\alpha}(u(t))\boldsymbol{e}_{i} \;=\; \frac{du^{\alpha}}{dt}\, \boldsymbol{a}_{\alpha}(u(t)) \quad . \tag{158}$$

Expression (158) shows nicely the fact that the vectors $\{\boldsymbol{a}(u)\}$ span locally the complete tangent space, as indeed an arbitrary tangent vector, tangent to an arbitrary curve in the surface, can be written as a linear combination of the vectors (141).

In section (16) we introduced the notion of the proper length, which we denoted by s (see formula 118) and which measures distances of line segments along the curve by means of the integral over the lengths of infinitesimal line elements in the directions of the local tangent vectors of the curve. Here, we repeat this procedure.

The length of the local tangent vector (158) is here, by the use of formula (142), given by

$$|\dot{\boldsymbol{x}}(t)|^{2} \;=\; \frac{du^{\alpha}}{dt}\frac{du^{\beta}}{dt}\, \boldsymbol{a}_{\alpha}(u(t)) \cdot \boldsymbol{a}_{\beta}(u(t)) \;=\; \frac{du^{\alpha}}{dt}\frac{du^{\beta}}{dt}\, g_{\alpha\beta}(u(t)) \quad . \tag{159}$$

So, following the procedure of section (16), we obtain for the proper length of the above defined curve the expression

$$ds^{2} \;=\; du^{\alpha}\, du^{\beta}\, g_{\alpha\beta}(u(t)) \quad . \tag{160}$$

Clearly, we were up to introduce the proper length s for parametrizing the curve.

As before (see section 16), we define here the local unit tangent vector $\boldsymbol{\tau}$ at the point $P(s)$ by

$$\boldsymbol{\tau}(s) \;=\; \boldsymbol{x}'(u(s)) \;=\; \frac{du^{\alpha}}{ds}\, \boldsymbol{a}_{\alpha}(u(s)) \quad . \tag{161}$$

Notice that, since $\boldsymbol{\tau}(s)$ has unit length and is, moreover, a vector in the tangent plane of the surface at point $P(s)$, all curves which have at $P(s)$ the same tangential direction, share the same tangent vector $\boldsymbol{\tau}(s)$ when parametrized by their proper lengths. Furthermore

$$1 \;=\; |\boldsymbol{\tau}(s)|^{2} \;=\; |\boldsymbol{x}'(u(s))|^{2} \;=\; \frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\, g_{\alpha\beta}(u(s)) \quad . \tag{162}$$

In section (17) we saw that the variations of the local tangent vector for small displacements, measure the curvature of the curve. From equation (161), also using formula (155) for the derivatives of the local basis vectors, we obtain for the derivatives of the local tangent vector the following

$$\begin{aligned}
\boldsymbol{\tau}'(s) &= \frac{d^2u^\alpha}{ds^2}\,\boldsymbol{a}_\alpha(u(s)) \,+\, \frac{du^\alpha}{ds}\,\boldsymbol{a}_\alpha{}'(u(s)) \\
&= \frac{d^2u^\alpha}{ds^2}\,\boldsymbol{a}_\alpha(u(s)) \,+\, \frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\,\boldsymbol{a}_{\alpha,\,\beta}(u) \\
&= \frac{d^2u^\alpha}{ds^2}\,\boldsymbol{a}_\alpha(u(s)) \,+\, \frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\,\left\{\Gamma^\mu_{\alpha\beta}(u)\,\boldsymbol{a}_\mu(u) \,+\, L_{\alpha\beta}(u)\,\boldsymbol{n}(u)\right\} \\
&= \left\{\frac{d^2u^\mu}{ds^2} \,+\, \Gamma^\mu_{\alpha\beta}(u)\,\frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\right\}\boldsymbol{a}_\mu(u) \,+\, L_{\alpha\beta}(u)\,\frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\,\boldsymbol{n}(u) \quad .
\end{aligned} \tag{163}$$

which is in general neither a vector of the tangent plane, tangent to the surface at $P(s)$, nor a vector normal to that plane.

A quantity of interest is the innerproduct of the local normal, $\boldsymbol{n}(u(s))$, normal to the tangent plane, and $\boldsymbol{\tau}'(s)$. We find, using formula (163) and the fact that $\boldsymbol{n}(u(s))$ is normal to the local tangent vectors $\{\boldsymbol{a}\}$, the following

$$\boldsymbol{\tau}'(s)\cdot\boldsymbol{n}(u(s)) \,=\, \frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\,L_{\alpha\beta}(u(s)) \quad . \tag{164}$$

22 The curvature of the curves on the surface

In the previous section we studied the local tangent vectors, $\boldsymbol{\tau}$, tangent to curves on the two-dimensional surface which pass at location P and their derivatives, $\boldsymbol{\tau}'$. From expression (163) it is clear that the derivatives are, in general, not normal to the local tangent plane. Furthermore, by its definition (143), the local normal vector, $\boldsymbol{n}$, normal to the local tangent plane, is unity. Moreover, from section (17) we learn, as expressed in formula (129), that the length of the derivative of the local tangent vector is equal to the inverse, κ, of the radius of curvature of the curve at location P. Consequently, the innerproduct of $\boldsymbol{\tau}'$ and $\boldsymbol{n}$ must be equal to κ times the cosine of the angle, say ϑ, which the two vectors make. Also using formula (164), we obtain for a specific curve, parametrized by its proper length parameter s, the following

$$k(\text{curve}) \;=\; \kappa\cos(\vartheta) \;=\; L_{\alpha\beta}(u(s))\,\frac{du^{\alpha}}{ds}\,\frac{du^{\beta}}{ds} \quad . \tag{165}$$

From its definition (154), we must conclude that the torsion matrix does not depend on a specific choice of curve at P on the two-dimensional surface, but only on the properties of the surface itself at location P. Now, as discussed before (see the text following formula 161), infinitely many curves on the surface share at P the same tangent vector $\boldsymbol{\tau}$. Consequently, when we restrict ourselves to all curves whith the same tangential direction in P, which, according to formula (161), means a particular choice for the two values of du/ds, then these curves share the same value for the quantity k as defined in formula (165). The curvature, κ, of such class of curves in P differs from one curve to the other. But, then also the angle ϑ is different for each curve, in such a way that k is constant.

Hence, we may conclude that k is a function of the tangential direction only, not of a particular choice of curve out of the above described class of curves at the point P. We write therefor:

$$k \;=\; k(\boldsymbol{\tau}) \quad . \tag{166}$$

Out of the class of curves which belong to a certain tangential direction, $\boldsymbol{\tau}$, we select one representative: the curve which in the vicinity of P is defined by the cross section between the two-dimensional surface and the plane defined by the local normal vector $\boldsymbol{n}$ and the local tangent vector $\boldsymbol{\tau}$. For that curve, when parametrized by its proper length parameter s, $\boldsymbol{\tau}'(s)$ must at $P(s)$ be parallel (or antiparallel) to the normal vector, since it will not have a component out of the above defined plane, when the curve is entirely inside that plane, and $\boldsymbol{\tau}'(s)$ also has to be perpendicular to $\boldsymbol{\tau}(s)$. This representative curve corresponds to a *geodesic* at the point P, as we will see later on.

So, for a geodesic at P, we have from formula (165) the following result

$$\vartheta \;=\; 0 \;\text{ or }\; \pi \;\;, \quad \text{and} \quad k(\boldsymbol{\tau}(s)) \;=\; \pm\,\kappa \quad . \tag{167}$$

23 The curvature of the surface

From the previous section we learn that the radius of curvature, $R(\boldsymbol{\tau}(s))$, of the geodesic in the direction $\boldsymbol{\tau}(s)$ at location P, is given by

$$\frac{\pm 1}{R(\boldsymbol{\tau}(s))} = k(\boldsymbol{\tau}(s)) = L_{\alpha\beta}(u(s))\,\frac{du^{\alpha}}{ds}\,\frac{du^{\beta}}{ds} \quad . \tag{168}$$

We have inserted the $\pm$ sign because at this stage it is not clear whether the angle ϑ of formula (165) equals 0 or π for our choice of the local normal $\boldsymbol{n}(s)$.

By inspection of all possible tangent directions, we find that the corresponding values of k have two extrema, which we denote by k_1 and k_2. We study this in the following. In order to simplify the necessary algebra, we define

$$r = \frac{du^1}{ds}\left(\frac{du^2}{ds}\right)^{-1} \quad , \tag{169}$$

$$L = L_{11}(u) \quad , \quad M = L_{12}(u) = L_{21}(u) \quad , \quad N = L_{22}(u) \quad , \tag{170}$$

and

$$E = g_{11}(u) \quad , \quad F = g_{12}(u) = g_{21}(u) \quad , \quad G = g_{22}(u) \quad . \tag{171}$$

When we substitute the above definitions (169- 171) in formula (168), using moreover relation (162), then we obtain

$$k(\boldsymbol{\tau}) = \frac{L_{\alpha\beta}(u(s))\,\frac{du^{\alpha}}{ds}\,\frac{du^{\beta}}{ds}}{g_{\alpha\beta}(u(s))\,\frac{du^{\alpha}}{ds}\,\frac{du^{\beta}}{ds}} = \frac{Lr^2 + 2Mr + N}{Er^2 + 2Fr + G} \quad . \tag{172}$$

The parameter r, defined in formula (169), parametrizes the various directions of tangent vector. So, apparently, our problem is now reduced to finding the extrema of expression (172). At the values of r, which we denote by ρ, for which those extrema occur, one has

$$\left.\frac{dk}{dr}\right|_{r=\rho} = 0 \quad ,$$

which leads to the following quadratic equations for ρ

(a) $\quad (L\rho + M)\left(E\rho^2 + 2F\rho + G\right) - (E\rho + F)\left(L\rho^2 + 2M\rho + N\right) = 0 \quad ,$

(b) $\quad (FL - EM)\,\rho^2 + (GL - EN)\,\rho + (GM - FN) = 0 \quad ,$

(c) $\quad \left(F\frac{L\rho + M}{E\rho + F} - M\right)^2 - \left(E\frac{L\rho + M}{E\rho + F} - L\right)\left(G\frac{L\rho + M}{E\rho + F} - N\right) = 0 \quad .$

When, next, we denote an extremum value for k by $\bar{k}$, then we have, by substituting a solution of the above equation in formula (172), the expression

$$\bar{k} \;=\; \frac{L\rho^2 \;+\; 2M\rho \;+\; N}{E\rho^2 \;+\; 2F\rho \;+\; G} \;=\; \frac{L\rho \;+\; M}{E\rho \;+\; F} \quad ,$$

and its inverse relation

$$\rho \;=\; -\frac{F\bar{k} \;-\; M}{E\bar{k} \;-\; L} \quad , \tag{173}$$

which, when inserted in the quadratic equation (c) for ρ, leads to a quadratic equation for the extrema of k, *i.e.*

$$(F\bar{k} \;-\; M)^2 \;-\; (E\bar{k} \;-\; L)(G\bar{k} \;-\; N) \;=\; 0 \quad . \tag{174}$$

The equation (174) has in general two solutions, k_1 and k_2, which, also using formulas (170) and (171), are characterized by

$$K \;=\; k_1 k_2 \;=\; \frac{LN - M^2}{EG - F^2} \;=\; \frac{\det(L)}{\det(g)} \;=\; \det\left(g^{-1}L\right) \quad , \tag{175}$$

and by

$$2H \;=\; k_1 + k_2 \;=\; \frac{GL - 2FM + EN}{EG - F^2} \;=\; g^{\alpha\beta}\, L_{\alpha\beta} \;=\; \mathrm{Tr}\left(g^{-1}L\right) \quad . \tag{176}$$

It is common practice to define the curvature of the surface at a point P by the product (175) of the two extremum values for k. The parameter H is called the average curvature at P.

For the tangential direction parameters r_1 and r_2, which correspond respectively to the extrema of curvature k_1 and k_2, we may proof the following relations:

$$Er_1r_2 \;+\; F\,(r_1 + r_2) \;+\; G \;=\; 0 \quad \text{and} \quad Lr_1r_2 \;+\; M\,(r_1 + r_2) \;+\; N \;=\; 0 \quad . \tag{177}$$

The proof of those relations is a matter of straightforward algebra: First, one substitutes relation (173) for r_1 and r_2 and then one uses the expressions (175) and (176) to obtain relations (177).

24 The local principle axes

The local principle axes of the surface at location P are defined by the local tangential directions, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, of those geodesics at P, which correspond to the extremum directions, r_1 and r_2, for which k has respectively the extremum values k_1 and k_2.

In this section we study those axes as well as the dependence of k on $\boldsymbol{\tau}$ at the basis $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ for the local tangent plane.

According to its definition in formula (161) we may write an arbitrary tangent vector of the tangent plane at P as a linear combination of the local tangent vectors (141). Also using formulas (142), (169) and (171), we obtain

$$\boldsymbol{\tau} = \mathcal{N}\left(r\boldsymbol{a}_1 + \boldsymbol{a}_2\right) \tag{178}$$

$$\text{with} \quad 1 = |\boldsymbol{\tau}|^2 = \mathcal{N}^2\left\{Er^2 + Fr + G\right\} \quad .$$

First, let us demonstrate that the principle axes are perpendicular, *i.e.*

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = 0 \quad . \tag{179}$$

Proof:

From formula (178) we deduce that the above innerproduct may be written in the form

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \mathcal{N}_1\mathcal{N}_2\left\{Er_1r_2 + F\left(r_1 + r_2\right) + G\right\} \quad ,$$

for which, by the use of the first of relations (177), we find vanishing result, which completes the proof.

