Mecânica dos Meios Continuos

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Contents

Chapter 1

Fluid flow patterns in two dimensions.

Although in principle restricted to the flow of incompressible fluids in cases where the Reynolds number is large, the subject to be studied in this chapter has also given an important contribution to the development of the shape of objects which move with high velocities in air or water. Consequently, the subject can be found in the literature under the alternative names: Hydrodynamics, Fluid Dynamics and Aerodynamics.

In this chapter we consider non-viscous or frictionless fluids and furthermore we only study flow patterns in two dimensions. The basic formulas will not be derived here with great rigor, but are introduced following an intuitive approach. At first we will introduce the stream function, the velocity potential and the complex potential for the study of flow patterns in two dimensions. Then we enter the subject of forces and moments excerted on obstacles emersed in a fluid flow. At that stage we introduce the law of Bernoulli for the relation between pressure and velocity in a fluid flow. Moreover subjects like analytic functions and complex contour integration will then be discussed. At the end of the chapter we give some practical applications of the theory for objects moving in air.

1.1 The equation of motion.

The quantities which represent the state of a fluid (liquid or gas) are its density (ρ) and its velocity ($\vec{v}$). The density can differ from place to place and thus is a function of the position vector, $\vec{r}$. Moreover, the density of a fluid at a given position can change with time. Consequently, the density is also a function of time, t. In general is the density ρ of a fluid represented by a function of position and time, according to:

$$\rho = \rho(\vec{r}, t). \tag{1.1}$$

The motion of a fluid can be characterized in each point of space by a definite direction, represented by a vector ($\vec{j}$), which in general might also be a function of time, *i.e.*:

$$\vec{j} = \vec{j}(\vec{r}, t). \tag{1.2}$$

This vector is called the *current* of the fluid flow. Its length $| \vec{j} |$ indicates the amount of fluid passing per unit of time through a unit surface perpendicular to the direction of the current.

It might be clear that there must exist a relation between the change in time of the density ρ at a given position $\vec{r}$ and the behavior of the current $\vec{j}$ in the vicinity of that position. When, for example, in a small domain the current vectors are all directed outward, then this indicates that the density at the position of this domain must decrease in time. Now, the amount of fluid which flows away from a given position is represented by the divergence of the current $(\nabla \cdot \vec{j})$, whereas the decrease in density is given by minus its derivative in time $(-\partial\rho/\partial t)$. The related equation is the *continuity equation*, given by:

$$\frac{\partial}{\partial t}\rho(\vec{r}, t) + \nabla \cdot \vec{j}(\vec{r}, t) = 0. \tag{1.3}$$

Later on we will discuss a more elegant derivation of this equation.

For incompressible fluids the density is constant in space and independent of time. In that case the continuity equation (1.3) reduces to the following differential equation:

$$\nabla \cdot \vec{v}(\vec{r}, t) = 0, \tag{1.4}$$

where $\vec{v}(\vec{r}, t)$ represents the velocity of the fluid at a certain position at a certain time. In the following we concentrate on incompressible fluids.

1.2 Examples of flow patterns in two dimensions.

In this and the following sections we concentrate on fluids for which the velocity pattern does not depend on one of the space dimensions (for example the vertical direction in the case a stream of water). The two remaining space directions will be indicated by x and y. In the case of incompressible fluids the equation of motion (1.4) reduces then to:

$$\frac{\partial v_x(x, y)}{\partial x} + \frac{\partial v_y(x, y)}{\partial y} = 0. \tag{1.5}$$

Here we ignore the time dependence, since it is not of much relevance because no derivatives in time appear in (1.5). The above equation (1.5) is the basic equation of the present chapter.

In general, the space of pairs of functions (v_x, v_y) which form a solution of equation (1.5) is infinite. Only after imposing a sufficient amount of boundary conditions, a specific solution can be well determined. Here, however, we will not specify those boundary conditions, but, with the help of several examples, demonstrate how one can construct solutions which have certain well defined properties.

A first inspection of equation (1.5) reveals that for any reasonable function $\psi(x, y)$ of the two variables x and y, a solution is given by:

$$v_x(x,y) = \frac{\partial \psi(x,y)}{\partial y} \qquad \text{and} \qquad v_y(x,y) = -\frac{\partial \psi(x,y)}{\partial x}. \tag{1.6}$$

For the moment we will not define "reasonable", but just discuss below some characteristic examples of fluid flows.

All properties of a given fluid flow can be derived from the knowledge of its related function ψ which for that reason is called the *stream function*.

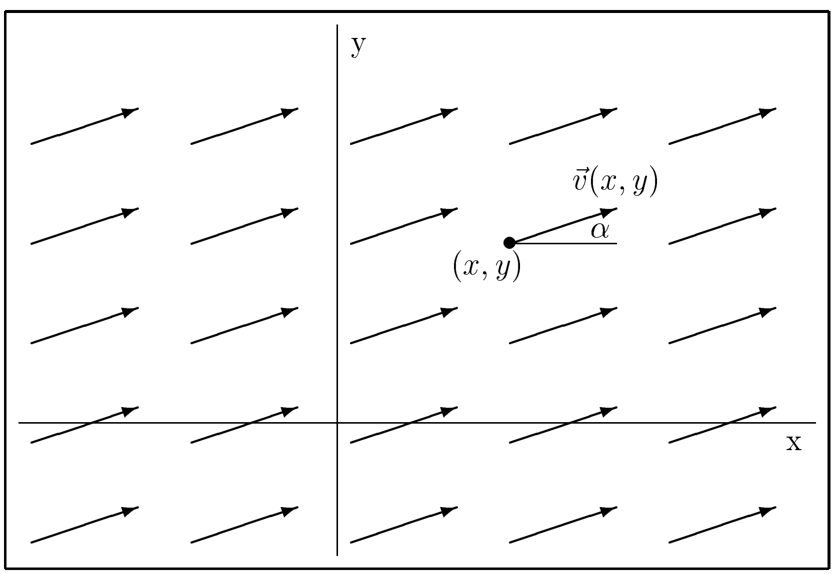
Example 1.

$$\psi(x,y) = -x\sin(\alpha) + y\cos(\alpha). \tag{1.7}$$

Using relation (1.6), we find in this case for the velocity vector field the expression:

$$\vec{v}(x,y) = (v_x(x,y), v_y(x,y)) = (\cos(\alpha), \sin(\alpha)),$$

which represents a constant unit vector field which makes a fixed angle α with the horizontal axis. The corresponding situation is shown in the figure below:



We observe that this case represents a constant fluid flow in a direction which makes an angle α with the x-axis or, alternatively, the flow pattern around an object which moves in the opposite direction (*i.e.* towards the left in the above figure) through a fluid at rest.

The representation of a vector field, like in the figure above, is intended to facilitate the discovery of the pecularities of such vector field. To each point in space (the xy-plane in this case) is associated a vector (the velocity at that point in this case). In the above figure we selected several points to depict the vector which represents the "value" of the vector field associated with the stream function (1.7), at that point. For example, the vector value $\vec{v}(x,y)$ of the field at the point (x,y) has in the figure the same point (x,y) as its point of application.

Example 2.

$$\psi(x,y) = 2xy. \tag{1.8}$$

Again using relation (1.6), we find here for the velocity vector field the expression:

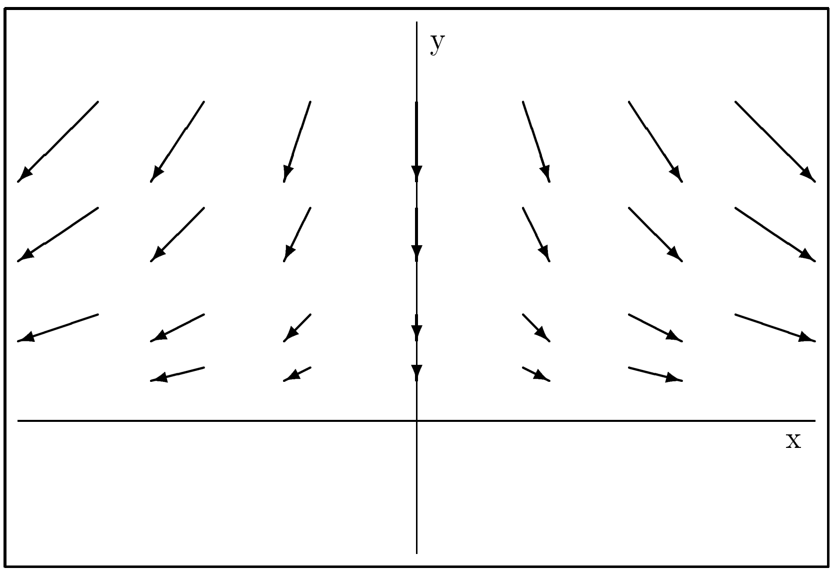
$$\vec{v}(x,y) = (v_x(x,y), v_y(x,y)) = (2x, -2y),$$

which represents a vector field which has different directions in different positions and of which the absolute values, given by:

$$|\ \vec{v}(x,y)\ | = 2\sqrt{x^2+y^2},$$

also vary with place.

The corresponding situation is shown in the figure below:



This case represents a fluid flow against a wall. For obvious reasons we have only shown the velocity vector field in the upper half (x,y) plane.

Notice that the scale for the velocity field differs from the scale for the position vectors: $1m/s$ is represented by 1/8 of the unit which represents $1m$.

At the point $\vec{r} = (x = 0, y = 0)$ the velocity vanishes. Such point is called a stagnation point of the fluid flow. In a river it is a place where objects collect which float in water, such as branches and leafs of trees and plastic material.

