Lecture Course

" Some Non-Perturbative Means in Quantum Physics"

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with accent on

- Bogoliubov Renormalization Group,
- Analyticity and Spectral Representations
- Analytic Perturbation Theory in QCD

Plan of the Course

"Non-Perturb. Means in Quantum Physics"

- Need in Non-Perturbation methods
- Simple Nature of Renormalization Group
- The Renormalization Group Method
- Analyticity and Dispersion Relations
- RG + Analyticity =APT

Lecture I - Need for Non-Perturb methods

Feynman Series $\sum c_k \alpha^k$ *isn't Convergent!*

Plan of the Lecture I:

- Dyson 1952 argument; The ill-posed Problem
- Functional (Path) Integral representation
- Singularity at $\alpha = o$; Factorial grouth $c_k \sim k!$
- Asymptotic Series; "Practical convergence" in QCD
- Possible solution for QCD – APT

Series $\sum c_k \alpha^k$ is not Convergent !

a. Dyson' 1952 argument;

In QED, change $\alpha \to -\alpha$ is equivalent to $e \to i\, e$.
As $S = T(\exp i \int L_{int}(x)\, dx) = T(e^{i\, e \int j_\mu A^\mu\, dx})$, such a change destroys Unitarity. Hence, in the complex $\alpha \to z$ plane,the origin $\alpha = 0$ can't be a regular point.

b. The ill-posed Problem

Small parameter g at highest nonlinearity - indispensable attribute of Quantum Perturbation:

- First, one quantize linear eq. (as a set of oscillators).
- 2nd, one takes into account non-linear term(s) $\sim g \ll 1$ as a small perturbation.

Non-linearity change equation seriously – new solutions !

Functional Integral

Functional Integral (FI) representation – general, powerful method for problems with huge Nos. degree of freedom - class.& quant. statistics, turbulence, QFT. It ascends to Dirac (mid30ies) and Feynman path integral for Quant Mechanics. The Functional Integral is a formal limit of a multiple one

$$\int \delta x e^{\frac{i}{\hbar} S} = \lim_{n \to \infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} dx_1 \ldots dx_n \, e^{\frac{i}{\hbar} S}, \quad S = \int L(t, x) dt$$

with S , the classical action along the trajectory.
The FI is a rather natural within quasi-classical limit of QM being useful for general analysis of Quant.Stat. and QFT amplitudes. There, $S = \int \mathcal{L} \left(\phi(x), \partial \phi(x) \right) dx, \;\; dx = dt \, \mathbf{dx}$.

Singularity at $g = 0$. Factorial growth $c_k \sim k!$

The most general way to analyze the issue - to use the Functional Integral (path integral) representation.

Instead, for illustration, consider the 0-dim analog

$$I(g) = \int_{-\infty}^{\infty} e^{-x^2 - gx^4}\, dx \qquad (1)$$

Expanding it in power-in-g series, one obtain

$$I(g) \sim \sum_{k=0} (-g)^k I_k ; \quad I_k = \frac{\Gamma(2k+1/2)}{\Gamma(k+1)} \to 2^k\, k! \qquad (2)$$

Meanwhile, $I(g)$ can be expressed via special, MacDonald function $I(g) = exp(1/8g)\, K_{1/4}(1/8g)\frac{1}{\sqrt{2g}}$ with known analytic properties in complex g plane.

Essential Singularity at $g = 0$

The $I(g)$ is a 4-sheeted function of the complex variable g, analytical in the whole complex plane with a cut from the origin $g = 0$. There, it has an essential singularity $e^{-1/8g}$ and can be written down in the Cauchy integral form

$$I(g) = \sqrt{\pi} - \frac{g}{\sqrt{\pi}} \int_0^\infty \frac{d\gamma \, exp(-1/8\gamma)}{\gamma(g+\gamma)} \quad (3)$$

As far as the origin is not an analytical point, the *power Taylor series has no convergence domain for real positive g* values – in concert with (2).

Also, the *series is not valid for negative g* values – in accordance with Dyson's reasoning.

Besides, via integral (1) one can study analytic properties of $I(g)$ in the complex g plane *by steepest-descent method*.

Factorial growth & Singularity at $g = 0$ in QFT

For the QFT case,
1. one can use, within **functional integral** representation, technique of saddle-point method. By this way, (*Lipatov '77)* it's possible to prove *factorial growth* of expansion coefficients in the ϕ^4 scalar and few other QFT models. These results have been anticipated in '52-'53 *(Hurst, Thirring, Peterman)* just after Dyson' paper.

2. The same singularity stricture $\sim exp(-1/g)$, like in (3), was established by different approach. By combining perturbation result with two other non-perturbative methods – **analyticity** and **Renormalization Invariance**.
As a result

$$f_{pert} = 1 + \beta_0\,\alpha\,\ln(Q^2) \to f(Q^2\,e^{-1/\beta_0\,\alpha})$$

Asymptotic Series; "Practical convergence"

The Henri Poincaré (end of XIX) analysis of Asymptotic (non-convergnt) Series (AS) can be summed as follows:

AS can be used for obtaining quantitative information on expanded function. Here, the error of approximation $F(K,g)$-

(first K terms of expansion) -
$$F(g) \to F(K,g) = \sum_{k=1}^{K} F_k(g)$$
is equal to the last detained term, $F_K(g)$.

For the power AS , $F_k(g) = f_k\, g^k$, with factorial growth $f_k \sim k!$, like in (1), the absolute values of expansion terms $F_k(g)$ cease to diminish at $k \sim 1/g$. This yields to natural the best possible accuracy of a given AS.

In contrast to convergent series ! [picture]

“Practical convergence” is bad for QCD

In QED, this “divergence menace” is not actual, as the real expansion parameter is quite small $(\alpha/\pi) \sim 1/420 \sim 2.10^{-3}$. At the same time, as it is well-known, in perturbative QCD

$$A_{QCD} = \Sigma_k A_k = \Sigma_k a_k (\bar{\alpha}_s)^k; \quad a_k \sim 1\,, \tag{4}$$

with expansion parameter below 5-10 GeV being *not very small*: $\bar{\alpha}_s(Q) \sim 0.2 - 0.3$. With critical order $K \sim 3-5$, the menace of “exploding" of the pQCD series is actual. Indeed

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Table. Relative size (in %) of 1- , 2- and 3-loop terms to observables

Process	Energy	1st	2nd	3rd
GLS sum rule	2-4 GeV	65	24	11
Bjorken. s.r.	2-3	55	26	**19**
Incl.τ-decay	0-2	55	29	**16**
$e^+e^- \to$ hadr.	10 GeV	96	8	- 4

Possible Remedy – Analytic Perturbation Theory

– Hence, a practical Non-Perturbation approach to perturbative QCD is of utmost importance.
– Below, we concentrate on Analytic Perturbation Theory (APT), a closed theoretical scheme that combines information from PT with two other non-perturb. methods – **Analyticity** and **Renormalization Group**.
Due to this,

- First, we outline the *Renorm. Group* approach and *Renorm Group Method*
- Then, ideas of *Dispersion Relation* method are presented
- On this basis, *APT method* and results are given.