Hence, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ form an orthonormal basis for the local tangent plane at P. Their directions are called the *local principle axes*. An arbitrary tangential direction, $\boldsymbol{\tau}$ at P can thus be written as

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1\cos(\varphi) + \boldsymbol{\tau}_2\sin(\varphi) \quad , \tag{180}$$

where the angle φ defines the direction of $\boldsymbol{\tau}$ with respect to the first principle axis.

The curvature of the geodesic in the direction of $\boldsymbol{\tau}$ is characterized by $k(\varphi)$ as defined in (168). Its relation with k_1 and k_2 is given by

$$k(\varphi) = k_1\cos^2(\varphi) + k_2\sin^2(\varphi) \quad . \tag{181}$$

Proof:

We start from expression (178), to find for (180) the form

$$\begin{aligned}\boldsymbol{\tau} &= \mathcal{N}_1\left(r_1\boldsymbol{a}_1 + \boldsymbol{a}_2\right)\cos(\varphi) + \mathcal{N}_2\left(r_2\boldsymbol{a}_1 + \boldsymbol{a}_2\right)\sin(\varphi) \\ &= \left[\mathcal{N}_1r_1\cos(\varphi) + \mathcal{N}_2r_2\sin(\varphi)\right]\,\boldsymbol{a}_1 + \left[\mathcal{N}_1\cos(\varphi) + \mathcal{N}_2\sin(\varphi)\right]\,\boldsymbol{a}_2 \quad .\end{aligned}$$

By comparing this expression to equation (178), we conclude the following relations

$$\mathcal{N}r \;=\; \mathcal{N}_1 r_1 \cos(\varphi) \;+\; \mathcal{N}_2 r_2 \sin(\varphi) \quad \text{and} \quad \mathcal{N} \;=\; \mathcal{N}_1 \cos(\varphi) \;+\; \mathcal{N}_2 \sin(\varphi) \quad . \tag{182}$$

The curvature parameter k for the tangential direction (180) can be obtained by substituting the norm $\mathcal{N}$ of formula (178) into equation (172), to give

$$k \;=\; \mathcal{N}^2 \left\{ Lr^2 + 2Mr + N \right\} \;=\; L\left(\mathcal{N}r\right)^2 + 2M\mathcal{N}\left(\mathcal{N}r\right) + N\mathcal{N}^2 \quad , \tag{183}$$

which, by substitution of moreover relations (182), leads to

$$\begin{aligned} k \;=\; & \left\{ L\left(\mathcal{N}_1 r_1\right)^2 + 2M\mathcal{N}_1\left(\mathcal{N}_1 r_1\right) + N{\mathcal{N}_1}^2 \right\} \cos^2(\varphi) \;+ \\ & + \left\{ L\left(\mathcal{N}_2 r_2\right)^2 + 2M\mathcal{N}_2\left(\mathcal{N}_2 r_2\right) + N{\mathcal{N}_2}^2 \right\} \sin^2(\varphi) \;+ \\ & + 2\mathcal{N}_1\mathcal{N}_2 \left\{ Lr_1 r_2 + M\left(r_1 + r_2\right) + N \right\} \cos(\varphi)\sin(\varphi) \quad . \end{aligned}$$

When we substitute here equation (183) for k_1 in the first term and for k_2 in the second term, and moreover the second relation of formula (177) in the third term, then we obtain result (181), which completes the proof.

25 Egregium theorem of Gauss

Relation (175) for the curvature of the surface can be entirely expressed in terms of the metrical tensor (142) and its derivatives, as we will study in this section.

In formula (155) we find the decomposition of the first derivatives of the local tangential vectors $\{\boldsymbol{a}(u)\}$, in terms of the two components in the local tangential plane and the component perpendicular to that plane. For the innerproduct of two such objects, also using formula (142), we find

$$\begin{aligned} \boldsymbol{a}_{\alpha,\,\beta}(u)\cdot\boldsymbol{a}_{\tau,\,\sigma}(u) &= \Gamma^{\mu}_{\alpha\beta}(u)\,\Gamma^{\nu}_{\tau\sigma}(u)\,\boldsymbol{a}_{\mu}(u)\cdot\boldsymbol{a}_{\nu}(u)\;+\;L_{\alpha\beta}(u)\,L_{\tau\sigma}(u)\,\boldsymbol{n}(u)\cdot\boldsymbol{n}(u) \\ &= \Gamma^{\mu}_{\alpha\beta}(u)\,\Gamma^{\nu}_{\tau\sigma}(u)\,g_{\mu\nu}(u)\;+\;L_{\alpha\beta}(u)\,L_{\tau\sigma}(u) \quad . \end{aligned}$$

From this result we deduce that

$$L_{\alpha\beta}(u)\,L_{\tau\sigma}(u)\;=\;\boldsymbol{a}_{\alpha,\,\beta}(u)\;\cdot\;\boldsymbol{a}_{\tau,\,\sigma}(u)\;-\;\Gamma^{\mu}_{\alpha\beta}(u)\,\Gamma^{\nu}_{\tau\sigma}(u)\,g_{\mu\nu}(u) \quad . \tag{184}$$

Relation (184) gives us the possibility to express the determinant of the local torsion tensor as follows

$$\begin{aligned} \det(L) &= L_{11}\,L_{22}\;-\;L_{12}\,L_{21} \\ &= \boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{2,2}\;-\;\boldsymbol{a}_{1,2}\cdot\boldsymbol{a}_{2,1}\;-\;\left\{\Gamma^{\mu}_{11}\Gamma^{\nu}_{22}\;-\;\Gamma^{\mu}_{12}\Gamma^{\nu}_{21}\right\}g_{\mu\nu} \end{aligned} \tag{185}$$

As easily can be verified, for instance by taking the double derivatives of the local metrical tensor (142) thereby remembering that $\boldsymbol{a}_{\alpha,\,\beta} = \boldsymbol{a}_{\beta,\,\alpha}$ (see definition 141 and formula 150), that the first two terms at the righthand side of formula (185) can be expressed as a linear combination of second order derivatives of the local metrical tensor, *i.e.*

$$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{2,2}\;-\;\boldsymbol{a}_{1,2}\cdot\boldsymbol{a}_{2,1}\;=\;-\frac{1}{2}\left\{g_{11,22}\;-\;2g_{12,12}\;+\;g_{22,11}\right\} \quad ,$$

Substitution of this result in formula (185) for the determinant of the local torsion tensor L, leads for the local curvature parameter (175) to the following expression:

$$K\;=\;\frac{\det(L)}{\det(g)}\;=\;\frac{1}{g}\left[-\frac{1}{2}\left\{g_{11,22}\;-\;2g_{12,12}\;+\;g_{22,11}\right\}\;-\;\left\{\Gamma^{\mu}_{11}\Gamma^{\nu}_{22}\;-\;\Gamma^{\mu}_{12}\Gamma^{\nu}_{21}\right\}g_{\mu\nu}\right] \tag{186}$$

This is an expression which only refers to the local surface coordinates $\{u^1, u^2\}$ via the local metrical tensor and its first and second order derivatives. As no reference to an embedding space is involved, expression (186) uses only the inner properties of the surface for the definition of curvature (*Egregium theorem of Gauss*).

Almost vanishing curvature is difficult to measure, as mankind struggled for millions of years to detect signs of the curvature of the Earth's surface and even then decided to only be really convinced after sailors made their trips around the Globe. However, those voyages were not necessary as the Greek civilization already had knowledge of the curvature of the Earth's surface and rough estimates of its radius. The metrical tensor can be composed by precise measurements on medium long distances and from that information alone, curvature can be determined. This is the practical result of the work of Gauss, Bólyai and Lobachevski.

26 The curvature tensor

Using the relation (155), we may determine the second derivatives of the local basis vectors $\{\boldsymbol{a}(u)\}$ of the local tangent plane, *i.e.*

$$\boldsymbol{a}_{\nu,\,\rho\sigma} \;=\; \Gamma^{\alpha}_{\nu\rho,\,\sigma}\,\boldsymbol{a}_{\alpha} \;+\; \Gamma^{\alpha}_{\nu\rho}\,\boldsymbol{a}_{\alpha,\,\sigma} \;+\; L_{\nu\rho,\,\sigma}\,\boldsymbol{n} \;+\; L_{\nu\rho}\,\boldsymbol{n}_{,\,\sigma} \quad .$$

When we take the innerproduct of this expression and one of the local tangential basis vectors, then, also using formulas (142), (151) and (156), and the fact that $\boldsymbol{n}$ is normal to the local tangent plane, then we obtain

$$\boldsymbol{a}_{\nu,\,\rho\sigma} \cdot \boldsymbol{a}_{\mu} \;=\; \Gamma^{\alpha}_{\nu\rho,\,\sigma}\,g_{\alpha\mu} \;+\; \Gamma^{\alpha}_{\nu\rho}\Gamma_{\mu\alpha\sigma} \;-\; L_{\nu\rho}\,L_{\sigma\mu} \quad .$$

When, next, we take the difference of the above equation with the same expression for ρ and σ interchanged, then, also using the symmetry properties of the afine connection and the Christoffel symbol, we find

$$L_{\nu\rho}(u)\,L_{\sigma\mu}(u) \;-\; L_{\nu\sigma}(u)\,L_{\rho\mu}(u) \;=\; R_{\mu\nu\rho\sigma}(u) \quad , \tag{187}$$

where the curvature is defined by

$$R_{\mu\nu\rho\sigma}(u) \;=\; g_{\mu\alpha}(u)\,R^{\alpha}_{\nu\rho\sigma}(u) \quad , \tag{188}$$

and where

$$R^{\alpha}_{\nu\rho\sigma} \;=\; \Gamma^{\alpha}_{\nu\rho,\,\sigma} \;-\; \Gamma^{\alpha}_{\nu\sigma,\,\rho} \;+\; \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\beta}_{\nu\rho} \;-\; \Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\nu\sigma} \quad . \tag{189}$$

The curvature tensor (188) has the following symmetry properties:

$$\begin{aligned}
R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho} \\
R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\sigma\rho} \\
R_{\mu\nu\rho\sigma} &= 0 \;\;,\quad \text{for } \rho=\sigma \\
R_{\mu\nu\rho\sigma} &= 0 \;\;,\quad \text{for } \mu=\nu \quad .
\end{aligned} \tag{190}$$

With those symmetry relations, we have for the curvature tensor at a two-dimensional surface only one independent nonzero element, *i.e.*

$$R_{1212} \;=\; -R_{1221} \;=\; R_{2121} \;=\; -R_{2112} \quad . \tag{191}$$

The local *curvature scalar* is in general defined by

$$R(u) \;=\; g^{\mu\rho}(u)\,g^{\nu\sigma}(u)\,R_{\mu\nu\rho\sigma}(u) \quad . \tag{192}$$

For a two-dimensional surface the only non-zero contributions come from the components collected in formula (191), which results for the curvature scalar then in

$$R(u) \;=\; \left\{ g^{11}\,g^{22} \;-\; g^{12}\,g^{21} \;+\; g^{22}\,g^{11} \;-\; g^{21}\,g^{12} \right\}\; R_{1212} \quad .$$

So, using also formulas (175) and (187), we find for the local curvature scalar in two dimensions

$$\begin{aligned} R(u) \;&=\; 2\mathrm{det}\left(g^{-1}\right)\; R_{1212} \;=\; 2\mathrm{det}\left(g^{-1}\right)\; \left(L_{21}\,L_{12} \;-\; L_{11}\,L_{22}\right) \\ &=\; -2\mathrm{det}\left(g^{-1}L\right) \;=\; -2K \quad . \end{aligned} \tag{193}$$

For the extension to arbitrary dimensions one chooses the curvature scalar as defined in (192) for the parameter which characterizes the deviation of the metrical space from Euclidean.

27 Geodesics

As previously discussed in the context of the curvature of curves on the two-dimensional surface embedded in Cartesian three dimensions (see section 22), one of the many curves which share the same tangential direction, $\boldsymbol{\tau}$, at a certain location P of the surface, is the geodesic curve. As before we denote the proper length at this curve by s. This curve is special, because its related derivative of the tangential direction, $\boldsymbol{\tau}'(s)$, is parallel to the local normal, $\boldsymbol{n}(u(s))$, of the surface at point $P(s)$ (see formula 167)

Since $\boldsymbol{\tau}'(s)$ is normal to the tangent plane in the case of a geodesic, the innerproduct of $\boldsymbol{\tau}'(s)$ with the local tangent basis vectors $\{\boldsymbol{a}\}$ vanishes. Using formulas (142) and (163) and the fact that $\boldsymbol{n}(u)$ is also normal to the tangent plane, we obtain for the innerproduct of $\boldsymbol{\tau}'(s)$ and $\boldsymbol{a}_\nu(u)$ the result:

$$0 \;=\; \boldsymbol{\tau}'(s)\;\cdot\;\boldsymbol{a}_\nu(u) \;=\; \frac{d^2u^\alpha}{ds^2}\,g_{\alpha\nu}(u) \;+\; \frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\,\Gamma^\sigma_{\alpha\beta}(u)\,g_{\sigma\nu}(u) \quad .$$

Notice that we have two equations here, one for each value of ν. For $\nu = 1$ we multiply this expression with the matrix element $g^{1\mu}$ of the inverse of the local metrical tensor, where μ can be any of the two possibilities, and for $\nu = 2$ we multiply the expression with $g^{2\mu}$. The sum of the two results

$$0 \;=\; \frac{d^2u^\alpha}{ds^2}\,g_{\alpha\nu}(u)\,g^{\nu\mu}(u) \;+\; \frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\,\Gamma^\sigma_{\alpha\beta}(u)\,g_{\sigma\nu}(u)g^{\nu\mu}(u) \quad ,$$

is called a *contraction*. When we moreover use formula (90), then we find

$$0 \;=\; \frac{d^2u^\mu}{ds^2} \;+\; \frac{du^\alpha}{ds}\,\frac{du^\beta}{ds}\,\Gamma^\mu_{\alpha\beta}(u) \quad , \tag{194}$$

which relation is called the *geodesic equation*.