Example 3.

$$\psi(x, y) = arctg\left(\frac{y}{x}\right). \tag{1.9}$$

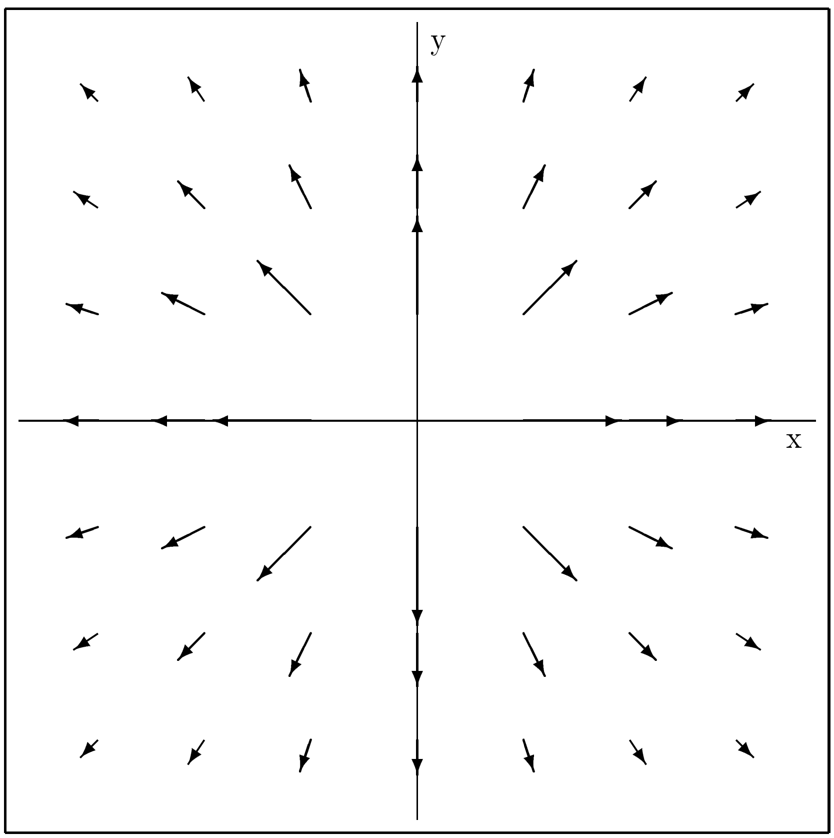
Another time using relation (1.6), we find here for the velocity vector field the expression:

$$\vec{v}(x, y) = (v_x(x, y), v_y(x, y)) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right).$$

The absolute values of the velocities are inverse proportional to their distances from the center, *i.e.*:

$$| \vec{v}(x, y) | = \frac{1}{\sqrt{x^2 + y^2}}.$$

The corresponding situation is shown in the figure below:



This field represents the flow of a fluid around a source situated in the center of the coordinate system.

Notice that here the scale for the velocity field is the same as the scale for the position vectors: $1m/s$ is represented by the same unit which represents $1m$.

Example 4.

$$\psi(x,y) = \frac{1}{2}\log(x^2+y^2). \tag{1.10}$$

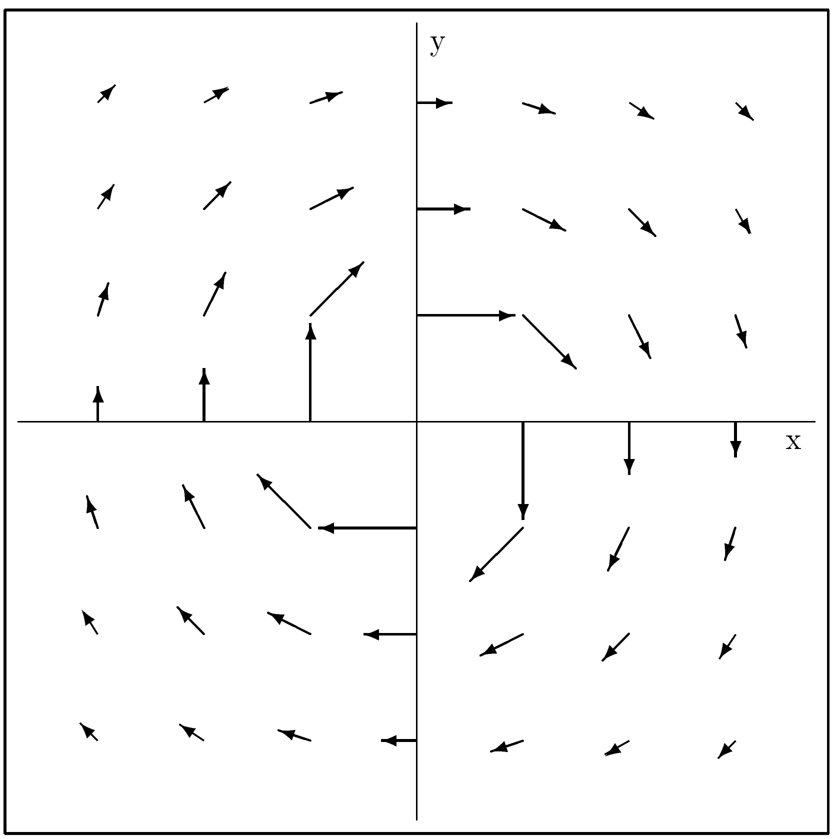
Once more using relation (1.6), we find here for the velocity vector field and its absolute values the expressions:

$$\vec{v}(x,y) = (v_x(x,y), v_y(x,y)) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right),$$

and

$$\mid \vec{v}(x,y) \mid = \frac{1}{\sqrt{x^2+y^2}}.$$

The corresponding situation is shown in the figure below:



The vector field of this example represents the flow of a fluid around a vortex which is situated in the origin of the coordinate system.

Notice that also here the scale for the velocity field is the same as the scale for the position vectors: $1m/s$ is represented by the same unit which represents $1m$.

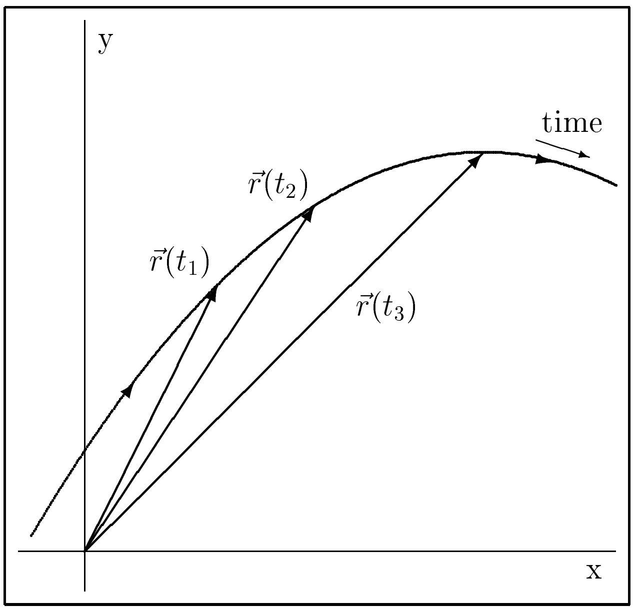
1.3 Stream lines.

The trajectory of a small object (point particle) which is taken by the stream of a fluid flow, is described by a curve in the two-dimensional space. Such curve is called a *stream line*.

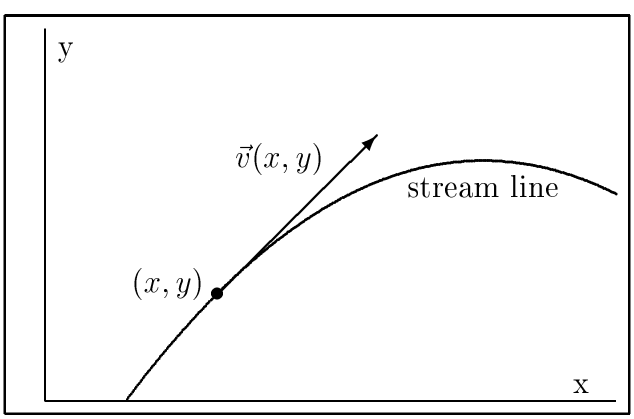
Let us denote the position vector of the object at a given instant, t, by:

$$\vec{r}(t) = (x(t), y(t)).$$

In the figure below we have illustrated of such object its trajectory as well as its position vectors at various different times ($t_1 < t_2 < t_3$):



At each point (x, y) of this curve, the direction of motion of the object is given by the local velocity vector $\vec{v}(x, y)$ of the fluid flow. Consequently, the velocity vector $\vec{v}(x, y)$ is tangential to the stream line at the position (x,y), as depicted in the figure below:



Let us assume that at instant t the position vector is given by $\vec{r}(t)$ and at instant $t + \Delta t$ by $\vec{r}(t + \Delta t)$. The components $x(t + \Delta t)$ and $y(t + \Delta t)$ of the position vector $\vec{r}(t + \Delta t)$ can be related to the components $x(t)$ and $y(t)$ of the position vector $\vec{r}(t)$ via the following Taylor expansions:

$$x(t + \Delta t) = x(t) + \left(\left.\frac{dx}{dt}\right|_{\text{at t}} \right) \Delta t + \cdots$$

and

$$y(t + \Delta t) = y(t) + \left(\left.\frac{dy}{dt}\right|_{\text{at t}} \right) \Delta t + \cdots .$$

