Center of Mass

Consider a system of two point–like objects of masses m_1 and m_2 which move, by mutual interaction, with velocities $\vec{v}_1$ and $\vec{v}_2$. Let their position vectors with respect to some inertial frame at instant t be given by $\vec{r}_1(t)$ and $\vec{r}_2(t)$. We define their relative position vector $\vec{r}$ by

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \tag{1}$$

The center–of–mass position vector $\vec{R}_{\text{CM}}$ is given by

$$\vec{R}_{\text{CM}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{2}$$

Then we have the following.
The position vector $\vec{r}_1$ is given by

$$\begin{aligned}\vec{r}_1 &= \vec{R}_{\text{CM}} + \vec{r}_1 - \vec{R}_{\text{CM}} \\ &= \vec{R}_{\text{CM}} + \frac{(m_1 + m_2)\,\vec{r}_1 - m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 + m_2} \\ &= \vec{R}_{\text{CM}} + \frac{m_2\vec{r}_1 - m_2\vec{r}_2}{m_1 + m_2} = \vec{R}_{\text{CM}} - \frac{m_2}{m_1 + m_2}\,\vec{r}\,,\end{aligned} \tag{3}$$

whereas the position vector $\vec{r}_2$ is given by

$$\begin{aligned}\vec{r}_2 &= \vec{R}_{\text{CM}} + \vec{r}_2 - \vec{R}_{\text{CM}} \\ &= \vec{R}_{\text{CM}} + \frac{(m_1 + m_2)\,\vec{r}_2 - m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 + m_2} \\ &= \vec{R}_{\text{CM}} + \frac{m_1\vec{r}_2 - m_1\vec{r}_1}{m_1 + m_2} = \vec{R}_{\text{CM}} + \frac{m_1}{m_1 + m_2}\,\vec{r}\,.\end{aligned} \tag{4}$$

We observe that

$\vec{r}_1 = \vec{R}_{\text{CM}} - \dfrac{m_2}{m_1 + m_2}\vec{r}$ and $\vec{r}_2 = \vec{R}_{\text{CM}} + \dfrac{m_1}{m_1 + m_2}\vec{r}$.

Hence, the center–of–mass of the system comes at the line which connects m_1 and m_2. Moreover, the distance of the center–of–mass to m_1 is proportional to m_2, namely equal to

$$\left|\vec{r}_1 - \vec{R}_{\text{CM}}\right| = \frac{m_2}{m_1 + m_2}\ |\vec{r}|\ , \tag{5}$$

whereas the distance of the center–of–mass to m_2 is proportional to m_1, namely equal to

$$\left|\vec{r}_2 - \vec{R}_{\text{CM}}\right| = \frac{m_1}{m_1 + m_2}\ |\vec{r}|\ . \tag{6}$$

The velocity $\vec{v}_1$ is given by

$$\vec{v}_1 = \dot{\vec{r}}_1 = \dot{\vec{R}}_{\text{CM}} - \frac{m_2}{m_1 + m_2}\,\dot{\vec{r}}\ , \tag{7}$$

whereas the velocity $\vec{v}_2$ is given by

$$\vec{v}_2 = \dot{\vec{r}}_2 = \dot{\vec{R}}_{\text{CM}} + \frac{m_1}{m_1 + m_2}\,\dot{\vec{r}}\ . \tag{8}$$

The total linear momentum $\vec{p}_{\text{tot}}$ of the system is given by

$$\begin{aligned} \vec{p}_{\text{tot}} &= \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \\ &= m_1\dot{\vec{R}}_{\text{CM}} - \frac{m_1 m_2}{m_1 + m_2}\,\dot{\vec{r}} + m_2\dot{\vec{R}}_{\text{CM}} + \frac{m_1 m_2}{m_1 + m_2}\,\dot{\vec{r}} \\ &= (m_1 + m_2)\,\dot{\vec{R}}_{\text{CM}} \end{aligned} \tag{9}$$

When no external forces act on the system, then we have, by Newton's first law, that the total linear momentum of the system is conserved, *i.e.*

$$\mathbf{0} = \dot{\vec{p}}_{\text{tot}} = (m_1 + m_2)\,\ddot{\vec{R}}_{\text{CM}} \quad \Longleftrightarrow \quad \ddot{\vec{R}}_{\text{CM}} = \mathbf{0} \tag{10}$$

This implies that the center–of–mass moves with constant velocity $\dot{\vec{R}}_{\text{CM}}$.

We define the relative linear momentum of the system $\vec{p}$ by

$$\begin{aligned}\vec{p} &= \vec{p}_2 - \vec{p}_1 = m_2\vec{v}_2 - m_1\vec{v}_1 \\ &= m_2\dot{\vec{R}}_{\text{CM}} + \frac{m_1m_2}{m_1+m_2}\,\dot{\vec{r}} - m_1\dot{\vec{R}}_{\text{CM}} + \frac{m_1m_2}{m_1+m_2}\,\dot{\vec{r}} \\ &= (m_2 - m_1)\,\dot{\vec{R}}_{\text{CM}} + 2\,\frac{m_1m_2}{m_1+m_2}\,\dot{\vec{r}} \qquad (11)\end{aligned}$$