Part IV
Examples of Riemann surfaces

In the following we study some properties of two-dimensional surfaces, embedded in three dimensions, or Riemann surfaces. The book of Coxeter [4] has a lot more details and examples. Here, we just restrict ourselves to the most obvious parametrizations for those surfaces, their tangent spaces, the geodesic equations and, when easily possible, the geodesics.

28 The cylinder

As a first example, let us study the two-dimensional surface of a cylinder, embedded in an Euclidean three-dimensional space. We let the axis of the cylinder coincide with the z-axis. For its radius we take a and for its local coordinates we select the azimuthal angle φ and z. The surface of the cylinder is then given by

$$\boldsymbol{x}(\varphi, z) = \begin{pmatrix} a\cos(\varphi) \\ a\sin(\varphi) \\ z \end{pmatrix} \quad . \tag{195}$$

Setting $u^1 = \varphi$ and $u^2 = z$ and using formula (141), we find for the local tangent vectors at location $P(\varphi, z)$

$$\boldsymbol{a}_1(\varphi, z) = \begin{pmatrix} -a\sin(\varphi) \\ a\cos(\varphi) \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{a}_2(\varphi, z) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad . \tag{196}$$

The local metrical tensor for the surface of the cylinder is then found by using the procedure which is given in formula (142). Hence

$$g(\varphi, z) = \begin{pmatrix} g_{\varphi\varphi} & g_{\varphi z} \\ g_{z\varphi} & g_{zz} \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{197}$$

From expression (197) for the local metric it follows, by the use of relations (105) and (106), that the local affine connection vanishes. Consequently, also using formulas (188) and (189), we must conclude that the cylinder surface has no curvature. This may be a surprising conclusion, but, after a little thinking it becomes evident that the result is correct: A flat piece of paper has no curvature. When we roll up this sheet of paper to a cylinder, then that procedure does not alter the intrinsic properties of the surface.

For the geodesic equations, using formula (194), we obtain accordingly

$$\frac{d^2\varphi}{ds^2} = \frac{d^2 z}{ds^2} = 0 \quad . \tag{198}$$

Hence, both $\varphi(s)$ and $z(s)$ are linear functions in s along a geodesic curve. Such solutions are the circular helices, represented by the vector $\boldsymbol{x}(s)$ of formula (123). Slightly more general solutions are given by

$$\boldsymbol{x}(s) = \begin{pmatrix} a\cos\left(\frac{s}{\sqrt{a^2+b^2}} + \varphi_0\right) \\ a\sin\left(\frac{s}{\sqrt{a^2+b^2}} + \varphi_0\right) \\ \frac{bs+c}{\sqrt{a^2+b^2}} \end{pmatrix} . \tag{199}$$

where a represents the radius of the cylinder and where b, the speed of the circular helix, c and φ_0 are constant parameters. c allows for translations in the vertical (z) direction and φ_0 for rotations around the z-axis. In the limit $b \to \infty$ one obtains

$$\boldsymbol{x}(s) \longrightarrow \begin{pmatrix} a\cos(\varphi_0) \\ a\sin(\varphi_0) \\ s \end{pmatrix} ,$$

which parametrizes straight vertical lines at the surface of the cylinder parallel to the z-axis.

It is illustrative to draw a skew straight line on a sheet of paper and roll up the sheet, to find that indeed one obtains a circular helix. This shows then that a geodesic is indeed the closest approximation to a straight line, or more precisely, the "shortest" connection along the surface between two points at the surface.

29 The Ellipsoid

Let us consider the two-dimensional surface of an ellipsoid embedded in an Euclidean three-dimensional space. The coordinates of three-space are the usual, given by x, y and z. An ellipsoid, centered at the origin, is most conveniently parametrized by the spherical angles ϑ and φ, according to

$$x \;=\; a\sin(\vartheta)\cos(\varphi) \;\;,\;\; y \;=\; a\sin(\vartheta)\sin(\varphi) \;\text{ and }\; z \;=\; b\cos(\vartheta) \quad . \tag{200}$$

These relations represent a surface which is rotationally symmetric around the z-axis, whereas its cross-section with the (x, z)-plane forms an ellips with principle axes of length $2a$ in the x-direction and $2b$ in the z-direction. For this two-dimensional surface we select, naturally, the following coordinates

$$u^1 \;=\; \vartheta \;\text{ and }\; u^2 \;=\; \varphi \quad . \tag{201}$$

Using formula (141), we find for the local tangential vectors at a certain point $P(\vartheta, \varphi)$, the result

$$\boldsymbol{a}_1(u) \;=\; \begin{pmatrix} a\cos(\vartheta)\cos(\varphi) \\ a\cos(\vartheta)\sin(\varphi) \\ -b\sin(\vartheta) \end{pmatrix} \;\text{ and }\; \boldsymbol{a}_2(u) \;=\; \begin{pmatrix} -a\sin(\vartheta)\sin(\varphi) \\ a\cos(\vartheta)\cos(\varphi) \\ 0 \end{pmatrix} \quad . \tag{202}$$

The metrical tensor for the surface of ellipsoid (200 is obtained by the procedure of formula (142). One finds

$$g(u) \;=\; \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \;=\; \begin{pmatrix} a^2\cos^2(\vartheta) \;+\; b^2\sin^2(\vartheta) & 0 \\ 0 & a^2\sin^2(\vartheta) \end{pmatrix} \quad . \tag{203}$$

The local afine connection follows from the relations (105) and (106). One finds from equation (203) for the non-zero components

$$\Gamma^1_{11} \;=\; \frac{(b^2 - a^2)\sin(\vartheta)\cos(\vartheta)}{a^2\cos^2(\vartheta) \;+\; b^2\sin^2(\vartheta)} \quad ,$$

$$\Gamma^1_{22} \;=\; \frac{-a^2\sin(\vartheta)\cos(\vartheta)}{a^2\cos^2(\vartheta) \;+\; b^2\sin^2(\vartheta)} \quad \text{and}$$

$$\Gamma^2_{12} \;=\; \Gamma^2_{12} \;=\; \frac{\cos(\vartheta)}{\sin(\vartheta)} \quad . \tag{204}$$

As we may observe from formula (203), the local basis vectors $\{\boldsymbol{a}(u)\}$ of the tangent plane are orthogonal to each other. Hence, when we consider a geodesic curve $\boldsymbol{x}(s)$ at $P(u)$, parametrized by its proper length, s, then we can parametrize the corresponding tangent vector, $\boldsymbol{\tau}(s)$, at $P(u)$ by the angle, α, it makes with $\boldsymbol{a}_1$, according to

$$\boldsymbol{\tau}(s) \;=\; \frac{\boldsymbol{a}_1}{|\boldsymbol{a}_1|}\cos(\alpha) \;+\; \frac{\boldsymbol{a}_2}{|\boldsymbol{a}_2|}\sin(\alpha) \quad . \tag{205}$$

Consequently, following definition (161), we must conclude that along the geodesic curve one has for the derivatives of the local coordinates, also using formula (202) the following

$$\frac{d\vartheta}{ds} = \frac{\cos(\alpha)}{|\boldsymbol{a}_1|} = \frac{\cos(\alpha)}{\sqrt{a^2\cos^2(\vartheta) + b^2\sin^2(\vartheta)}} \quad \text{and}$$

$$\frac{d\varphi}{ds} = \frac{\sin(\alpha)}{|\boldsymbol{a}_2|} = \frac{\sin(\alpha)}{a\sin(\vartheta)} \quad . \tag{206}$$

Now, we also know that the geodesics follow the geodesic equation (194), which allows us to find the second order derivatives of the local coordinates along the geodesic curve, from the first order derivatives (206), also using equations (204) as follows

$$\begin{aligned}\frac{d^2\vartheta}{ds^2} &= -\Gamma^1_{11}\left(\frac{d\vartheta}{ds}\right)^2 - \Gamma^1_{22}\left(\frac{d\varphi}{ds}\right)^2 \\ &= \frac{\operatorname{cotg}(\vartheta)\sin^2(\alpha)}{a^2\cos^2(\vartheta) + b^2\sin^2(\vartheta)} - \frac{(b^2-a^2)\sin(\vartheta)\cos(\vartheta)\cos^2(\alpha)}{\left\{a^2\cos^2(\vartheta) + b^2\sin^2(\vartheta)\right\}^2}\end{aligned}$$

and

$$\frac{d^2\varphi}{ds^2} = -2\Gamma^2_{12}\frac{d\vartheta}{ds}\frac{d\varphi}{ds} = \frac{-2\cos(\vartheta)\sin(\alpha)\cos(\alpha)}{a\sin^2(\vartheta)\sqrt{a^2\cos^2(\vartheta) + b^2\sin^2(\vartheta)}} \quad . \tag{207}$$

The relation of $\boldsymbol{\tau}(s)$ with the local basis vectors of the tangent plane is given by formulas (205) and (206). Hence for the derivative of $\boldsymbol{\tau}(s)$, we may write:

$$\boldsymbol{\tau}'(s) = \frac{d}{ds}\left\{\frac{d\vartheta}{ds}\boldsymbol{a}_1 + \frac{d\varphi}{ds}\boldsymbol{a}_2\right\} \tag{208}$$

$$= \frac{d^2\vartheta}{ds^2}\boldsymbol{a}_1 + \frac{d\vartheta}{ds}\left(\frac{d\vartheta}{ds}\boldsymbol{a}_{1,1} + \frac{d\varphi}{ds}\boldsymbol{a}_{1,2}\right) + \frac{d^2\varphi}{ds^2}\boldsymbol{a}_2 + \frac{d\varphi}{ds}\left(\frac{d\vartheta}{ds}\boldsymbol{a}_{2,1} + \frac{d\varphi}{ds}\boldsymbol{a}_{2,2}\right) \quad .$$

After some algebra, by substituting formulas (202), (206) and (207) into the above expression (208), we find for the derivative of $\boldsymbol{\tau}$ the following

$$\boldsymbol{\tau}'(s) = \frac{b}{a}\,\frac{(a^2-b^2)\sin^2(\vartheta)\sin^2(\alpha) - a^2}{\left\{a^2\cos^2(\vartheta) + b^2\sin^2(\vartheta)\right\}^2}\begin{pmatrix} b\sin(\vartheta)\cos(\varphi) \\ b\sin(\vartheta)\sin(\varphi) \\ a\cos(\vartheta)\end{pmatrix} \quad . \tag{209}$$

The length of this vector is, according to formulas (126) to (129) of section (17), equal to the inverse of the local radius of curvature in the direction of $\boldsymbol{\tau}$. Hence, we find

$$k\left(\boldsymbol{\tau}(s)\right) = \boldsymbol{\tau}'(s) = \left|\frac{b}{a}\,\frac{(b^2-a^2)\sin^2(\vartheta)\sin^2(\alpha) + a^2}{\left\{a^2\cos^2(\vartheta) + b^2\sin^2(\vartheta)\right\}^{3/2}}\right| \quad . \tag{210}$$

The extremum values of $k\left(\boldsymbol{\tau}(s)\right)$ for different choices of the tangential direction parameter α, are

$$k_1 \; = \; \frac{ab}{\left\{ a^2\cos^2(\vartheta) \; + \; b^2\sin^2(\vartheta)\right\}^{3/2}} \quad , \tag{211}$$

for $\alpha = 0$, which corresponds to the tangential direction $\delta\varphi = 0$, and

$$k_2 \; = \; \frac{b/a}{\sqrt{a^2\cos^2(\vartheta) \; + \; b^2\sin^2(\vartheta)}} \quad , \tag{212}$$

for $\alpha = \pi$, which corresponds to the tangential direction $\delta\vartheta = 0$.

The curvature parameter $K(u) = K(\vartheta, \varphi)$ of the local curvature of the surface at location $P(\vartheta, \varphi)$ is defined in formula (175) by the product of k_1 and k_2. Consequently, applying the results (211) and (212), one finds

$$K(u) \; = \; k_1 k_2 \; = \; \frac{b^2}{\left\{ a^2\cos^2(\vartheta) \; + \; b^2\sin^2(\vartheta)\right\}^{2}} \quad . \tag{213}$$

Notice that the curvature which follows from formula (213) for the case $a = b$, resulting in $K = 1/a^2$, corresponds to the naïve expectation for the curvature of a sphere.

The average local curvature parameter $H(u)$ is defined in formula (176) by half the sum of k_1 and k_2. Hence, from the results (211) and (212), one deduces

$$2H(u) \; = \; k_1 \; + \; k_2 \; = \; \frac{b}{a} \, \frac{(b^2 - a^2)\sin^2(\vartheta) \; + \; 2a^2}{\left\{ a^2\cos^2(\vartheta) \; + \; b^2\sin^2(\vartheta)\right\}^{3/2}} \quad . \tag{214}$$

The local normal vector $\boldsymbol{n}(u)$ is defined in formula (143) by the outer product, normalized to unity, of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. So, from the results shown in formula (202), one obtains

$$\boldsymbol{n}(u) \; = \; \frac{1}{\sqrt{a^2\cos^2(\vartheta) \; + \; b^2\sin^2(\vartheta)}} \begin{pmatrix} b\sin(\vartheta)\cos(\varphi) \\ b\sin(\vartheta)\sin(\varphi) \\ a\cos(\vartheta) \end{pmatrix} \quad . \tag{215}$$

Notice, by comparing with expression (209), that $\boldsymbol{\tau}'(s)$ is indeed parallel to the normal of the local tangent plane.