Rewriting the above expressions, one finds the components Δx and Δy of the displacement vector $\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$, *i.e.*

$$\Delta x = x(t + \Delta t) - x(t) = \left(\left.\frac{dx}{dt}\right|_{\text{at t}} \right) \Delta t + \cdots$$

and

$$\Delta y = y(t + \Delta t) - y(t) = \left(\left.\frac{dy}{dt}\right|_{\text{at t}} \right) \Delta t + \cdots .$$

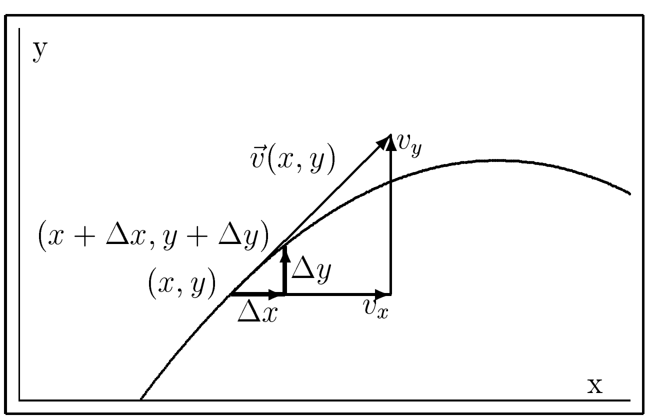
However, the derivative with respect to time of $x(t)$ at the given instant t represents the x-component v_x of the velocity vector at the position $(x(t), y(t))$. And similar for the derivative of $y(t)$. So, we obtain for the components of the displacement vector $\vec{\Delta r}$, the following expressions:

$$\Delta x = v_x(x, y)\Delta t + \cdots \quad \text{and} \quad \Delta y = v_y(x, y)\Delta t + \cdots .$$

In the limit for $\Delta t \to 0$, one finds:

$$\left.\frac{dx}{dy}\right|_{(x,y)} = \frac{v_x(x, y)}{v_y(x, y)}. \tag{1.11}$$

In the figure below a geometrical derivation of the above relation (1.11) is shown:



At each point of the stream line the stream function ψ has a definite value depending on the position (x, y). Now, because x and y at the stream line are parametrized by t, the stream function becomes at the stream line a function of time according to:

$$\psi(t) = \psi(x(t), y(t)).$$

Its derivative with respect to time is given by:

$$\left.\frac{d\psi}{dt}\right|_t = \left.\frac{dx}{dt}\right|_t \left.\frac{\partial\psi}{\partial x}\right|_{(x(t),\, y(t))} + \left.\frac{dy}{dt}\right|_t \left.\frac{\partial\psi}{\partial y}\right|_{(x(t),\, y(t))}.$$

Using the definition (1.6) of the stream function as well as the general definition of velocity $\vec{v} = d\vec{r}/dt$ one finds:

$$\left.\frac{d\psi}{dt}\right|_t = v_x(x(t), y(t))\ \{-v_y(x(t), y(t))\} + v_y(x(t), y(t))\ v_x(x(t), y(t)) = 0. \quad (1.12)$$

The derivative of the stream function with respect to the stream line parameter time t, vanishes. Consequently, the stream function is constant along a stream line. Or, in other words: Stream Lines are lines which connect points of constant value for the stream function. Below we show some examples:

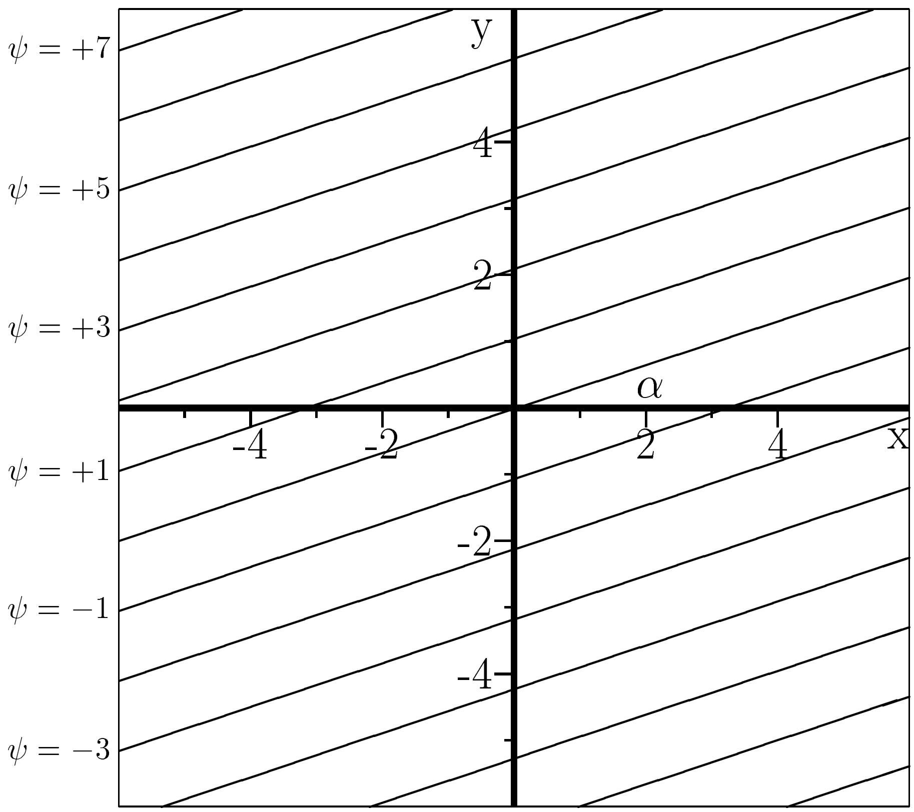
Example 1.

$$\psi(x,y) = -x\sin(\alpha) + y\cos(\alpha). \tag{1.13}$$

The lines which connect points for constant values of ψ are given by:

$$y = x\,tg(\alpha) + \frac{\psi}{\cos(\alpha)},$$

which relation represents a set of straight lines, parametrized by ψ, which all make a definite angle α with the horizontal axis. The corresponding situation is shown in the figure below (compare example (1.7)):



For the angle between the stream lines and the x-direction in the above example, $tg(\alpha) = 1/3$ ($\alpha \approx 18.4°$) has been chosen. The various stream lines in the figure correspond to the values $-7, -6, \cdots, +6, +7$ for ψ.

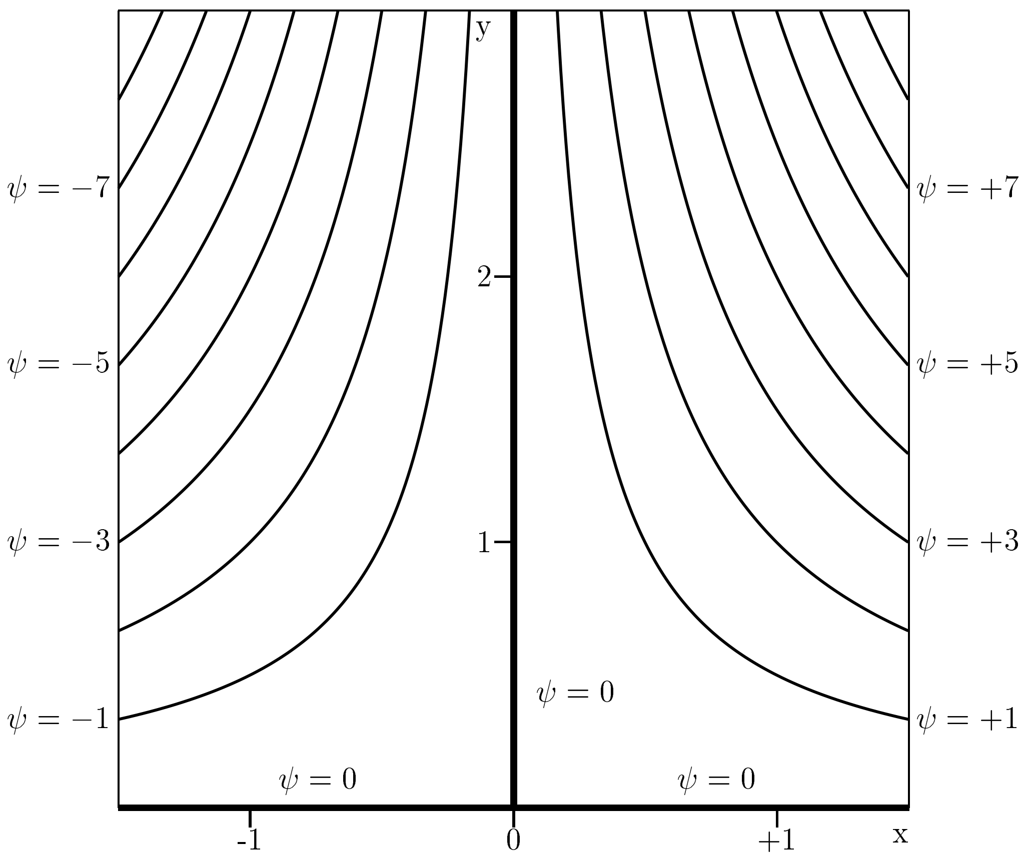
Example 2.

$$\psi(x, y) = 2xy. \tag{1.14}$$

The lines which connect points for constant values of ψ are given by:

$$y = \frac{\psi}{2x},$$

which relation represents a set of hyperbolas, parametrized by ψ, visualizing the flow of a fluid against a wall (x-axis) as shown in the figure below (compare example (1.8)):



The various stream lines in the figure correspond to the values $-8, -6, \cdots, +6, +8$ for ψ. Notice that the stream function is constant ($\psi = 0$ in this case) along the surface of the wall (x-axis). In general, stream lines are smoothly varying functions of position, except at the stagnation points where two different stream lines intersect at 90° angles.