When we, furthermore, define the reduced mass μ of the system by

$$\mu = \frac{m_1m_2}{m_1+m_2} \qquad (12)$$

then we obtain for the relative momentum

$$\vec{p} = (m_2 - m_1)\,\dot{\vec{R}}_{\text{CM}} + 2\mu\dot{\vec{r}} \qquad (13)$$

For its time derivative, also using relation (10), we find

$$\dot{\vec{p}} = (m_2 - m_1)\,\ddot{\vec{R}}_{\text{CM}} + 2\mu\ddot{\vec{r}} = 2\mu\ddot{\vec{r}} \qquad (14)$$

On the other hand, when $\vec{F}_{12}$ represents the force which is acted upon object m_2 by object m_1 and $\vec{F}_{21}$ the force which is acted upon object m_1 by object m_2, also assuming that no external forces act on the system, then we have

$$\dot{\vec{p}} = \dot{\vec{p}}_2 - \dot{\vec{p}}_1 = \vec{F}_{12} - \vec{F}_{21} \qquad (15)$$

However, from Newton's third law (reaction equals and is opposite to action) we know that $\vec{F}_{21} = -\vec{F}_{12}$. Hence,

$$2\mu\ddot{\vec{r}} = \dot{\vec{p}} = 2\vec{F}_{12} \quad\Longleftrightarrow\quad \mu\ddot{\vec{r}} = \vec{F}_{12} \qquad (16)$$

The internal dynamics is given by equation (16).

When external forces, $\vec{F}_{1,\text{ext}}$ on object m_1 and $\vec{F}_{2,\text{ext}}$ on object m_2, are relevant, we find the following.
For equation (10) we obtain

$$\begin{aligned} \vec{F}_{1,\text{ext}} + \vec{F}_{2,\text{ext}} &= \vec{F}_{21} + \vec{F}_{1,\text{ext}} + \vec{F}_{12} + \vec{F}_{2,\text{ext}} = \dot{\vec{p}}_1 + \dot{\vec{p}}_2 = \dot{\vec{p}}_{\text{tot}} \\ &= (m_1 + m_2)\, \ddot{\vec{R}}_{\text{CM}} \end{aligned} \tag{17}$$

Hence, as expected, the center–of–mass dynamics is only determined by the external forces on the two–body system and the total mass of the system.
For equation (16) we obtain

$$\begin{aligned} \ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 &= \frac{\dot{\vec{p}}_2}{m_2} - \frac{\dot{\vec{p}}_1}{m_1} \\ &= \frac{\vec{F}_{12} + \vec{F}_{2,\text{ext}}}{m_2} - \frac{\vec{F}_{21} + \vec{F}_{1,\text{ext}}}{m_1} \\ &= \left(\frac{1}{m_2} + \frac{1}{m_1}\right) \vec{F}_{12} + \frac{\vec{F}_{2,\text{ext}}}{m_2} - \frac{\vec{F}_{1,\text{ext}}}{m_1} \\ &= \frac{m_1 + m_2}{m_1 m_2}\, \vec{F}_{12} + \frac{\vec{F}_{2,\text{ext}}}{m_2} - \frac{\vec{F}_{1,\text{ext}}}{m_1} \\ &= \frac{1}{\mu}\, \vec{F}_{12} + \frac{\vec{F}_{2,\text{ext}}}{m_2} - \frac{\vec{F}_{1,\text{ext}}}{m_1} \end{aligned}$$

Hence,

$$\mu \ddot{\vec{r}} = \vec{F}_{12} + \frac{\mu \vec{F}_{2,\text{ext}}}{m_2} - \frac{\mu \vec{F}_{1,\text{ext}}}{m_1} \tag{18}$$

Consequently, only for gravitational forces near the Earth's surface where $\vec{F}_{2,\text{ext}}$ is about equal to $-m_2 g \hat{z}$ and $\vec{F}_{1,\text{ext}}$ about equal to $-m_1 g \hat{z}$, we find again equation (16). However, for the Moon-Earth system we clearly find an extra tidal force for the internal dynamics of that system.

The total kinetic energy of the system can be calculated by the use of equations (7) and (8) as follows (notice that the cross terms cancel).

$$\begin{aligned}
E_{\mathrm{kin}} &= \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \\
&= \tfrac{1}{2} m_1 \left(\dot{\vec{R}}_{\mathrm{CM}} - \frac{m_2}{m_1 + m_2}\, \dot{\vec{r}} \right)^2 + \\
&\quad + \tfrac{1}{2} m_2 \left(\dot{\vec{R}}_{\mathrm{CM}} + \frac{m_1}{m_1 + m_2}\, \dot{\vec{r}} \right)^2 \\
&= \tfrac{1}{2} \left(m_1 + m_2 \right) \dot{\vec{R}}_{\mathrm{CM}}^{\,2} + \tfrac{1}{2} \frac{m_1 m_2^2 + m_2 m_1^2}{\left(m_1 + m_2 \right)^2}\, \dot{\vec{r}}^{\,2} \\
&= \tfrac{1}{2} \left(m_1 + m_2 \right) \dot{\vec{R}}_{\mathrm{CM}}^{\,2} + \tfrac{1}{2} \mu \dot{\vec{r}}^{\,2} \qquad (19)
\end{aligned}$$

We find that the total kinetic energy consists of a term which depends on the total mass of the system and the center–of–mass velocity and a term which represents the internal kinetic energy and which depends on the reduced mass and the relative velocity.