The local torsion tensor $L(u)$ is defined in formula (153) by the innerproduct of $\boldsymbol{a}_{\alpha,\,\beta}(u)$ and $\boldsymbol{n}(u)$. Consequently, exploiting the results shown in formulas (202) and (215), one gets

$$L(u) \; = \; \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \; = \; \frac{-ab}{\sqrt{a^2\cos^2(\vartheta) \; + \; b^2\sin^2(\vartheta)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{216}$$

Consequently, we can verify that the expressions (213) and (214) are in ageement with the formulas (175) and (176). Using formulas (203) and (216), we find

$$K(u) \; = \; \frac{L_{11}L_{22}}{g_{11}g_{22}} \; = \; \det\left(g^{-1}L\right)$$

and

$$2H(u) \;=\; -\,\frac{L_{11}}{g_{11}} \;-\; \frac{L_{22}}{g_{22}} \;=\; -\,\mathrm{Tr}\left(g^{-1}L\right) \quad .$$

Except for a minus sign in the latter result, we find perfect agreement. This sign difference is caused by the angle ambiguity which we discussed in formula (167). By comparing formulas (209) and (215), we observe that $\boldsymbol{\tau}'(s)$ and $\boldsymbol{n}(u)$ are antiparallel and hence the referred angle is here π rather than 0, which causes a minus sign in formula (165) for the definition of the curvature. We may also verify formula (193) for this particular surface. By the use of formulas (188), (189), (203) and (204), we find for the curvature tensor

$$\begin{aligned} R_{1212}(\vartheta,\varphi) \;&=\; g_{11}\left\{\Gamma^{1}_{21,2} \;-\; \Gamma^{1}_{22,1} \;+\; \Gamma^{1}_{22}\,\Gamma^{2}_{21} \;-\; \Gamma^{1}_{11}\,\Gamma^{1}_{22}\right\} \\ &=\; \frac{-a^2b^2\sin^2(\vartheta)}{a^2\cos^2(\vartheta) \;+\; b^2\sin^2(\vartheta)} \end{aligned} \tag{217}$$

When we substitute this expression in formula (193), then we find indeed

$$K(u) \;=\; -\frac{R_{1212}(\vartheta,\varphi)}{\det(g)} \quad .$$

30 Geodesics on the sphere

Since a sphere with radius R centered at the origin can be considered as a special case of the ellipsoid defined in formula (200), we may use all of the material developed in section (29), by setting $a = b = R$.

The geodesic equations can be obtained from formula (194). Using also relations (204), we find for the coordinates $\vartheta(s)$ and $\varphi(s)$ of points at a geodesic on the sphere, which is parametrized by its proper length s, the following differential equations:

$$\frac{d^2\vartheta}{ds^2} - \sin(\vartheta)\cos(\vartheta)\left(\frac{d\varphi}{ds}\right)^2 = 0 \quad \text{and} \quad \frac{d^2\varphi}{ds^2} + 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{d\vartheta}{ds}\,\frac{d\varphi}{ds} = 0 \quad . \tag{218}$$

One possible strategy of solving the equations (218), is to express one of the coordinates as a function of the other along the geodesic curve, for example let

$$\varphi(s) = \varphi(\vartheta(s)) \quad .$$

In order to simplify the formulas to come, we define

$$\varphi' = \frac{d\varphi}{d\vartheta} \quad \text{and} \quad \varphi'' = \frac{d^2\varphi}{d\vartheta^2} \quad ,$$

in which notation we obtain for the derivatives of φ with respect to the proper length parameter s, the expressions:

$$\frac{d\varphi}{ds} = \frac{d\vartheta}{ds}\,\varphi' \quad \text{and} \quad \frac{d^2\varphi}{ds^2} = \frac{d^2\vartheta}{ds^2}\,\varphi' + \left(\frac{d\vartheta}{ds}\right)^2 \varphi'' \quad ,$$

and hence for the equations (218)

$$\begin{aligned} &\frac{d^2\vartheta}{ds^2} - \sin(\vartheta)\cos(\vartheta)\left(\frac{d\vartheta}{ds}\,\varphi'\right)^2 = 0 \quad \text{and} \\ &\frac{d^2\vartheta}{ds^2}\,\varphi' + \left(\frac{d\vartheta}{ds}\right)^2 \varphi'' + 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\left(\frac{d\vartheta}{ds}\right)^2 \varphi' = 0 \quad . \end{aligned} \tag{219}$$

When we substitue moreover the first of the geodesic equations (219) into the second, then we find the differential equation

$$\left(\frac{d\vartheta}{ds}\right)^2 \left\{\varphi'' + \sin(\vartheta)\cos(\vartheta)\,(\varphi')^3 + 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\varphi'\right\} = 0 \quad . \tag{220}$$

One type of solutions can readily be found. Namely those for which

$$\frac{d\vartheta}{ds} = 0 \quad i.e. \quad \vartheta(s) = \vartheta \quad \text{constant} \quad ,$$

and for which, following the equations (218), one has moreover

$$\sin(\vartheta)\cos(\vartheta)\left(\frac{d\varphi}{ds}\right)^2 = 0 \quad \text{and} \quad \frac{d^2\varphi}{ds^2} = 0 \quad .$$

The solutions of the latter equation can be classified by **singular points** on the sphere:

$$\vartheta = 0 \, , \, \pi \quad , \qquad i.e. \text{ North and South poles on the sphere;}$$
$$\vartheta \text{ and } \varphi \text{ constant} \quad , \qquad i.e. \text{ singular points on the sphere;} \tag{221}$$

and by the **Equator**, corresponding to the circumference of the circle in the (x, y)-plane (see also example 122), given by

$$\vartheta = \frac{\pi}{2} \quad \text{and} \quad \varphi(s) = \frac{s}{R} \quad . \tag{222}$$

So, besides singular points, which obviously are solutions of the equations (218), we only found one non-trivial geodesic curve, the Equator, actually the only solution of (218) for which the coordinate ϑ is constant along the curve.

Other solutions of (220) are found from the second piece of this equation, *i.e.*

$$\varphi'' + \sin(\vartheta)\cos(\vartheta) \ (\varphi')^3 + 2 \frac{\cos(\vartheta)}{\sin(\vartheta)} \varphi' = 0 \quad . \tag{223}$$

In order to solve this equation, we will study here an arbitrary intersection of the sphere with a plane through the origin. Such intersections are called large circles. We will show then that all large circles on the sphere are geodesic curves.

Any plane through the origin can be characterized by its orientation, for which one may select its normal vector, say $\boldsymbol{N} = p\hat{x} + q\hat{y} + r\hat{z}$. The position vector $\boldsymbol{x}_P = x\hat{x} + y\hat{y} + z\hat{z}$ of a point P in the plane and the normal to the plane are perpendicular by definition. Hence, one has

$$0 = \boldsymbol{x}_P \cdot \boldsymbol{N} = px + qy + rz \quad . \tag{224}$$

When P is moreover on the surface of the sphere, then we obtain, using relations (200) with $a = b = R$, for equation (224) the form

$$p\sin(\vartheta)\cos(\varphi) + q\sin(\vartheta)\sin(\varphi) + r\cos(\vartheta) = 0 \quad ,$$

or, since the cases $\sin(\vartheta) = 0$ and $\cos(\vartheta) = 0$ have been studied in formulas (221) and (222), for $\sin(\vartheta) \neq 0$ the form

$$p \ \cos(\varphi) + q \ \sin(\varphi) = -r \frac{\cos(\vartheta)}{\sin(\vartheta)} \quad . \tag{225}$$

When we determine the first and second order derivatives of equation (225) with respect to ϑ, then we find

$$\varphi' \ \{-p \ \sin(\varphi) + q \ \cos(\varphi)\} = \frac{r}{\sin^2(\vartheta)} \tag{226}$$

and

$$\varphi'' \ \{-p \ \sin(\varphi) + q \ \cos(\varphi)\} - (\varphi')^2 \ \{p \ \cos(\varphi) + q \ \sin(\varphi)\} = -2r \frac{\cos(\vartheta)}{\sin^3(\vartheta)}$$

By multiplying the second equation of formula (226) with φ' and by substitution of relation (225) and the first equation of formula (226) in the resulting expression, we obtain

$$\varphi'' \frac{r}{\sin^2(\vartheta)} \; + \; (\varphi')^3 \; r \, \frac{\cos(\vartheta)}{\sin(\vartheta)} \; + \; 2\varphi' \, r \, \frac{\cos(\vartheta)}{\sin^3(\vartheta)} \; = \; 0 \quad , \tag{227}$$

which is, for $\sin(\vartheta) \neq 0$ is completely equivalent to the geodesic equation (223). This proofs that large circles are geodesic curves of the surface of the sphere.

31 The torus

Another such example, which is often referred to in the literature, is the torus, the two-dimensional surface of a doughnut shaped object. Let us consider the surface of a wedding-ring with a circular cross section of radius r. The radius of the ring, which is measured from the center of the ring to the center of the circular cross section, will be denoted by R. From this picture of a torus it might be clear that r is considered to be smaller than R. In order to define coordinates on the surface of our torus, we let the center of the wedding-ring coincide with the origin of the three-dimensional Euclidean embedding space. The centers of the circular cross sections of the wedding-ring, which together form a circle of radius R, now centered at the origin, we suppose to all be in the (x, y)-plane. This way we obtain for a possible parametrization of the surface of a torus the following expression:

$$\boldsymbol{x}(\alpha,\beta) = \begin{pmatrix} R\cos(\beta) \\ R\sin(\beta) \\ 0 \end{pmatrix} + \begin{pmatrix} r\cos(\alpha)\cos(\beta) \\ r\cos(\alpha)\sin(\beta) \\ r\sin(\alpha) \end{pmatrix} \ ,$$

$$\text{where} \quad r < R \ , \quad \alpha, \beta \in (0, 2\pi) \ . \tag{228}$$

Apparently, we have chosen α and β for the coordinates at the surface of the torus. Notice, that the cross sections of this surface with planes which contain the z-axis, are circles centered at a distance R from the origin. Such planes are characterized by $\beta =$*constant* and their cross sections with the torus, which are called *meridians*, by the usual relations for circles, *i.e.*

$$(x\cos(\beta) + y\sin(\beta) - R)^2 + z^2 = r^2 \ . \tag{229}$$

For the local tangent plane, tangent to the surface at the point $P(\alpha, \beta)$, we find then, using also formula (141), the basis vectors given by

$$\hat{\boldsymbol{\alpha}}(\alpha,\beta) = \frac{\partial \boldsymbol{x}(\alpha,\beta)}{\partial\alpha} = \begin{pmatrix} -r\sin(\alpha)\cos(\beta) \\ -r\sin(\alpha)\sin(\beta) \\ r\cos(\alpha) \end{pmatrix} \quad \text{and}$$

$$\hat{\boldsymbol{\beta}}(\alpha,\beta) = \frac{\partial \boldsymbol{x}(\alpha,\beta)}{\partial\beta} = \left\{\frac{R}{r} + \cos(\alpha)\right\} \begin{pmatrix} -r\sin(\beta) \\ r\cos(\beta) \\ 0 \end{pmatrix} \ . \tag{230}$$

Inserting the result (230) into formula (142) yields for the local metric

$$g(\alpha,\beta) = \begin{pmatrix} g_{\alpha\alpha} & g_{\alpha\beta} \\ g_{\beta\alpha} & g_{\beta\beta} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{\alpha}} & \hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{\alpha}} & \hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{\beta}} \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos(\alpha))^2 \end{pmatrix} \ . \tag{231}$$

The non-vanishing local affine connections are, by the use of formulas (151) and (152), collected below

$$\Gamma^{\beta}_{\alpha\beta} = \Gamma^{\beta}_{\beta\alpha} = g^{\beta\beta}\Gamma_{\beta\beta\alpha} = \frac{1}{2}g^{\beta\beta}g_{\beta\beta,\,\alpha} = \frac{-\sin(\alpha)}{\frac{R}{r}+\cos(\alpha)} \tag{232}$$
$$\text{and } \Gamma^{\alpha}_{\beta\beta} = g^{\alpha\alpha}\Gamma_{\alpha\beta\beta} = -\frac{1}{2}g^{\alpha\alpha}g_{\beta\beta,\,\alpha} = \sin(\alpha)\left\{\frac{R}{r}+\cos(\alpha)\right\} \quad ,$$

from which, using formula (194), follow the geodesic equations

$$\frac{d^2\alpha}{ds^2} = -\left(\frac{d\beta}{ds}\right)^2 \sin(\alpha)\left\{\frac{R}{r}+\cos(\alpha)\right\} \quad \text{and} \quad \frac{d^2\beta}{ds^2} = 2\left(\frac{d\alpha}{ds}\right)\left(\frac{d\beta}{ds}\right)\frac{\sin(\alpha)}{\frac{R}{r}+\cos(\alpha)} \quad . \tag{233}$$

Two types of solutions can be discovered without any difficulty. Those are solutions for which

$$\begin{gathered}\beta = \text{constant} \;\Rightarrow\; \frac{d\beta}{ds} = 0 \;\Rightarrow\; \frac{d^2\alpha}{ds^2} = 0 \;\Rightarrow\; \alpha = \frac{s}{r} \quad , \\ \text{or} \\ \alpha = 0 \text{ or } \pi \;\Rightarrow\; \frac{d^2\beta}{ds^2} = 0 \;\Rightarrow\; \beta = \frac{s}{R+r} \text{ or } \frac{s}{R-r} \quad .\end{gathered} \tag{234}$$

The first type of geodesics described by formula (234) are the meridian circles, represented by formula (229). The second type are the equatorial circles in the (x, y)-plane and centered at the z-axis with extremal radii, respectively maximum, $R+r$, and minimum, $R-r$. When both conditions $d\alpha/ds$ and $d\beta/ds$ vanish, one has isolated points at the surface of the torus.