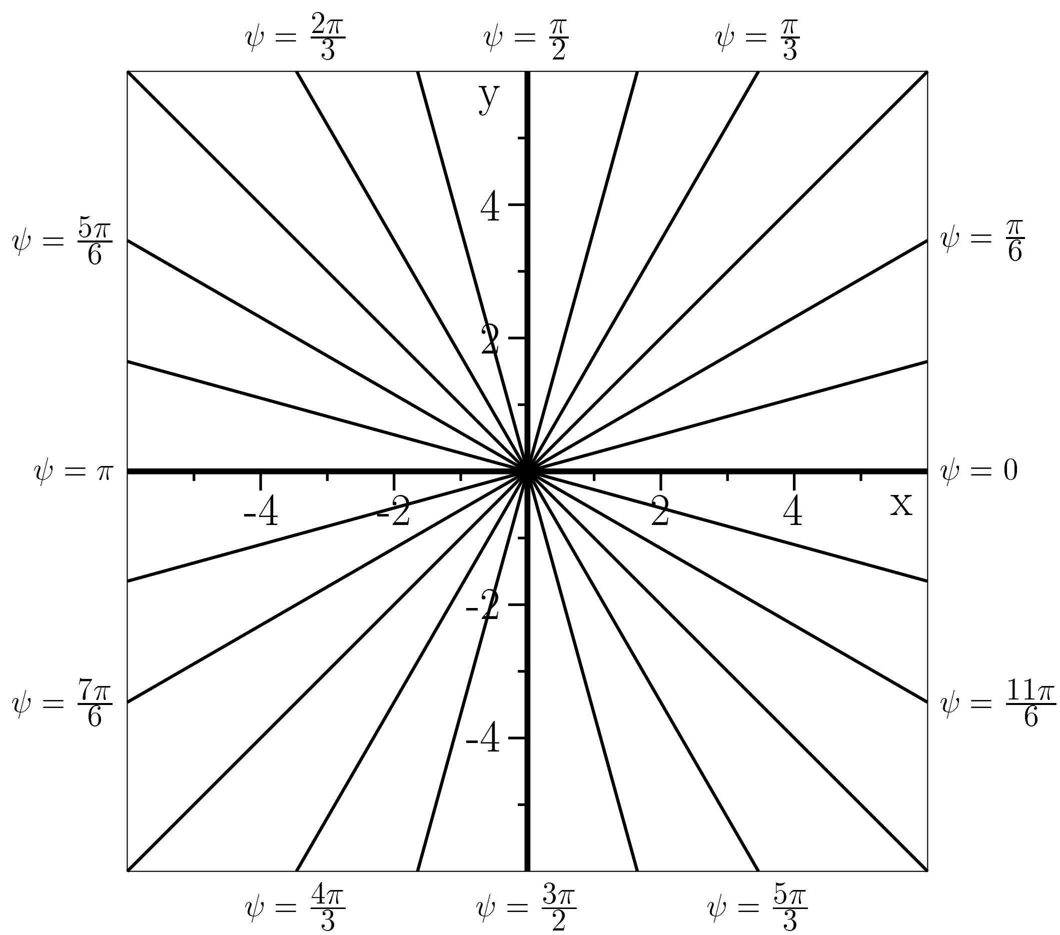
Example 3.

$$\psi(x,y) = arctg(\frac{y}{x}). \tag{1.15}$$

The lines which connect points for constant values of ψ are given by:

$$y = x\ tg(\psi).$$

This relation represents a set of straight lines starting in the origin under different angles with the x-axis and which are parametrized by ψ. As is shown in the figure below, it corresponds to the flow around a source of fluid in the origin (compare example (1.9)):



The various stream lines which are shown in the above figure correspond to the values 0, $\pi/12$, $2\pi/12$, $\cdots$, $23\pi/12$ for ψ.

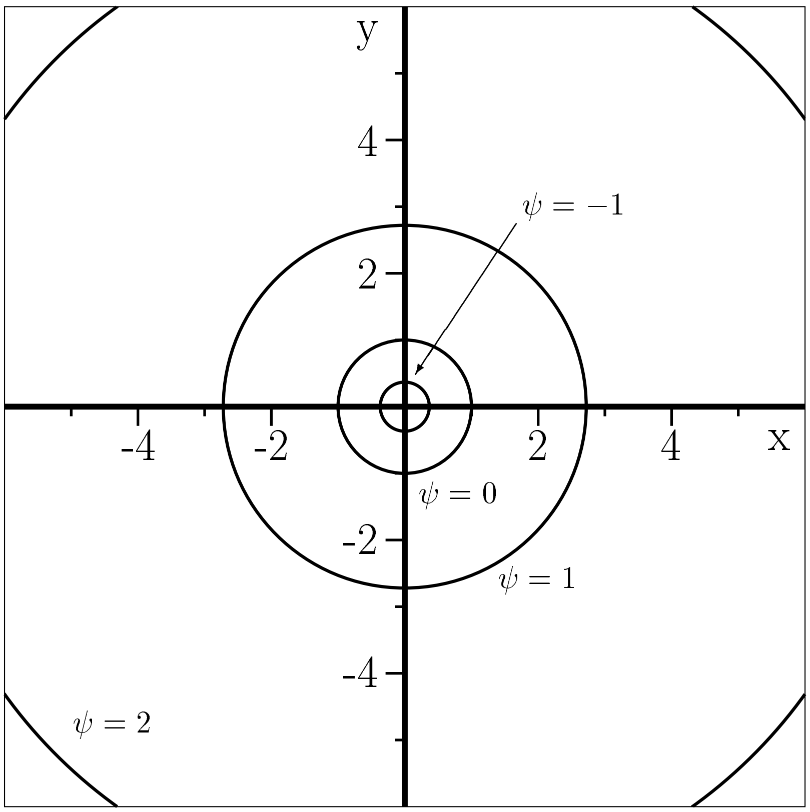
Example 4.

$$\psi(x,y) = \frac{1}{2}\ \log(x^2 + y^2). \tag{1.16}$$

The lines which connect points for constant values of ψ are given by:

$$x^2 + y^2 = \{\exp(\psi)\}^2.$$

This relation represents a set of circles around the origin with different radii $R = \exp(\psi)$ parametrized by ψ. This case corresponds to the fluid flow around a vortex in the origin (compare example (1.10)):



The stream lines in the above figure correspond to the values -1, 0, 1 and 2 for ψ. The radii, R, grow exponentially for increasing values of ψ. In the limit $\psi \to -\infty$, the radius vanishes.

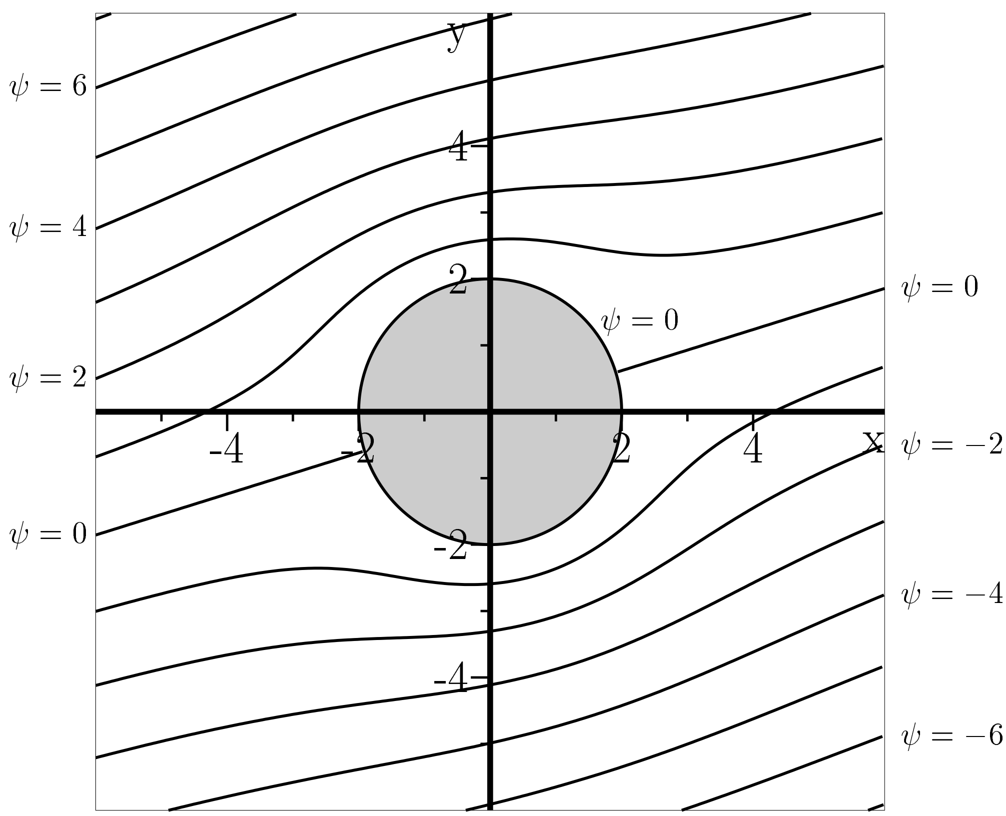
Example 5.

$$\psi(x,y) = \{1 - \frac{R^2}{x^2+y^2}\}\{-x\sin(\alpha) + y\cos(\alpha)\}, \quad \text{for} \quad x^2+y^2 \geq R^2. \qquad (1.17)$$

An expression of y in terms of x for the lines which connect points for constant values of ψ is rather difficult in this case. However, for some special choices of coordinates it is reasonably simple to determine the values of ψ:

In the first place we might consider points (x,y) at the circle given by $x^2+y^2 = R^2$. At those locations we find $\psi = 0$. Consequently, the circle with radius R represents a stream line of the fluid flow. Next we observe that ψ also vanishes at the (stream)line given by $y = xtg(\alpha)$. Furthermore, we might notice that for large values of x^2+y^2, the expression (1.17) tends towards the relation (1.13). So, we may expect that at large distances from the origin the stream lines in the present case tend towards the stream lines of example 1, (1.13). Finally, it is also relatively easy albeit tedious, to determine the coordinates of points for definite values of ψ at the line given by $y = xtg(\alpha + \pi/2)$.