In the following we assume that neither α, nor β, is constant. One may notice then that the second of the geodesic equations (233) can also be written in the form

$$\frac{d}{ds}\left[\left\{\frac{R}{r}+\cos(\alpha)\right\}^2\left(\frac{d\beta}{ds}\right)\right] = 0 \quad ,$$

which implies that the expression within the brackets is constant, say J, along a geodesic curve. When this result is inserted in the first of the geodesic equations (233), one finds for $\alpha(s)$ the equation

$$\frac{d^2\alpha}{ds^2} = \frac{-J^2\sin(\alpha)}{\left\{\frac{R}{r}+\cos(\alpha)\right\}^3} \quad .$$

We may simplify the first of equations (233), by considering α directly a function of β. In that case one has

$$\frac{d\alpha}{ds} = \frac{d\beta}{ds}\frac{d\alpha}{d\beta} \quad , \quad \frac{d^2\alpha}{ds^2} = \frac{d^2\beta}{ds^2}\frac{d\alpha}{d\beta} + \left(\frac{d\beta}{ds}\right)^2\frac{d^2\alpha}{d\beta^2}$$
$$\text{and } \frac{d^2\beta}{ds^2} = 2\left(\frac{d\alpha}{d\beta}\right)\left(\frac{d\beta}{ds}\right)^2\frac{\sin(\alpha)}{\frac{R}{r}+\cos(\alpha)} = \frac{2J^2\sin(\alpha)}{\left\{\frac{R}{r}+\cos(\alpha)\right\}^5}\left(\frac{d\alpha}{d\beta}\right) \quad ,$$

which formulas can be put together, resulting in a second order differential equation for α as a function of β

$$\frac{d^2\alpha}{d\beta^2} \;+\; \frac{2\sin(\alpha)}{\frac{R}{r}+\cos(\alpha)}\left(\frac{d\alpha}{d\beta}\right)^2 \;+\; \sin(\alpha)\left\{\frac{R}{r}+\cos(\alpha)\right\} \;=\; 0 \quad . \tag{235}$$

which remains to be solved.

The curvature tensor is defined in formulas (188) and (189). By the use of formula (232) we find for the torus

$$R_{\beta\alpha\beta\alpha} \;=\; -r\cos(\alpha)\left\{R+r\cos(\alpha)\right\} \quad ,$$

and, hence, for the curvature scalar, which is defined in formula (192) and in particular for a two-dimensional surface in formula (193), one obtains

$$R(\alpha,\beta) \;=\; -\frac{2\cos(\alpha)}{r\left\{R+r\cos(\alpha)\right\}} \quad . \tag{236}$$

Notice that at the equatorial circles one finds for the curvature scalar indeed, as expressed by formulas (168), (175) and (193), -2 divided by the product of the radii of the equatorial circle and the meridian circle. Notice moreover that at the outside ($\alpha = 0$) the result is positive (curved towards the origin), but at the inside ($\alpha = \pi$) negative (curved away from the origin). At the top ($\alpha = \pi/2$) and at the bottom ($\alpha = 3\pi/2$), where the curvature is vanishing, the surface of the torus has locally the aspect of a cylinder.

Part V
The description of gravitation by curvature

32 Rectilinear motion

The rectilinear motion of a freely moving object in ordinary three dimensions can, at any instant t, be parametrized by

$$\boldsymbol{x}(t) \;=\; \boldsymbol{a}\, t \;+\; \boldsymbol{b} \quad , \tag{237}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ depend on the initial conditions of the motion: The constant velocity of the particle is represented by the vector $\boldsymbol{a}$. Whereas the vector $\boldsymbol{b}$ represents the position of the particle at instant $t = 0$.

The equations of motion for the particle's movement can be characterized by the well-known relation

$$\frac{d^2\, x^i(t)}{dt^2} \;=\; 0 \quad \text{for} \quad i = 1, 2, 3 \quad . \tag{238}$$

Now, in the local coordinates of section (9), the equations of motion (238) take the form

$$0 \;=\; \frac{d}{dt}\left(\frac{d\, x^i}{dt}\right) \;=\; \frac{d}{dt}\left\{ x^{i}_{\,,j}\, \frac{d\, x'^j}{dt} \right\}$$

$$= \; x^{i}_{\,,jk}\, \frac{d\, x'^k}{dt}\, \frac{d\, x'^j}{dt} \;+\; x^{i}_{\,,j}\, \frac{d^2\, x'^j}{dt^2} \quad ,$$

or, equivalently

$$\frac{\partial\, x'^\ell}{\partial\, x^i}\, x^{i}_{\,,jk}\, \frac{d\, x'^k}{dt}\, \frac{d\, x'^j}{dt} \;+\; \frac{\partial\, x'^\ell}{\partial\, x^i}\, x^{i}_{\,,j}\, \frac{d^2\, x'^j}{dt^2} \;=\; 0 \quad ,$$

which, by the use of the definition of the affine connection in formula (102), can be written as follows:

$$\Gamma^{\ell}_{jk}\, \frac{d\, x'^j}{dt}\, \frac{d\, x'^k}{dt} \;+\; \frac{d^2\, x'^\ell}{dt^2} \;=\; 0 \quad . \tag{239}$$

The non-zero components of the affine connections for the 3*D spherical coordinates are given in formula (107). Substitution in the equations (239) gives*

$$
\begin{aligned}
0 &= r'' + \Gamma^{r}_{\vartheta\vartheta}(\vartheta')^2 + \Gamma^{r}_{\varphi\varphi}(\varphi')^2 = r'' - r(\vartheta')^2 - r\ \sin^2(\vartheta)(\varphi')^2 \\
0 &= \vartheta'' + 2\Gamma^{\vartheta}_{r\vartheta}r'\vartheta' + \Gamma^{\vartheta}_{\varphi\varphi}(\varphi')^2 = \vartheta'' + \frac{2}{r}r'\vartheta' - \sin(\vartheta)\cos(\vartheta)(\varphi')^2 \\
0 &= \varphi'' + 2\Gamma^{\varphi}_{r\varphi}r'\varphi' + 2\Gamma^{\varphi}_{\vartheta\varphi}\vartheta'\varphi' = \varphi'' + \frac{2}{r}r'\varphi' + 2\frac{\cos(\vartheta)}{\sin(\vartheta)}\vartheta'\varphi' \quad .
\end{aligned}
\tag{240}
$$

In order to solve those equations for $r = r(t)$, $\vartheta = \vartheta(t)$ *and* $\varphi = \varphi(t)$, *we first observe that the third equation can be written in the form*

$$0 = \frac{d}{dt}\left(r^2\ \varphi'\ \sin^2(\vartheta)\right) \quad , \tag{241}$$

which implies that $r^2\varphi'\sin^2(\vartheta)$ *is a constant along the curve. Hence, without much loss of generality we may select curves in the* (x, y)*-plane, for which* $\vartheta = \pi/2$. *The equations of motion reduce then to*

$$0 = r'' - r\ (\varphi')^2 \quad \text{and} \quad 0 = \varphi'' + \frac{2}{r}r'\ \varphi' \quad . \tag{242}$$

It it easy to verify that those equations are solved by

$$r(t) = \sqrt{(at)^2 + b^2} \quad \text{and} \quad \varphi(t) = \text{arctg}\left(\frac{b}{at}\right) \quad ,$$

which, for two perpendicular vectors $\boldsymbol{a}$ *and* $\boldsymbol{b}$, *just represent the parametrization of a straight line, as, indeed, parametrizing* 3*D space by spherical coordinates does not imply any curvature.*

On a curved surface we consider relation (239) as the definition of *local rectilinear* motion. Instead of the time-parameter t, we use then the proper length parameter s, to obtain for "rectilinear" motion at the curved surface (140) the equation of motion

$$\Gamma^{\mu}_{\alpha\beta}(u)\,\frac{d\,u^{\alpha}}{ds}\,\frac{d\,u^{\beta}}{ds} + \frac{d^2\,u^{\mu}}{ds^2} = 0 \quad . \tag{243}$$

This is actually the same equation as the geodesic equation (194), as indeed geodesic curves are the closest approximation to straight lines on a curved surface.

Locally, geodesics form exact straight lines with a full Euclidean geometry. At a global scale, geodesics are of course not straight, however the most natural generalization of the "straight line" concept to a curved surface. We may say that a "free" particle which is confined to move at a curved surface, will follow a geodesic, with equation (243) being the equation to describe its kinematics.

In the absence of curvature, we return to equation (238).

33 Parallel transport

In section (14) the derivatives of vector fields with respect to the local coordinates are studied. Which is equivalent to the study of variations of a vector field for infinitesimal displacements in space. It led us to the definition of the concept of the covariant derivative, formulated in equations (112) and (113).

Now, suppose that a certain vector field $\boldsymbol{v}(x')$ is constant. Then, in the N-dimensional Euclidean space defined in section (9), one has, for the derivatives of that vector field with respect to the local coordinates, the following

$$\frac{d\,\boldsymbol{v}(x')}{dx'^{i}} \;=\; 0 \quad . \tag{244}$$

A constant vector field in space means that at any point in space the vector $\boldsymbol{v}(x')$ which represents the field at the point $P(x')$, is parallel to the corresponding vector at any other point in space. We could however, equally well consider the vector associated with the field at one point in space, as the vector of the field from another point in space transported parellel to itself. Hence, the vector field which is the result of transporting parallel to itself a given vector at a given position to all other positions in space, is represented by equation (244). Moreover, in view of equation (112), one has for the components of the vector field, when decomposed at the local basis dictated by the local coordinates, the relation

$$v'^{k}_{\;,i}(x') \;=\; 0 \quad . \tag{245}$$

At a curved surface we might like to consider equation (245) as the definition of parallel transport. Which means that we assume then that that equation is the closest analogue to the global flat space definition.

Hence, we are then assuming that we are capable to define a constant vector field at the surface (140) as the result of parallel transporting one given vector at the surface at a certain point to all points at the surface. And, consequently, such vector field should satisfy

$$0 \;=\; v^{\mu}_{\;;\alpha}(u) \;=\; v^{\mu}_{\;,\alpha}(u) \;+\; \Gamma^{\mu}_{\alpha\beta}(u)\,v^{\beta}(u) \quad . \tag{246}$$

For parallel transport along a curve, parametrized by its proper length s, we have moreover

$$\frac{d\,v^{\mu}(s)}{ds} \;=\; v^{\mu}_{\;,\alpha}(u)\,\frac{d\,u^{\alpha}(s)}{ds} \;=\; -\Gamma^{\mu}_{\alpha\beta}(u)\,v^{\beta}(s)\,\frac{d\,u^{\alpha}(s)}{ds} \quad . \tag{247}$$

However, it can be shown that at a curved surface one cannot construct a vector field which satisfies equation (246), since parallel transport depends at the curved surface on the path one chooses (see examples in section 34).

Hence, *we cannot extend the notion of a constant vector field to a curved surface*, but only the concept of parallel transport along a given path, as formulated in expression (247)

34 Parallel transport along curves at the sphere

The surface of a sphere with radius R and centered at the origin is parametrized by ϑ and φ, as a special case of relations (200) for $R = a = b$. Let a curve at the surface of the sphere be parametrized by its proper length parameter s, according to $\vartheta(s)$ and $\varphi(s)$ and let us moreover consider the vector field $\boldsymbol{A}(\vartheta, \varphi)$ at the surface of the sphere. Restricted to the curve $\boldsymbol{A}$ turns a function of s.

Now, let us assume that the vector field $\boldsymbol{A}(s)$ is the result of parallel transporting the vector $\boldsymbol{A}(0)$, which is defined at the point of the curve parametrized by $s = 0$. Then, our vector field satisfies relations (247), which, also using formula (204) for $R = a = b =$, in components reads

$$\begin{aligned} \frac{d}{ds}\, A^1(s) &= \sin(\vartheta)\cos(\vartheta)\; \varphi'\; A^2(s) \quad \text{and} \\ \frac{d}{ds}\, A^2(s) &= -\frac{\cos(\vartheta)}{\sin(\vartheta)} \left\{ \varphi'\; A^1(s) \;+\; \vartheta'\; A^2(s) \right\} \quad , \end{aligned} \tag{248}$$

where, as before, $\varphi' = d\varphi/ds$ and $\vartheta' = d\vartheta/ds$.

Let us consider the special case of parallel transport along curves for which $\vartheta(s) = \vartheta$ is constant. In that case it is easy to relate φ and s. Using equations (118) and (203) for $R = a = b$, we find

$$ds \;=\; R\; \sin(\vartheta)\; d\varphi \quad , \tag{249}$$

which equation and the fact that $d\vartheta/ds = 0$, reduce the differential equations (248) to

$$\frac{d}{d\varphi}\, A^1(\varphi) \;=\; \sin(\vartheta)\cos(\vartheta)\; A^2(\varphi) \quad \text{and} \quad \frac{d}{d\varphi}\, A^2(\varphi) \;=\; -\frac{\cos(\vartheta)}{\sin(\vartheta)}\; A^1(\varphi) \quad . \tag{250}$$

Equations (250) have solutions of the form

$$\begin{aligned} A^1(\varphi) &= V \sin\left[\varphi\cos(\vartheta)\right] \;+\; W \cos\left[\varphi\cos(\vartheta)\right] \quad \text{and} \\ A^2(\varphi) &= \frac{1}{\sin(\vartheta)} \left\{ V \cos\left[\varphi\cos(\vartheta)\right] \;-\; W \sin\left[\varphi\cos(\vartheta)\right] \right\} \quad , \end{aligned}$$

with V and W constants.

Let us study two typical cases:

1. The case $\boldsymbol{\vartheta = \pi/2}$

For parallel displacement along the Equator, which serves as an example for any geodesic curve at the sphere, one finds for the vector field (251):

$$\boldsymbol{A}(\varphi) \;=\; \begin{pmatrix} A^1(\varphi) \\ A^2(\varphi) \end{pmatrix} \;=\; \begin{pmatrix} W \\ V \end{pmatrix} \quad ,$$

which is constant. We may thus conclude that **for parallel displacement along the Equator and, hence, along any geodesic curve, the displaced vector remains constant**.

2. The case $\boldsymbol{\vartheta = \pi/3}$

For displacement on the minor circle for $\vartheta = \pi/3$, we find for the vector field (251) the following

$$\boldsymbol{A}(\varphi) \;=\; \left(\begin{array}{c} V \;\sin\left(\frac{\varphi}{2}\right) \;+\; W \;\cos\left(\frac{\varphi}{2}\right) \\ \frac{2}{\sqrt{3}}\left\{ V \;\cos\left(\frac{\varphi}{2}\right) \;-\; W \;\sin\left(\frac{\varphi}{2}\right)\right\} \end{array} \right) \quad ,$$

which is obviously not constant along the curve. Let $\boldsymbol{A}(0)$ be the original vector, which had to be displaced parallel to itself. Below we write the original and the final vectors of the vector field after a complete roundtrip, *i.e.* $\boldsymbol{A}(0)$ and $\boldsymbol{A}(2\pi)$. For the case $W = 1$ and $V = 0$ we find:

$$\boldsymbol{A}(0) \;=\; \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \quad \text{and} \quad \boldsymbol{A}(2\pi) \;=\; \left(\begin{array}{c} -1 \\ 0 \end{array} \right) \quad .$$

We find that after a complete roundtrip the vector appears upside down at the initial location.

The general conclusion is that under parallel transport along non-geodesic curves a vector field does not remain constant. And, consequently, that constant vector fields cannot be defined.

For completeness let us mention that for not too large roundtrips there is a relation between the amount of variation of the parallel displaced vector, the enclosed area and the curvature. Take for example the second case, where the vector changed over 180°. The enclosed area by the roundtrip is given by

$$\text{area} \;=\; \int_0^{\pi/3} d\vartheta \int_0^{2\pi} d\varphi \; R^2 \;\sin(\vartheta) \;=\; \pi R^2 \quad .$$

When we devide this area by the amount of variation, $= \pi$, of the displaced vector, then we find the square of the radius of curvature of the sphere.

35 Local Euclidean geometry

When we make a map of a small village, then we will probably not notice that the Earth surface is curved. The geometry, at the accuracy with which we can measure distances and angles, looks perfectly Euclidean. But, what do we precisely mean by this last statement? A possible answer to this question could for instance be that the sum of the angles of any triangle within the small village equals perfectly 180^0 at the accuracy of our measurements. This latter statement was proven to be equivalent to Euclid's fifth postulate, by Adrien Marie Legendre (1752-1833).

However, if we go to a larger scale at the Earth' surface, then we might encounter triangles which give results, different from 180^0, for the sums of their angles. As an example, let us take the triangle which makes at the North Pole of the sphere a right angle and has two more right angles at the Equator. This triangle encloses one octant of the Earth' surface. Its angles apparantly sum up to 270^0, which implies that at this scale the geometry of the Earth is noticably not Euclidean. At the much smaller scale of the little village it is also not Euclidean, but within the accuracy of measurement we do not notice the resulting deviations.

In Descartes' analytic geometry, Euclidean geometry is related to the definition of the distance between any two points, P and Q, in space, given by:

$$\left\{d\left[P\left(x^1,x^2\right)\ ,\ Q\left(X^1,X^2\right)\right]\right\}^2\ =\ \left(X^1-x^1\right)^2\ +\ \left(X^2-x^2\right)^2\quad , \tag{251}$$

i.e. the distance follows the law of Pythagoras.

In the previous paragraphs we studied a two-dimensional surface endowed with coordinates u and metrical tensor, g, such that infinitesimally a distance is given by formula (160). It was Gauss who assumed that in a sufficiently small neighborhood around any point P on the surface, we can define new coordinates $\{\Sigma(P)\}$, such that distances in this region of space are given by:

$$ds^2\ =\ \delta_{ab}\, d\Sigma^a(P)\, d\Sigma^b(P)\quad a,b=1,2\ . \tag{252}$$

Consequently, in the above defined region of space, given a certain accuracy of measurement (or phrased more formal: to lowest order), distances obey Pythagoras' law, which implies that the geometry in that region is Euclidean.

A way to find these local Euclidean coordinates $\{\Sigma(P)\}$ is to compare with each other formulas (160) and (252), and, also using formula (89), to conclude that consequently we must have:

$$g_{ab}(P)\ =\ \delta_{ab}\,\frac{\partial\Sigma^a(P)}{\partial u^\alpha}\,\frac{\partial\Sigma^b(P)}{\partial u^\beta}\quad . \tag{253}$$

So, we must find two such coordinates $\{\Sigma(P)\}$ which satisfy (253) at the point P of space, in order to be allowed to forget about curvature in a sufficiently small neighboring region of space around P. It was Gauss who demonstrated that this procedure is always possible.

Notice at this point that the local Euclidean space has the geometry of the local tangent space.

For example, at the sphere with radius a (use formula (203) for a = b) the metrical tensor is given by:

$$ds^2 \;=\; a^2\,d\vartheta^2 \;+\; a^2\sin^2(\vartheta)\,d\varphi^2 \quad . \tag{254}$$

Now, at a certain point $P(\vartheta,\varphi)$ on the surface of the sphere, we define new coordinates $\{\Sigma(P)\}$, according to:

$$\Sigma^1(P) \;=\; a\left\{\vartheta-\vartheta(P)\right\} \quad\text{and}\quad \Sigma^2(P) \;=\; a\sin\left(\vartheta(P)\right)\,\left\{\varphi-\varphi(P)\right\} \quad . \tag{255}$$

This choice of coordinates implies

$$d\Sigma^1(P) \;=\; a\,d\vartheta \quad\text{and}\quad d\Sigma^2(P) \;=\; a\sin\left(\vartheta(P)\right)\,d\varphi \quad . \tag{256}$$

At the point P we find thus for an infinitesimal distance

$$ds^2 \;=\; \left\{\Sigma^1(P)\right\}^2 \;+\; \left\{\Sigma^2(P)\right\}^2 \quad .$$

And in small region around the point P, we find for an infinitesimal distance to lower order to expression

$$\begin{aligned} ds^2(\vartheta,\varphi) \;&=\; a^2\,d\vartheta^2 \;+\; a^2\sin^2(\vartheta)\,d\varphi^2 \\ &\approx\; a^2\,d\vartheta^2 \;+\; a^2\sin^2\left(\vartheta(P)\right)\,d\varphi^2 \\ &\approx\; \left\{\Sigma^1(P)\right\}^2 \;+\; \left\{\Sigma^2(P)\right\}^2 \quad . \end{aligned} \tag{257}$$

The latter relation represents indeed Pythagoras' law.

The choice of coordinates (256) can, in a more formal way, be written according to

$$\Sigma^a(P) \;=\; V^{a}_{\;\alpha}(P)\,\left\{u^\alpha - u^\alpha(P)\right\} \quad ,$$

where the two objects $\{V(P)\}$ are given by:

$$V_1^1 \;=\; a \quad , \quad V_2^1 \;=\; V_1^2 \;=\; 0 \quad\text{and}\quad V_2^2 \;=\; a\sin\left(\vartheta(P)\right) \quad . \tag{258}$$

The objects $\{V(P)\}$ are called "Zweibein" (in German "Zwei" means two, and "Bein" means leg), and their generalization to four dimensions will show extremely useful for the handling of particles with spin in curved space-time.

In terms of the Zweibein $\{V(P)\}$, relations (256) read in general

$$d\Sigma^a(P) \;=\; V^{a}_{\;\alpha}(P)\,du^\alpha \quad . \tag{259}$$

Under a general coordinate transformation, $\{u\} \to \{u'\}$, we obtain for the above expression the form

$$d\Sigma^a(P) \;=\; V^{a}_{\;\alpha}(P)\,\frac{\partial u^\alpha}{\partial u'^\beta}\,du'^\beta \quad , \tag{260}$$

from which equation we might extract for the Zweibein the transformation rule

$$V'^{a}_{\ \beta}(P) \;=\; V^{a}_{\ \alpha}(P)\,\frac{\partial u^{\alpha}}{\partial u'^{\beta}} \quad . \tag{261}$$

This, by the use of formulas (83) and (86), is subsequently found to be the transformation rule for covariant components. So, we may interpret the Zweibein as the covariant components of a set of two vectors.

36 Tidal forces

In the following we will study our four dimensional space-time. We will discover that the effects of a gravitational field can be described by assuming that the metric of space-time differs from the usual Minkowskian metric. This implies that we assume that the effects of gravitation are equivalent to space-time being curved. However, at this point we have to be very careful. We only find an equivalence in the description. We do not assume that our space-time is embedded in some higher dimensional space (neither exclude that this is possible). Because of this reason, we will not refer to an embedding space. Which means that we only have at our disposal the coordinates of space-time and the metric induced by gravitation, no tangent plane and neither a normal to the tangent plane. However, the analogy with the local Euclidean space will still be present in this case.

At Earth we observe the effects of gravitation from the fact that objects tend to fall towards the center of the Earth, which is sometimes rather disturbing. At the surface of the Earth, the effect of gravitation can be characterized by its constant of acceleration

$$g = \frac{GM}{R^2} \quad , \tag{262}$$

where G, M and R represent respectively the Gravitational constant and the Earth' mass and radius.

Let $\boldsymbol{x}(t)$ be the position vector of a particle with mass m, which is freely falling near the surface of the Earth, with respect to me being comfortably seated in my armchair at the surface of the Earth. The equation of motion for the position vector in my coordinate system is in this situation given by

$$m\,\frac{d^2\boldsymbol{x}(t)}{dt^2} = m\boldsymbol{g} \quad . \tag{263}$$

Another situation to study the motion of this freely falling particle, might to be put my armchair in a equally freely falling elevator. Let us suppose that I am comfortably seated again, and that the particle's motion takes place inside the same elevator.

The new coordinates $\{x'\}$ of the particle's position vector with respect to me, are given by

$$\boldsymbol{x}'(t) = \boldsymbol{x}(t) - \frac{1}{2}\boldsymbol{g}t^2 \quad \text{and} \quad t' = t \quad . \tag{264}$$

Consequently, I find, by substitution of the new coordinates (264) in formula (263), for the equation of motion of the position vector of the particle

$$m\,\frac{d^2\boldsymbol{x}'(t')}{dt'^2} = 0 \quad . \tag{265}$$

The particle appears to be completely at rest with respect to me inside the freely falling elevator. I might thus conclude, that no forces act on the particle. Apparently, it seems that the coordinate transformation (264) makes the effect of the gravitational field of the Earth disappear.

For a system of two particles, A and B, with masses $m(A)$ and $m(B)$ and with mutual interactions described by some force, $\boldsymbol{F}$, which only depends on the relative positions of A and B with respect to each other, one finds a similar result as we will see below.

In the coordinate system with respect to the surface of the Earth, I find the equations of motions

$$
\begin{aligned}
m_A \frac{d^2\boldsymbol{x}_A(t)}{dt^2} &= m_A\boldsymbol{g} + \boldsymbol{F}(\boldsymbol{x}_A - \boldsymbol{x}_B) \quad \text{and} \\
m_B \frac{d^2\boldsymbol{x}_B(t)}{dt^2} &= m_B\boldsymbol{g} + \boldsymbol{F}(\boldsymbol{x}_B - \boldsymbol{x}_A) \quad . \qquad (266)
\end{aligned}
$$

Whereas, inside the freely falling elevator I obtain, using the transformations (264), for the equations of motion

$$
\begin{aligned}
m_A \frac{d^2\boldsymbol{x}'_A(t')}{dt'^2} &= \boldsymbol{F}(\boldsymbol{x}'_A - \boldsymbol{x}'_B) \quad \text{and} \\
m_B \frac{d^2\boldsymbol{x}'_B(t')}{dt'^2} &= \boldsymbol{F}(\boldsymbol{x}'_B - \boldsymbol{x}'_A) \quad . \qquad (267)
\end{aligned}
$$

As we may notice, in the latter equations the effects of gravitation have again completely disappeared.

However, we should not conclude that the effects of gravitation are absent in the freely falling elevator. At a larger scale, i.e. involving larger distances and longer periods of time, the effects of gravitation can easily be observed inside the freely falling elevator, as we will explain below for the case that the two particles do not mutually interact.

Suppose that initially the two particles have a certain distance with respect to each other. Then,

1 when the two particles have the same distance with respect to the center of the Earth, after a while we find that their relative distance has shrunk. The reason is of course, that both particles are falling towards the center of the Earth along different radial directions, which approach each other when their distance to the Earth becomes smaller.

2 when, also initially, the two particles have different distances with respect to the center of the Earth, but move along the same radial direction, we find that after a while their relative distance has grown. Here, the reason is that the particle which is closer to the center of the Earth feels a slightly larger attraction all the time and thus is all the time slightly more accelerated.

So, at a global scale we observe a horizontal attractive force which reduces the distances of particles, and a vertical repulsive force which increases the distances of particles. These effects are called the *tidal forces*, and no transformation can make them disappear.

The name tidal forces stems from the tidal motion of the oceans due to the presence of the Moon. If we consider the Earth as a freely falling object in the Moon's gravitational field, then the oceanic waters feel repulsive forces with respect to the Earth surface at the sides towards and opposite to the Moon (high-tide) and attractive forces in between (low tide). Due to the Earth rotation these positions change in time, which causes the tidal motion of the sea waters.