Below we show the computer result for $\alpha = 0,3$ ($\approx 17,2°$) and $R = 2$:



The above figure corresponds to the flow of a fluid around a circular obstacle. Notice that the surface of the obstacle itself represents a stream line.

Problem 1:

Consider in example (1.17) the points at the line given by $y = x\ tg(\alpha + \pi/2)$, which might be parametrized by:

$$(x, y) = (-q\sin(\alpha),\ q\cos(\alpha)\ )\ \ ,\quad |\ q\ | \geq R.$$

Show that the point(s) of intersection of this line with the stream line given by ψ, is determined by:

$$2q = \begin{cases} \psi + \sqrt{\psi^2 + 4R^2} & , \quad \psi \geq 0 \quad \text{, and} \\ \\ \psi - \sqrt{\psi^2 + 4R^2} & , \quad \psi \leq 0 \quad . \end{cases}$$

Verify the above formula, selecting $R = 2$, for the values $\psi = -6, \cdots, +6$ with the figure of example (1.17).

1.4 Velocity potential and equipotential lines.

The velocity potential $\phi(x, y)$ is a function of the two variables x and y which is related to the velocity vector field $\vec{v}(x, y)$ in the following way:

$$\vec{v}(x, y) = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right). \tag{1.18}$$

As a consequence of the above definition for the velocity potential ϕ and the definition (1.6) for the stream function ψ there exists the following relation between those two functions:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{1.19}$$

Problem 2:

Verify that the velocity potentials corresponding to the preceding examples (1.13), (1.14), (1.15), (1.16) and (1.17), are respectively given by:

$$\phi(x, y) = x\cos(\alpha) + y\sin(\alpha), \tag{1.20}$$

$$\phi(x, y) = x^2 - y^2, \tag{1.21}$$

$$\phi(x, y) = \frac{1}{2}\ \log(x^2 + y^2), \tag{1.22}$$

$$\phi(x, y) = -arctg\left(\frac{y}{x}\right), \tag{1.23}$$

and

$$\phi(x, y) = \{1 + \frac{R^2}{x^2 + y^2}\}\{x\cos(\alpha) + y\sin(\alpha)\}, \quad \text{for} \quad x^2 + y^2 \geq R^2. \tag{1.24}$$

Lines which connect the points of constant velocity potential in the (x,y)-plane are called *velocity equipotential lines*, or just *equipotential lines*, and can be found from the relation:

$$0 = d\phi = \frac{\partial\phi}{\partial x}\,dx \;+\; \frac{\partial\phi}{\partial y}\,dy = v_x dx + v_y dy,$$

which leads to the following equation:

$$\frac{dx}{dy} = -\frac{v_y(x, y)}{v_x(x, y)}. \tag{1.25}$$

When we compare the above formula with the relation given in formula (1.11), then we find that the tangential directions for respectively a stream line and an equipotential line in any point of the (x,y)-plane are perpendicular. In some of the preceding examples it is relatively easy to demonstrate this property:

Problem 3:

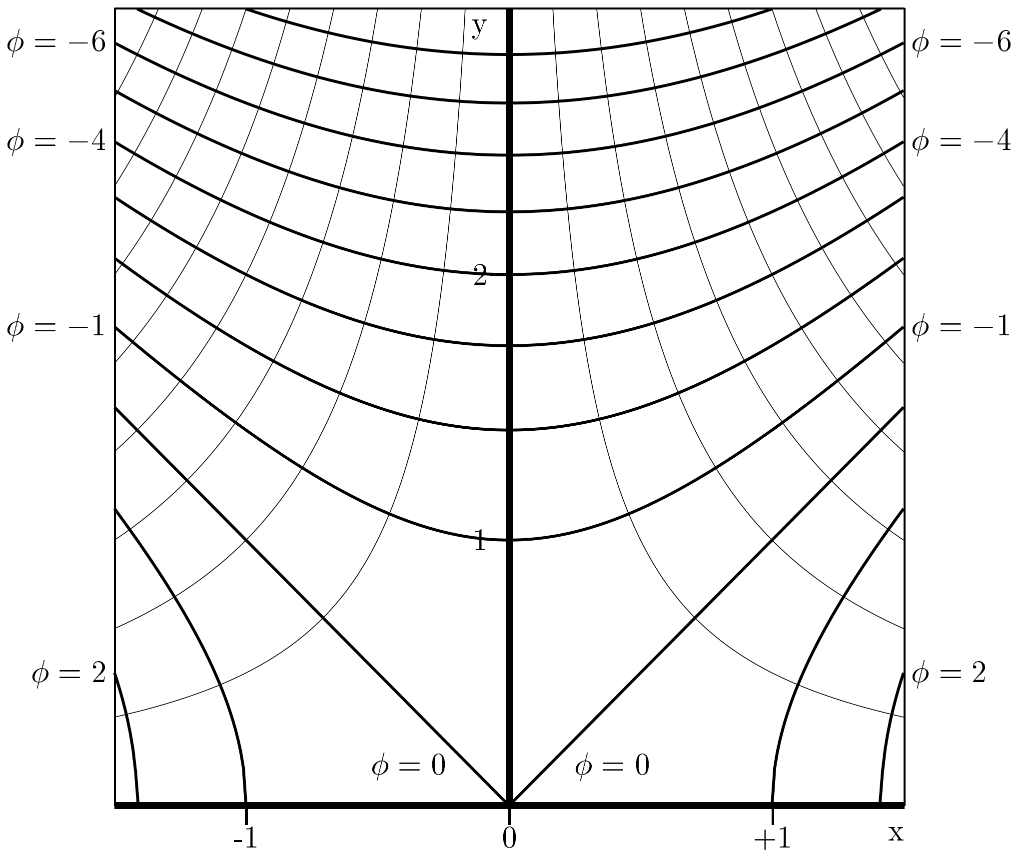
Verify that the velocity equipotential lines and stream lines corresponding to the preceding examples numbered by 1, given in (1.13) and (1.20), by 3, given in (1.15) and (1.22) and by 4, given in (1.16) and (1.23), are perpendicular in their points of intersection.

Example 2.

For the case of the fluid flow against a wall, (1.14) and (1.21), the lines which connect points for constant values of ϕ are given by:

$$y^2 = x^2 - \phi.$$

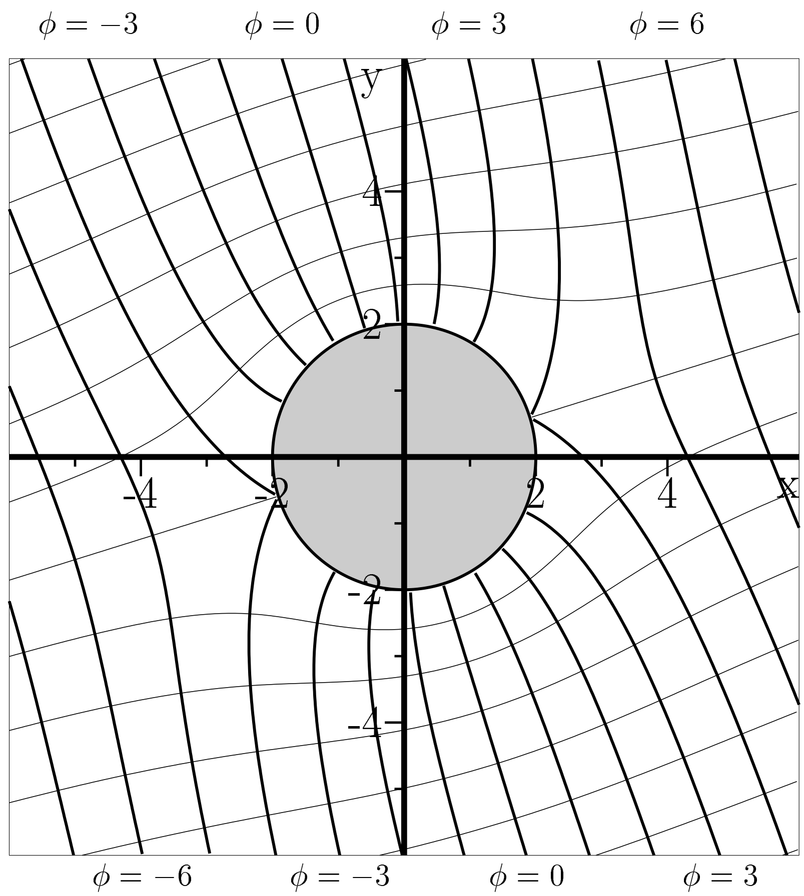
The corresponding equipotential lines are shown below.



In the above figure we also show the stream lines in this case. Notice that stream lines and equipotential lines all intersect under angles of 90°, except in the origin, *i.e.* in the stagnation points of this example.

Example 5.

For the case of the fluid flow around a circular obstacle, (1.17) (1.24), the lines which connect points for constant values of ϕ are shown below:



The above figure is the computer result for $\alpha = 0,3$ ($\approx 17,2^\circ$) and $R = 2$. We also show the stream lines in this case. Notice that stream lines and equipotential lines all intersect under angles of 90°, except in the two stagnation points at the surface of the obstacle. These latter points are for (1.17) given by $(R\cos(\alpha), R\sin(\alpha))$ and $(-R\cos(\alpha), -R\sin(\alpha))$, which are just opposite of each other.

1.5 The complex potential and analyticity.

Points in two dimensions can be represented by vectors or alternatively by complex numbers. For example, a point which is characterized by the position vector $\vec{r} = (x, y)$, can equally well be represented by a complex number, *i.e.*

$$z = x + iy.$$

In the literature one finds the velocity potential ϕ and the stream function ψ combined into a complex function Ω, defined as follows:

$$\Omega(z = x + iy) = \phi(x, y) + i\psi(x, y). \tag{1.26}$$

This function is called the *complex potential* and like the stream function, it contains all information about the system of a two-dimensional steady-state flow of an incompressible non-viscous fluid.

Problem 4:

Verify that for the five examples (1.20 to 1.24), the respective complex potentials are given by:

$$\Omega(z) = ze^{-i\alpha}, \tag{1.27}$$

$$\Omega(z) = z^2, \tag{1.28}$$

$$\Omega(z) = \log(z), \tag{1.29}$$

$$\Omega(z) = i\log(z), \tag{1.30}$$

and

$$\Omega(z) = ze^{-i\alpha} + \frac{R^2}{ze^{-i\alpha}}, \quad \text{for} \quad |\, z \,| \geq R. \tag{1.31}$$

In section 1.1 we introduced the stream function $\psi(x, y)$ as any "reasonable" function of the two variables x and y. In the following we will specify the term "reasonable" in more detail.

An important class of complex functions of complex variables is formed by the class of *analytic functions*:

A complex function $f(z)$ of the complex variable z, is called analytic in a region $\mathcal{R}$, of the complex z-plane, if the derivative of the function with respect to z, $f'(z)$, exists in all points of the region $\mathcal{R}$.

The derivative of the complex function $f(z)$ with respect to z, is defined in the usual way:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}. \tag{1.32}$$

However, $\Delta z = \Delta x + i\Delta y$ can be any complex number. One might for instance select real values for Δz, *i.e.* $\Delta z = \Delta x$. In that case one finds for the derivative (1.32) the expression:

$$f'(z = x + iy) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x + iy) - f(x+iy)}{\Delta x} = \left.\frac{\partial f(x+iy)}{\partial x}\right|_{y \text{ constant}}. \tag{1.33}$$

But one might equally well select Δz purely imaginary, *i.e.* $\Delta z = i\Delta y$. In that case one obtains for the derivative (1.32) the following:

$$f'(z = x + iy) = \lim_{i\Delta y \to 0} \frac{f(x + iy + i\Delta y) - f(x+iy)}{i\Delta y} = \left.\frac{\partial f(x+iy)}{\partial iy}\right|_{x \text{ constant}}. \tag{1.34}$$

The two expressions (1.33) and (1.34) do not necessarily give the same result for an arbitrary function f of z. Let us for example consider the function $f(z)$ defined by:

$$f(z = x + iy) = Re(z) = x.$$

For the two partial derivatives (1.33) and (1.34), one finds respectively the results:

$$\left.\frac{\partial f(z)}{\partial x}\right|_{y \text{ constant}} = 1 \quad \text{and} \quad \left.\frac{\partial f(z)}{\partial iy}\right|_{x \text{ constant}} = 0.$$

In such cases the limit (1.32) cannot be well defined and consequently, the derivative of $f(z)$ does not exist.

In general, one says that the derivative of $f(z)$ can be defined if:

1. the limit (1.32) exists, and
2. is independent of the choice of Δz.

Such functions are called analytic. A function $f(z)$ is said to be *analytic* in a point $z = z_0$ of the complex z-plane, when there exists a region $\mathcal{R}$ around that point where the function is analytic.

A necessary condition that $f(z = x + iy) = h(x, y) + ig(x, y)$ be analytic in a region $\mathcal{R}$ of the complex z-plane, is that its real part $h(x, y)$ and its imaginary part $g(x, y)$ satisfy the *Cauchy-Riemann equations* in that region, *i.e.*

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial h}{\partial y} = -\frac{\partial g}{\partial x}. \tag{1.35}$$

If moreover, these partial derivatives are continuous in $\mathcal{R}$, then the Cauchy-Riemann equations (1.35) are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$.

Problem 5:
Using the relation (1.19), show that the complex potentials $\Omega(z)$ defined by (1.27), (1.28), (1.29), (1.30) and (1.31) are analytic everywhere in the complex z-plane, except the point $z = 0$ for (1.31).

Any analytic complex function $\Omega(z)$ can serve as the complex potential which describes the two-dimensional steady-state flow of an incompressible non-viscous fluid. This way the term "reasonable" of section (1.1) has found a more precise definition.

The velocity vector in two dimensions can also be represented by a complex number, *i.e.*

$$v(z = x + iy) = v_x(x, y) + iv_y(x, y). \tag{1.36}$$

Its relation with the complex potential can be established by using the fact that $\Omega(z)$ is an analytic function which satisfies the Cauchy-Riemann equations (1.35 and 1.19). We obtain:

$$\Omega'(z = x + iy) = \frac{d\Omega}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = v_x(x, y) - iv_y(x, y) = v^*(z),$$

or equivalently:

$$v(z) = \{\Omega'(z)\}^*, \tag{1.37}$$

where the superscript * stands for complex conjugation.

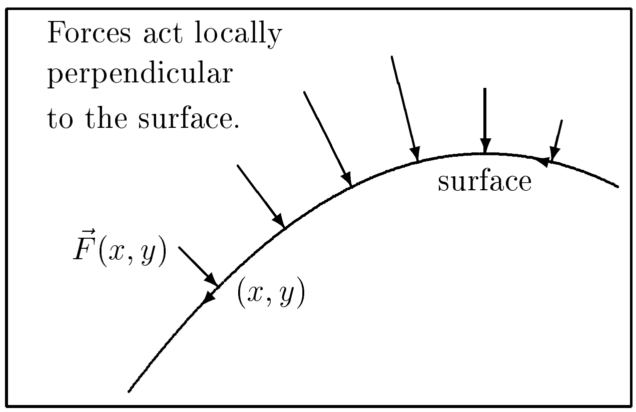
Problem 6:
Verify that the velocity vector fields of the examples (1.7) to (1.10) can also be derived from the complex potentials given in respectively the formulas (1.27) to (1.30), using the above relation (1.37).

1.6 Obstacles in a fluid flow.

In the following we study the forces excerted on obstacles which are emersed in a fluid. The scalar quantity related to forces in fluids is the force per unit area, or pressure p. In general this quantity will depend on position and thus becomes in two dimensions a function of the coordinates x and y or equivalently of the complex position z, *i.e.*

$$p = p(x, y) = p(z = x + iy). \tag{1.38}$$

However, different from the previously defined quantities, pressure is just a scalar local constant of proportionality and is therefore always represented by a *real* number. The force $\vec{F}(x, y)$ excerted by the fluid on a given surface element ΔS of the obstacle, is in absolute value equal to the product of the area of ΔS and the local pressure $p(x, y)$. Its orientation is perpendicular to the surface and directed towards the inside of the obstacle (see figure below).



In two dimensions a surface element ΔS is represented by a line element, let us say $\Delta z = \Delta x + i\Delta y$, whereas the whole surface of the obstacle is represented by a closed line. Let us assume that we go around the obstacle in counterclockwise direction. In that case, the vector which has the same absolute value as Δz and which is perpendicular to the surface and directed inward is represented by $-\Delta y + i\Delta x$. Consequently, the *complex* force at the above surface element is given by:

$$\Delta F(x, y) = p(x, y)\{-\Delta y + i\Delta x\} = ip(x, y)\{\Delta x + i\Delta y\} = ip(z)\Delta z.$$

The above result represents the contribution to the total force of the force at the surface element Δz. The total complex force excerted by the fluid at the obstacle is consequently given by:

$$F = F_x + iF_y = \oint_{\text{closed line counterclockwise}} dz\ ip(z). \tag{1.39}$$

Next, we need a relation between the pressure $p(z)$ and the complex potential $\Omega(z)$ or, equivalently, the complex velocity $v(z)$. Such relation exists and is referred to as *Bernoulli's theorem*, which for a horizontal fluid flow reduces to the following relation:

$$p(z) + \frac{\rho}{2} \mid \vec{v}(x,y) \mid^2 = p_0, \tag{1.40}$$

where ρ represents the fluid density and where p_0 is a constant along any stream line. In the next section Bernoulli's theorem will be discussed. But for the moment we will anticipate on the result of that section. Moreover, as we have seen in the previous examples, the boundary of an obstacle coincides with a stream line of the fluid flow. Consequently, p_0 is constant along the surface of the obstacle.