In conclusion, we can say that locally (i.e. at small spatial distances and in short periods of time) we can find a transformation which makes the effects of gravitation disappear for a given accuracy of measurement. Globally, this is impossible.

37 The principle of equivalence

The transformation (264) is based on the equivalence of inertial and gravitational mass. The first person who reported the discovery that objects fall at a rate independent of their masses, was of course Galileo Galilei (1564-1642). Since his initial measurements various experimental improvements have been carried out to finally come to the same conclusion. The two most famous results come from Roland von Eötvös and from R.H. Dicke. Von Eötvös concluded in 1889 that the inertial and gravitational masses for all materials are equal up to his experimental precision of nine decimals. Dicke improved this precision in 1964 to eleven decimals.

The above experimental results, combined with the results of the section (36), lead to the following generalization:

At every point in space-time in an arbitrary gravitational field, it is possible to select a locally inertial coordinate system, such that locally the laws of nature take the same form as in an inertial system in the absence of gravitation.

Notice the resemblance of the above statement with what is said in section (35) about the local Euclidean coordinates at a curved two-dimensional space.

For further reading on this subject, I recommand the paragraphs 1.2, 1.3 and 3.1 of the book of Steven Weinberg [5].

38 Minkowskian space-time

From the previous paragraphs we might have understood, that instead of the somewhat boring case of a two-dimensional arbitrarely curved surface embedded in three dimensions, we will study in the following the more exciting case of a four dimensional curved surface without referring to an embedding space.

We define a set of four coordinates $\{x\}$ which characterize uniquely events, $P(x)$, in this space by its components

$$x^{\mu} \quad , \mu \; = \; 0, \; 1, \; 2, \; 3 \quad .$$

Furthermore, we equip this space with a metric which represents the gravitational field and is characterized by the metrical tensor

$$g_{\mu\nu} \quad , \mu \, , \nu \; = \; 0, \; 1, \; 2, \; 3 \quad .$$

We associate the coordinates of this space with the space-time coordinates (in units where the light velocity is taken unity, i.e. $c = 1$), of events, i.e.

$$x^0 \; = \; t \; , \quad x^1 \; = \; x \; , \quad x^2 \; = \; y \; \text{ and } \; x^3 \; = \; z \quad .$$

A specific example of such space is the Minkowskian space-time which represents space-time in the absence of a gravitational field. Its metric tensor is given by

$$\eta \; = \; \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \; = \; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad . \tag{268}$$

The position vector of a certain particle moving with a constant velocity $\boldsymbol{\beta}$ in space, defines a trajectory in the above space-time which in components is specified by

$$x^0 \; = \; t \quad \text{and} \quad \boldsymbol{x}(t) \; = \; \boldsymbol{\beta} t \quad . \tag{269}$$

In the rest frame of the particle we might define coordinates $\{x'\}$, which according to the related Lorentz transformations are associated with the $\{x\}$ coordinates, by the usual expressions

$$\begin{aligned} t' \; &= \; \gamma \; \{t - \boldsymbol{\beta} \cdot \boldsymbol{x}\} \\ \text{and} \quad \boldsymbol{x}' \; &= \; \boldsymbol{x} \; + \; \gamma \left\{ \frac{\gamma}{1+\gamma} \, \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \boldsymbol{x}) \; - \; \boldsymbol{\beta} t \right\} \quad , \end{aligned} \tag{270}$$

where $\gamma^2 = 1/\left(1 - \beta^2\right)$.
The spatial position of the particle itself is, in its own restframe, evidently given by

$$\boldsymbol{x}'(t') \; = \; 0 \quad .$$

which result in agreement with (269).

For its proper time we find consequently, using formula(270), the following

$$t'^2 \;=\; \gamma^2 \left\{ t^2 - 2(\boldsymbol{\beta}\cdot\boldsymbol{x})t + (\boldsymbol{\beta}\cdot\boldsymbol{x})^2 \right\} \;=\; t^2 - \boldsymbol{x}^2 \;=\; \eta_{\mu\nu}\, x^\mu x^\nu \quad .$$

So, instead of using the time parameter, t, for the characterization of the trajectory, we could equally well select the proper time parameter, s, which we define by

$$ds^2 \;=\; \eta_{\mu\nu}\, dx^\mu\, dx^\nu \quad . \tag{271}$$

The above expression reminds us of a similar definition given in formula (118) for the proper lenght in ordinary space. In Minkowski space the related quantity is called the proper time. However, we must be a bit careful in using this parameter, because of the peculiar form of the metric (268). For particles which move with the velocity of light the proper time vanishes. So, for such particles we are forced to use a different parameter for the characterization of its trajectory.

At this point it might be useful to read the chapter on "Special Relativity" (chapter 2) of Steven Weinberg's book [5], which deals with Lorentz transformations, particle dynamics, electromagnetism and the Energy-Momentum tensor. But, be careful enough to notice that Weinberg's definition (2.1.3) of the metric in Minkowskian space-time differs a minus sign from the definition (268) in these notes.

39 Gravitational forces

According to the Principle of Equivalence (see section 37), there exists a local freely falling coordinate system, $\{\Sigma\}$ (compare section 35), in which the equation of motion of a particle which moves freely under the influence of purely gravitational forces, is given by (compare formula 265)

$$\frac{d^2\Sigma^\mu(s)}{ds^2} = 0 \quad . \tag{272}$$

and where the proper time, s, can be defined in the same way as the proper time of Minkowskian space-time (i.e. in the absence of gravitational forces, see formula 271).

In a different coordinate system $\{u\}$, we define the metric in analogy with formula (253), by

$$g_{\mu\nu}(u) = \eta_{\alpha\beta} \frac{\partial\Sigma^\alpha(u)}{\partial u^\mu} \frac{\partial\Sigma^\beta(u)}{\partial u^\nu} \quad . \tag{273}$$

In terms of this metric, the proper time is given by

$$ds^2 = g_{\mu\nu}\, du^\mu\, du^\nu \quad . \tag{274}$$

The equation of motion (272) in terms of the coordinates $\{u\}$, through the definition of the afine connection (formulas (105) and (106)), obtains the form

$$0 = \frac{d^2u^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}(u) \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \quad , \tag{275}$$

for which we recognize the geodesic equation (see formula 194).

For massless particles, which move at the speed of light, we have

$$ds^2 = 0 \quad . \tag{276}$$

So, for massless particles we are forced to use a different parametrization, say σ, for the geodesics. In general, in terms of a different parameter, σ, we find for the equation of motion (275), the expression

$$0 = \frac{d^2u^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}(u) \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} \quad , \tag{277}$$

and for the proper time (274), the expression:

$$ds^2 = g_{\alpha\beta}(u) \frac{du^\alpha(\sigma)}{d\sigma} \frac{du^\beta(\sigma)}{d\sigma} \quad . \tag{278}$$

40 The Schwarzschild metric

As an example of how the effect of a gravitational field can be represented by curvature, we study the Schwarzschild metric. Consider a three-dimensional space with spherical coordinates $\{u\} = \{r, \theta, \varphi\}$ which are related to the usual coordinates $\{x, y, z\}$, by

$$\begin{aligned} x &= x(r,\vartheta,\varphi) = r\sin(\vartheta)\cos(\varphi) \quad , \\ y &= y(r,\vartheta,\varphi) = r\sin(\vartheta)\sin(\varphi) \quad , \\ \text{and} \quad z &= z(r,\vartheta,\varphi) = r\cos(\vartheta) \end{aligned} \tag{279}$$

Concentrated in the center of this space we assume a source of gravitational field (e.g. the Sun) of mass M. Space-time is in this case characterized by the coordinates

$$u^0 = t \ , \ u^1 = r \ , \ u^2 = \vartheta \ , \ u^3 = \varphi \ . \tag{280}$$

Shortly after Albert Einstein formulated his general theory of relativity, in 1916, Karl Schwarzschild found a solution to Einstein's equations which describes precisely the above sketched situation. The Schwarzschild metric is given by

$$ds^2 = A(r)\,dt^2 - \frac{dr^2}{A(r)} - r^2\left\{d\vartheta^2 + \sin^2(\vartheta)\,d\varphi^2\right\} \quad , \tag{281}$$

where $A(r) = 1 - \frac{2MG}{r}$.

A first inspection of the above formula for the proper time indicates us that at large spatial distances the expression tends towards the Minkowskian proper time relation (271) for spherical coordinates. So, at large spatial distances we expect locally results similar to motion in the absence of a gravitational field.

Let us study here the geodesics of the space-time characterized by the coordinates (280) and the metric (281). For this purpose we first determine the non-zero elements of the afine connection (see formulas (105) and (106)), to find

$$\begin{aligned} &\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = -\Gamma^{r}_{rr} = \frac{A'(r)}{2A(r)} \ , \ \Gamma^{r}_{tt} = \frac{1}{2}A(r)A'(r) \ , \ \Gamma^{r}_{\vartheta\vartheta} = -rA(r) \ , \\ &\Gamma^{r}_{\varphi\varphi} = -r\sin^2(\vartheta)A(r) \ , \ \Gamma^{\vartheta}_{r\vartheta} = \Gamma^{\vartheta}_{\vartheta r} = \Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r} \ , \\ &\Gamma^{\vartheta}_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta) \ \text{ and } \ \Gamma^{\varphi}_{\vartheta\varphi} = \Gamma^{\varphi}_{\varphi\vartheta} = \text{cotg}(\vartheta) \ , \end{aligned} \tag{282}$$

where $A'(r) = \dfrac{dA(r)}{dr}$.

According to formula (277) the geodesic equations give here a set of four coupled differential equations, for μ =0, 1, 2 and 3. However, without much loss of generality we may, because of the spherical symmetry of (281), reduce this to a set of three coupled differential equations by selecting $\theta = \pi/2$, which implies $d\theta/d\sigma = 0$. It might however be instructive, to consider first the general case and then, after being convinced that no additional solutions are surpressed, reduce the set of geodesic equations by a specific choice for ϑ. This is left as an exercise for the reader.

For $\mu = 0$, we find, using formulas (277) and (282), the equation

$$\frac{d^2t}{d\sigma^2} + \frac{A'(r)}{A(r)}\frac{dt}{d\sigma}\frac{dr}{d\sigma} = 0 \quad , \tag{283}$$

which equation for $A(r) \neq 0$ can be rewritten in the form

$$\frac{d}{d\sigma}\left\{A(r)\frac{dt}{d\sigma}\right\} = 0 \quad . \tag{284}$$

Consequently, we discovered a *constant of motion*, for which, again without much loss of generality, we may choose unity, *i.e.*

$$\frac{dt}{d\sigma} = \frac{1}{A(r)} \quad . \tag{285}$$

For $\mu = 3$ the geodesic equation (277), using once more formula (282), yields

$$\frac{d^2\varphi}{d\sigma^2} + \frac{2}{r}\frac{dr}{d\sigma}\frac{d\varphi}{d\sigma} = 0 \quad , \tag{286}$$

which equation for $r \neq 0$ can be rewritten in the form

$$\frac{d}{d\sigma}\left\{r^2\frac{d\varphi}{d\sigma}\right\} = 0 \quad . \tag{287}$$

We find a second *constant of motion* to which we will give the name *angular momentum per unit mass*, and for which we will introduce the symbol, j, *i.e.*

$$\frac{d\varphi}{d\sigma} = \frac{j}{r^2} \quad . \tag{288}$$

Finally, for $\mu = 1$, also using formulas (285) and (288), we reach at the geodesic equation for the radial distance r as a function of σ:

$$\begin{aligned} 0 &= \frac{d^2r}{d\sigma^2} + \frac{1}{2}A(r)A'(r)\left(\frac{dt}{d\sigma}\right)^2 - \frac{A'(r)}{2A(r)}\left(\frac{dr}{d\sigma}\right)^2 - rA(r)\left(\frac{d\varphi}{d\sigma}\right)^2 \\ &= \frac{d^2r}{d\sigma^2} + \frac{A'(r)}{2A(r)}\left\{1-\left(\frac{dr}{d\sigma}\right)^2\right\} - rA(r)\left(\frac{j}{r^2}\right)^2 \quad , \end{aligned} \tag{289}$$

which expression, for $A(r) \neq 0$, can be rewritten in the form

$$\frac{d}{d\sigma}\left\{\frac{1}{A(r)}\left[\left(\frac{dr}{d\sigma}\right)^2 - 1\right] + \frac{j^2}{r^2}\right\} = 0 \quad . \tag{290}$$

A third *constant of motion*, for which we will use the symbol ϵ, given by:

$$\epsilon \;=\; \frac{1}{A(r)} \left[1 \;-\; \left(\frac{dr}{d\sigma} \right)^2 \right] \;-\; \frac{j^2}{r^2} \quad . \tag{291}$$

Instead of the geodesic parameter σ we could have used the ordinary time parameter t to parametrize the orbit of a geodesic. This leads for equation (290), using formula (285), to the following expression

$$\frac{d}{dt} \left\{ \frac{1}{A^3(r)} \left(\frac{dr}{dt} \right)^2 \;-\; \frac{1}{A(r)} \;+\; \frac{j^2}{r^2} \right\} \;=\; 0 \quad . \tag{292}$$

Also, we could have used the azimuthal angle, φ, to parametrize the geodesic. A similar procedure has been studied in section (30). With this choice of parameter we find for equation (290), using the result (288), the relation

$$\frac{d}{d\varphi} \left\{ \frac{1}{A(r)} \left(\frac{j}{r^2} \right)^2 \left(\frac{dr}{d\varphi} \right)^2 \;-\; \frac{1}{A(r)} \;+\; \frac{j^2}{r^2} \right\} \;=\; 0 \quad ,$$

which also can be written in the form

$$2 \, \frac{j^2}{r^3} \, \frac{1}{A(r)} \left(\frac{dr}{d\varphi} \right) \left\{ \frac{1}{r} \, \frac{d^2 r}{d\varphi^2} \;-\; \frac{2}{r^2} \left(\frac{dr}{d\varphi} \right)^2 \left[1 + \frac{1}{4} r \frac{A'(r)}{A(r)} \right] \;+\; \frac{1}{2} \, \frac{r^3}{j^2} \, \frac{A'(r)}{A(r)} \;-\; A(r) \right\} \;=\; 0 \quad ,$$

Hence, for $A(r) \neq 0$, $dr/d\varphi \neq 0$ and $r \neq 0$, moreover introducing the notation r' and r'' for respectively $dr/d\varphi$ and $d^2r/d\varphi^2$, and substituting the expression of formula (281) for $A(r)$, we obtain

$$\frac{r''}{r} \;-\; 2 \left(\frac{r'}{r} \right)^2 \left[1 + \frac{MG}{2r} \right] \;+\; \frac{MGr}{j^2} \;-\; 1 \;+\; \frac{2MG}{r} \;=\; 0 \quad . \tag{293}$$

41 Planetary orbits

In the following we will make some very mild approximations (only affecting the solutions to less than one part out of a million), in order to be capable to study without much effort and in more detail the results of the previous section.