So, using the equations (1.37), (1.39) and (1.40), we find for the total force F on an obstacle which is emersed in a fluid, the following:

$$F = \oint_{\text{closed stream line counterclockwise}} dz\, i\{p_0 - \frac{\rho}{2} \mid \Omega'(z) \mid^2\}. \tag{1.41}$$

In order to proceed with this complex integration, we must be familiar with the properties of closed contour integrals in the complex plane. To this aim, we will study in the following the necessary theory of complex integration. However, first we still have to discuss Bernoulli's theorem. So, the coming sections are devoted to these two subjects. After this preparation we will return to the issue of forces and moments on obstacles emersed in a fluid.

Problem 7:
Let the boundary of the circular obstacle in example (1.17) be described by:

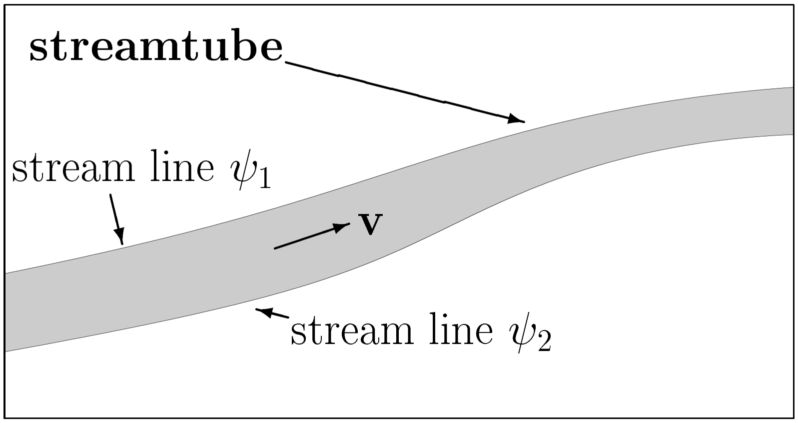
$$z = R\, e^{i\varphi} \quad ,$$

and let moreover the density of the fluid be equal to $\rho = 1$ and the constant of formula (1.40) be given by $p_0 = 1$. Show that the pressure $p(\varphi)$ along the surface of the obstacle as a function of the angle φ, yields:

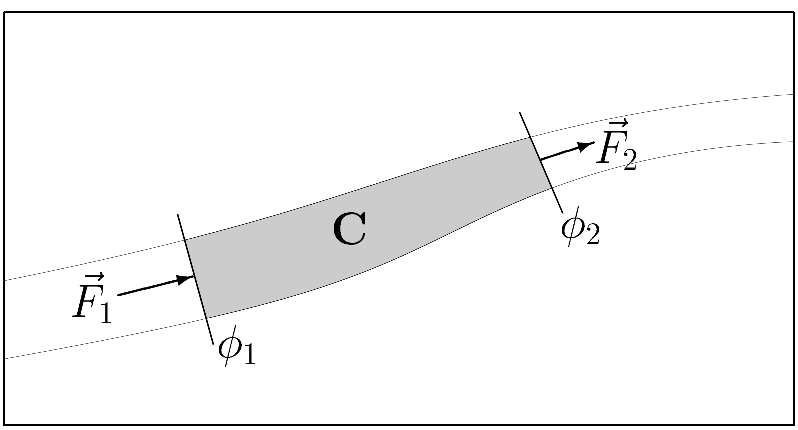
$$p(\varphi) = \cos(2\alpha - 2\varphi).$$

1.7 Bernoulli's theorem.

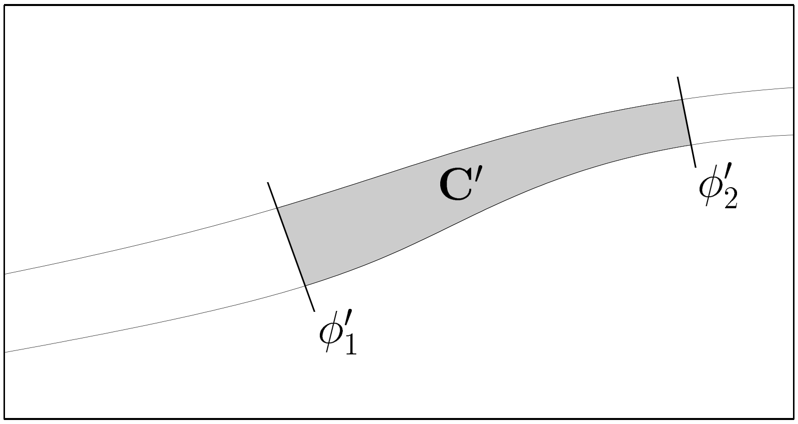
Let us consider a tube of fluid which is enclosed in a surface of stream lines. Such tube of fluid is called a *streamtube*. In two dimensions it is represented by the area in between two stream lines ψ_1 and ψ_2 as is shown in the figure below:



In the above represented streamtube we select a certain quantity of fluid $\mathcal{M}$ which at a given instant of time occupies a certain area of the streamtube, indicated by $\mathbf{C}$ in the figure below. For this quantity of fluid we will study its gain in kinetic energy ΔK when it moves through the streamtube. The area $\mathbf{C}$ is bounded by two equipotential lines, indicated by ϕ_1 and ϕ_2, as depicted in the figure below:

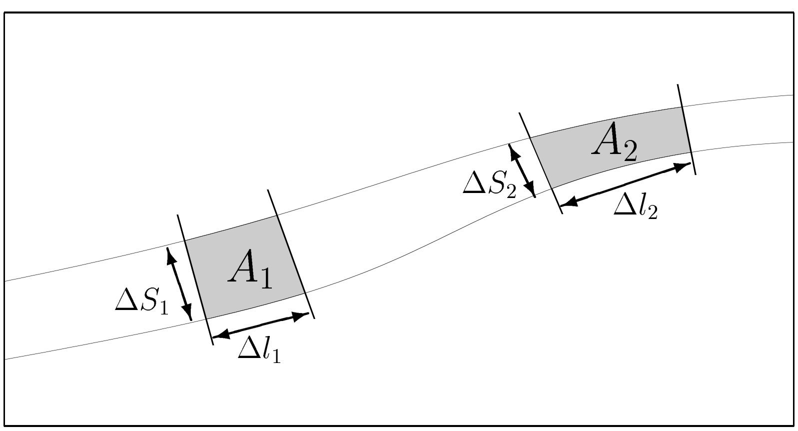


We assume that the direction of the fluid flow is from the left to the right. At the lefthand side of the area **C** acts a force indicated by $\vec{F}_1$ on the quantity of fluid $\mathcal{M}$. One might consider that this force pushes through the streamtube all fluid which encounters itself to the right of the equipotential line ϕ_1. At the righthandside of **C** acts a force $\vec{F}_2$ on all fluid which is in the streamtube to the right of the area **C**. The difference in work done by $\vec{F}_1$ and $\vec{F}_2$ must be equal to the gain in kinetic energy ΔK of the quantity of fluid $\mathcal{M}$ in **C**.



In the above figure we show the position of the quantity of fluid $\mathcal{M}$ at a later time. It ocupies a different area, $\mathbf{C}'$, still in the same streamtube as before, but in between two different equipotential lines, ϕ'_1 and ϕ'_2.

The difference between the areas $\mathbf{C}'$ and **C** is characterized by the two areas A_1 and A_2 as indicated in the figure below:



Furthermore shows this figure that in moving from the area $\mathbf{C}$ to the area $\mathbf{C}'$, the boundary of $\mathcal{M}$ at the lefthand side is displaced by a distance Δl_1 and at the righthand side by a distance Δl_2. From the figure above we also learn that at the lefthand side of $\mathcal{M}$ the distance in between boundaries ψ_1 and ψ_2 of the streamtube is given by ΔS_1, and at the righthand side by ΔS_2.

The quantity of fluid $\mathcal{M}$ in $\mathbf{C}$ is the same as in $\mathbf{C}'$. Moreover, is the fluid incompressible. Consequently, the areas A_1 and A_2 must be the same. This implies the following:

$$\Delta S_1 \times \Delta l_1 = \{area \;\; A_1\} = \{area \;\; A_2\} = \Delta S_2 \times \Delta l_2. \tag{1.42}$$

The difference in work ΔW done by $\vec{F}_1$ and $\vec{F}_2$ is, according to the definitions of the quantities represented in the above figures, expressed by:

$$\Delta W = \mid \vec{F}_1 \mid \Delta l_1 - \mid \vec{F}_2 \mid \Delta l_2.$$

Using moreover the relation between force and pressure, the above expression takes the form:

$$\Delta W = p(1)\Delta S_1 \Delta l_1 - p(2)\Delta S_2 \Delta l_2, \tag{1.43}$$

where $p(1)$ and $p(2)$ represent the pressures in respectively the areas A_1 and A_2.