At planetary distances from the Sun we have $r \ll 2MG$. For example, for the mass M of the Sun $2MG \approx 3 \times 10^3$ m, whereas for the distance of the Sun to the Earth one has $r \approx 1,5 \times 10^{11}$ m, which is a difference of eight orders of magnitude. Consequently, we may expand $A(r)$ in MG/r for planetary distances, keep only the lowest order terms and yet still obtain very accurate solutions for planetary motion.

In this approximation we observe from (285) that the time parameter, t, and the geodesic parameter, σ, coincide, since $A(r) \approx 1$, which leads for (288) to the relation

$$r^2 \, \frac{d\varphi}{dt} \;=\; j \quad , \tag{294}$$

which is the well-known second law of Johannes Keppler about *sweeping* equal areas in equal time-intervals.

For the geodesic equation (293) the above approximation has the consequence that terms MG/r, can be neglected with respect to unity. We obtain then to an accuracy of eight decimals the following “geodesic” equation for r:

$$\frac{r''}{r} - 2\left(\frac{r'}{r}\right)^2 + \frac{MGr}{j^2} - 1 = 0 \quad , \tag{295}$$

which has the well-known solutions

$$r(\varphi) = \frac{j^2/MG}{1 - e\cos(\varphi)} \quad , \quad e \geq 0 \quad . \tag{296}$$

The various orbits which are represented by formula (296) can be classified by the excentricity parameter, e, according to:

	1.	$e = 0$	circles
	2.	$0 < e < 1$	ellipses
	3.	$e = 1$	parabolas
and	4.	$e > 1$	hyperbolas

In order to compare easily relation (296) with geometric figures, one may define $R = j^2/MG$, $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$. Relation (296) turns then into

$$\sqrt{x^2 + y^2} - ex = r - er\cos(\varphi) = R \quad ,$$

which leads to the following relation for x and y

$$\left(\frac{x - \frac{eR}{1-e^2}}{\frac{R}{1-e^2}}\right)^2 + \left(1 - e^2\right)\left(\frac{y}{R}\right)^2 = 1 \quad .$$

We find at planetary distances, where $A(r) \approx 1$, that the geodesics of Schwarzschild's metric (281) represent the orbits of the planets as observed by Keppler. Newton's equations of motion are here represented by the geodesic equations (283), (286) and (289) for the "straight" lines at the curved surface, which are the **shortest connections between points in curved space-time**.

The conserved quantity ϵ, which has been defined in formula (291), is, in the approximation (295), related to the local energy, E/m, of the system per unit mass of the planet. From the Newtonian gravitation formula we find for the total energy per unit mass

$$\frac{E}{m} = \frac{1}{m}\left\{E(\text{kinetic}) + E(\text{gravitational field})\right\} = \frac{1}{2}\boldsymbol{v}^2 - \frac{MG}{r} \quad ,$$

which, by the relation (294), takes the form

$$\frac{E}{m} = \left(\frac{j}{r^2}\right)^2 \left\{r'^2 + r^2\right\} - \frac{2MG}{r} \quad . \tag{297}$$

Let us compare this expression with the approximation (295). The latter could have been formulated as follows

$$\frac{d}{d\varphi}\left\{\left(\frac{j}{r^2}\right)^2 \left(\frac{dr}{d\varphi}\right)^2 - 1 - \frac{2MG}{r} + \frac{j^2}{r^2}\right\} = 0 \quad , \tag{298}$$

which is the approximation at large distances of the differential equation (290) for the conserved quantity (291). We find then the relation

$$\epsilon = 1 - \frac{2E}{m} \quad . \tag{299}$$

For the specific form of the solutions (296) we may determine a relation with the excentricity parameter, e, and the above quantities. Also using (298) and (297), we find

$$\frac{2E}{m} = \left(\frac{j}{r^2}\right)^2 \left(e^2 - 1\right) \quad . \tag{300}$$

We learn then that for free particles, i.e. $E > 0$, we need $e^2 > 1$, which gives the hyperbolic solutions of (296). For bound planets, i.e. $E < 0$ we obtain from (300) that $e^2 < 1$, which gives the circles and ellipses. In the latter case also ϵ is positive, but this is generally true for material particles in the Schwarzschild metric, as we will show below.

Relation (291) can be rewritten. Using formulas (285) and (288), we obtain the following form for this constant of motion

$$\epsilon = A(r)\left(\frac{dt}{d\sigma}\right)^2 - \frac{1}{A(r)}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\varphi}{d\sigma}\right)^2 \quad ,$$

which expression, using the definition (281) of the proper time, takes, for the case $\theta = \pi/2$, the form

$$\epsilon = \frac{ds^2}{d\sigma^2} \geq 0 \quad . \tag{301}$$

Hence, we come in general to the conclusion that $\epsilon = 0$, for massless particles, and $\epsilon > 0$, for massive particles.

The Schwarzschild metric (281) not only reproduces the orbits of planets around the Sun, but also the deflection of light by the Sun as observed during Sun eclipses for stars behind the Sun, the perihelion precession of the planet Mercury and more (see for example Weinberg's book [5], chapter 8, *Classic tests of Einstein's Theory*). Moreover, gravitational collapse and black holes can be studied using the Schwarzschild metric. This is well described in chapter 11 of Weinberg's book.

Part VI
The Einstein equations

42 Preliminaries

The field equations for gravity must be more complex than the similar equations for electromagnetism, for a very simple reason. Namely, the electromagnetic field interacts with electric charge, but does not carry charge itself, whereas the gravitational field interacts with energy and momentum, but also carries energy and momentum itself. Therefor, the Maxwell equations are just linear differential equations. However, for the gravitational field we must expect more complicated and non-linear expressions.

The principle of equivalence states that inertial mass and gravitational mass are equal. In Newtonian gravity that has as a consequence that the mass m of a test particle, with

$$m = m_{\text{inertial}} = m_{\text{gravitational}} \quad , \tag{302}$$

does not come in the equations of motion when it moves (with low velocity) in the gravitational field of a distant massive source of gravity with mass M, namely

$$m_{\text{inertial}} \frac{d^2\vec{x}}{dt^2} = G\frac{m_{\text{gravitational}} M}{d^2} \quad \Longleftrightarrow \quad \frac{d^2\vec{x}}{dt^2} = \frac{GM}{d^2} \quad . \tag{303}$$

Here d represents the distance of the test particle from the source, whereas $\vec{x}$ are its local coordinates.

In terms of the metrical tensor, the principle of equivalence is expressed by the fact that, at any point X in an arbitrary strong gravitational field, one can define locally inertial coordinates x, such that

$$g_{\mu\nu}(x = X) = \eta_{\mu\nu} \quad \text{and} \quad \left(\frac{\partial g_{\mu\nu}(x)}{\partial x^\sigma}\right)_{x = X} = 0 \quad , \tag{304}$$

where $\eta_{\mu\nu}$ represents the in-the-absence-of-any-gravitational-field Minkowskian metric. Hence, in the locally inertial coordinates x, all derivatives of $g_{\mu\nu}(x)$ vanish at the point $x = X$. Consequently, on expanding the metric around the point X, one has

$$\begin{aligned} g_{\mu\nu}(x) \ &= \\ &= g_{\mu\nu}(X) + (x-X)^\sigma \left(\frac{\partial g_{\mu\nu}(x)}{\partial x^\sigma}\right)_{x = X} + \frac{1}{2}(x-X)^\sigma (x-X)^\rho \left(\frac{\partial^2 g_{\mu\nu}(x)}{\partial x^\sigma \partial x^\rho}\right)_{x = X} + \dots \\ &= \eta_{\mu\nu} + \mathcal{O}\left((x-X)^2\right) \quad . \end{aligned} \tag{305}$$

In the weak gravitational field approximation, we have Newton's theory to guide us. We assume that the gravitational field is somehow described by the metrical field $g_{\mu\nu}(x)$. Hence, for a slowly moving test particle, near $x = X$, in a weak stationary gravitational field, we assume that its motion is described by the geodesic equation (194), given by

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\ \alpha\beta}(x) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \quad , \tag{306}$$

where τ represents the proper time of the test particle along its geodesic.

Since, furthermore, the test particle is sufficiently slow (*i.e.* for its velocity $\vec{v}$ one has $v \ll 1$), we find

$$\left|\frac{d\vec{x}}{d\tau}\right| \ll \left|\frac{dt}{d\tau}\right| \quad . \tag{307}$$

As a consequence, one obtains for the equation of motion (306) of the test particle, the expression

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{00}(x)\left(\frac{dt}{d\tau}\right)^2 \approx 0 \quad . \tag{308}$$

Moreover, since the gravitational field is stationary, all time derivatives of $g_{\mu\nu}(x)$ vanish, which results for the affine connections in

$$\Gamma^{\mu}_{00}(x) \approx -\frac{1}{2}g^{\mu i}(x)g_{00,i}(x) \qquad (i = 1, 2, 3) \quad . \tag{309}$$

Furthermore, since the gravitational field is weak, one may write

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad |h_{\mu\nu}(x)| \ll 1 \quad . \tag{310}$$

We obtain for (309) the expression

$$\Gamma^{\mu}_{00}(x) \approx -\frac{1}{2}\eta^{\mu i}(x)\frac{\partial h_{00}(x)}{\partial x^i} \qquad (i = 1, 2, 3) \quad ,$$

or

$$\Gamma^{j}_{00}(x) \approx -\frac{1}{2}\eta^{ji}(x)\frac{\partial h_{00}(x)}{\partial x^i} \quad (i, j = 1, 2, 3) \quad \text{and} \quad \Gamma^{0}_{00}(x) \approx 0 \quad . \tag{311}$$

Substitution of the result (311) in equation (308), gives

$$\frac{d^2\vec{x}}{d\tau^2} \approx \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \nabla h_{00}(x) \quad \text{and} \quad \frac{d^2t}{d\tau^2} \approx 0 \quad . \tag{312}$$

The second equation implies $dt/d\tau$ is constant. We may divide out that constant in the first equation of (312), to obtain

$$\frac{d^2\vec{x}}{dt^2} \approx \frac{1}{2}\nabla h_{00}(x) \quad . \tag{313}$$

The corresponding Newtonian equation for a slowly moving particle in a stationary gravitational field, is given by

$$\frac{d^2\vec{x}}{dt^2} = -\nabla\frac{GM}{r} \quad , \tag{314}$$

thereby assuming that the locally inertial coordinates are chosen such that the source of the gravitational field is situated in the origin of the coordinate system, and $r = |\vec{x}\,|$. Hence, we could adopt the following identification

$$h_{00}(x) = -2\frac{GM}{r} \quad \Longleftrightarrow \quad g_{00}(x) = 1 - 2\frac{GM}{r} \quad . \tag{315}$$

You may compare the above expression to the g_{00} element of the Schwarzschild solution (281).

43 The energy-momentum density tensor

In section 42, we have assumed that the source of gravity is a pointlike massive object, with mass M, situated in the origin of the locally inertial coordinate system. Here, we will consider more general mass distributions.

For a point particle, with mass M and situated in the origin of the locally inertial coordinate system, it makes sense to define its energy-momentum density tensor by

$$T^{00}(x) = M\delta^{(3)}(\vec{x}\,) \quad \text{and} \quad T^{\mu\nu}(x) = 0 \quad \text{for} \quad \mu \neq 0 \quad \text{or} \quad \nu \neq 0 \quad , \tag{316}$$

since, in that case, one obtains for the total energy

$$E = \int d^3x\, T^{00}(x) = M \quad . \tag{317}$$

For a moving point particle along a path given by $x^\mu(t)$, with velocity $\vec{v} = d\vec{x}/dt$, one could define the following energy-momentum tensor

$$T^{\mu\nu}(x) = M\,\frac{dx^\mu(t)}{dt}\,\frac{dx^\nu(t)}{dt}\,\delta^{(3)}\left(x - x(t)\right) \quad . \tag{318}$$

That gives for the total energy of the point particle

$$E = \int \frac{d^3x}{\sqrt{1-\vec{v}^{\,2}}}\,T^{00}(x) = \frac{M}{\sqrt{1-\vec{v}^{\,2}}} \quad , \tag{319}$$

and for the i-th component of its momentum

$$p^i = \int \frac{d^3x}{\sqrt{1-\vec{v}^{\,2}}}\,T^{0i}(x) = \frac{M}{\sqrt{1-\vec{v}^{\,2}}}\,\frac{dx^i(t)}{dt} = \frac{Mv^i}{\sqrt{1-\vec{v}^{\,2}}} \quad . \tag{320}$$