The total kinetic energy, K, of the quantity of fluid $\mathcal{M}$ when it ocupies the area $\mathbf{C}$ is equal to the sum of the kinetic energies of all of its volume elements $\Delta x \Delta y$. The mass of such "volume" element equals $\Delta x \Delta y \rho$. So, in agreement with the usual definition of kinetic energy of a point particle with given mass and velocity, we find for the total kinetic energy of the fluid in the area $\mathbf{C}$, the expression:

$$K(\mathbf{C}) = \int_{\mathbf{C}} \frac{1}{2} dx \, dy \, \rho \mid \vec{v}(x,y) \mid^2 . \tag{1.44}$$

The difference in kinetic energy between the situation where the quantity of fluid $\mathcal{M}$ ocupies the area $\mathbf{C}'$ and the situation where it ocupies the area $\mathbf{C}$, is thus given by the difference of two "volume" integrals, one over $\mathbf{C}'$ and one over $\mathbf{C}$. From the quantities defined in the above figures and using equation (1.44), one can easily deduce that this leads to the expression:

$$\Delta K = K(\mathbf{C}') - K(\mathbf{C}) = \int_{A_2} dx \, dy \, \frac{\rho}{2} \mid \vec{v}(x,y) \mid^2 - \int_{A_1} dx \, dy \, \frac{\rho}{2} \mid \vec{v}(x,y) \mid^2 .$$

For infinitesimally small distances $\Delta S_{1,2}$ and $\Delta l_{1,2}$ this gives the following:

$$\Delta K = \{area \;\; A_2\} \frac{\rho}{2} \mid \vec{v}(2) \mid^2 - \{area \;\; A_1\} \frac{\rho}{2} \mid \vec{v}(1) \mid^2, \tag{1.45}$$

where $\vec{v}(1)$ and $\vec{v}(2)$ represent the velocities in respectively the areas A_1 and A_2.

In combining the equations (1.43) and (1.45), using also the relation (1.42), one finds Bernoulli's equation (1.40) in the form:

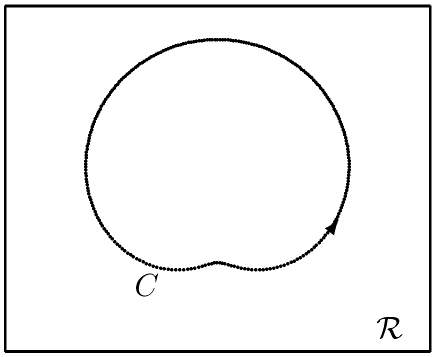
$$p(1) + \frac{\rho}{2} \mid \vec{v}(1) \mid^2 = p(2) + \frac{\rho}{2} \mid \vec{v}(2) \mid^2 .$$

1.8 Complex contour integration.

With respect to complex contour integration, we only have to know the following properties for analytic functions:

1. If C is a closed contour in the complex z-plane and if $f(z)$ is an analytic function in a domain $\mathcal{R}$ which includes the contour and its interior (see figure below), then:

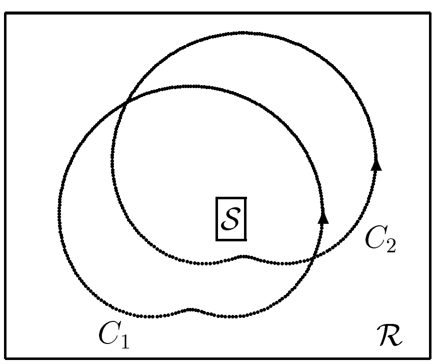
$$\oint_C dz\, f(z) = 0. \tag{1.46}$$



Property (1.46) is in *analysis* known as *the theorem of Cauchy.*

2. If C_1 and C_2 are two closed contours in the complex z-plane and if $f(z)$ is an analytic function in a domain $\mathcal{R}$ which includes both contours and the part of the complex z-plane which is in between C_1 and C_2 (see figure below), then:

$$\oint_{C_1} dz\, f(z) = \oint_{C_2} dz\, f(z). \tag{1.47}$$



The above property (1.47) is even true when $f(z)$ is not analytic in the domain indicated by $\mathcal{S}$ in the above figure.

Problem 8:

Consider the set of complex functions:

$$f_n(z) = z^n, \quad n = \cdots, -2, -1, 0, 1, 2, \cdots \ (i.e.\ \text{integer}).$$

For the integral contour C take a circle around the origin in the complex z-plane, defined by:

$$z = Re^{i\varphi}, \quad R \text{ arbitrary, but fixed.}$$

Prove that, independent of the radius R, holds:

$$\oint_{C\left(z = Re^{i\varphi}\right)} dz\, z^n = \begin{cases} 0 & \text{for } n \neq -1 \\ \pm 2\pi i & \text{for } n = -1 \end{cases} . \tag{1.48}$$

Show moreover that the result for $n = -1$ depends on the direction of integration, *i.e.* $+2\pi i$ for counterclockwise integration and $-2\pi i$ for clockwise integration.

The functions $f_n(z) = z^n$ are analytic in the whole complex z-plane for integer values of n, except for the point $z = 0$ when n is a negative integer. This point is called a *singularity* for those functions. In case the singularity is of first order (*i.e.* $n = -1$) then the complex integration at a closed contour around the singularity does not vanish, as we have seen in the previous problem (1.48).

Problem 9:

Determine the contour integrations, in counterclockwise direction, of the set of functions $g_n(z) = (z-2)^n$ (n integer), at circles around the point $z = 2$, which are given by $z = 2 + Re^{i\varphi}$.

In general, when an analytic function $f(z)$, has a singularity at the point $z = z_0$ in the complex plane, then for the integration of $f(z)$ over a closed contour around that singularity, we first determine the Laurent series expansion of $f(z)$ in the neighborhood of $z = z_0$, given by:

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots . \tag{1.49}$$

The contour integral is then readily determined by:

$$\oint_{C(\text{around } z_0)} dz\, f(z) = \sum_{n=-\infty}^{\infty} \oint_{C(\text{around } z_0)} dz\, a_n z^n = \pm 2\pi i a_{-1}, \qquad (1.50)$$

because, according to formula (1.48), the contributions of all other terms in the sum vanish.

The coefficient a_{-1} of the term which is proportional to $(z - z_0)^{-1}$ in the Laurent expansion (1.49) is called the *residue* of the function $f(z)$ at the singularity $z = z_0$.

Problem 10:

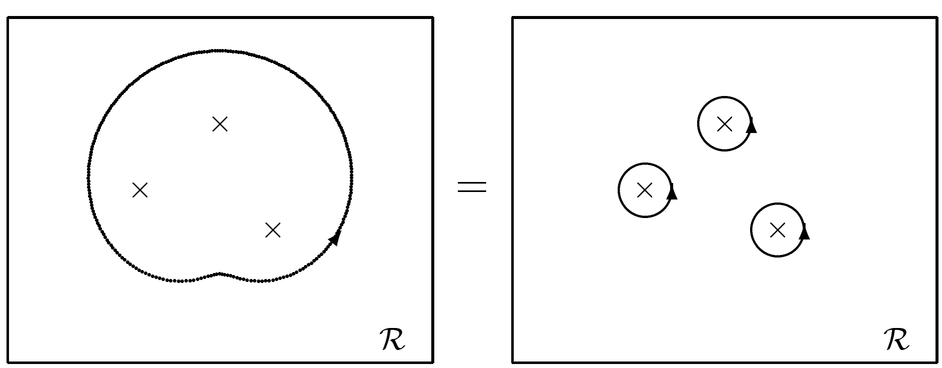
Determine the complex contour integral, in counterclockwise direction, at a circle around the point $z = 2$, given by $z = 2 + Re^{i\varphi}$, of the function $f(z)$ defined by:

$$f(z) = \frac{3z^2 - 7z + 4}{(z - 2)^2}.$$

Finally, one should also know the following property of complex contour integrations:

3. When a closed contour contains in its interior region more than one singularities of a function $f(z)$, then the contour integral of $f(z)$ over this contour equals $\pm 2\pi i$ times the sum of the residues at each of the singularities inside the contour.

Below we show a symbolic representation of this property:



(1.51)

1.9 Forces and moments on obstacles in a fluid flow.

In section (1.6) we found for the force on an obstacle emersed in a fluid flow, the following expression (see 1.41):

$$F = \oint_{\text{surface}} dz\; i\{p_0 - \frac{\rho}{2} \mid \Omega'(z) \mid^2\}.$$

By now we know from the results of section (1.8) that the integration over p_0 does not contribute (see formula 1.48). So,we are left with:

$$F = \oint_{\text{surface}} dz\; i\{-\frac{\rho}{2} \mid \Omega'(z) \mid^2\}. \tag{1.52}$$

The surface of an obstacle represents a stream line (see 1.41). Consequently, the imaginary part of $\Omega(z)$ is constant (see equation 1.26 and section 1.3). Therefore we have that $d\Omega$ is real. As a consequence we obtain for the integrand in (1.52) the following expression:

$$dz \left|\frac{d\Omega}{dz}\right|^2 = dz\frac{d\Omega}{dz}\left(\frac{d\Omega}{dz}\right)^* = dz^*\frac{d\Omega^*}{dz^*}\left(\frac{d\Omega}{dz}\right)^* = dz^*\left\{\left(\frac{d\Omega}{dz}\right)^2\right\}^*,$$

which leads for (1.52) to:

$$F = -\frac{i}{2}\rho \oint_{\text{surface}} dz^* \left\{\left(\frac{d\Omega}{dz}\right)^2\right\}^*,$$

or equivalently to:

$$F^* = \frac{i}{2}\rho \oint_{\text{surface}} dz \left(\frac{d\Omega}{dz}\right)^2. \tag{1.53}$$

Below we will apply this formula to calculate the forces on various obstacles in a fluid flow. But first we derive a similar expression for the moment of the forces on an obstacle.

The moment of a force $\vec{F}$ which acts at position $\vec{r}$ on a material point of a body, is in general defined by:

$$\vec{M} = \vec{r} \times \vec{F}.$$

In two dimensions is the moment perpendicular to the xy-plane, according to:

$$\vec{M} = M\hat{z} \;\;,\;\; \text{where} \;\; M = xF_y - yF_x. \tag{1.54}$$

In terms of the complex force F defined in (1.39) and the complex position variable z, we obtain:

$$M = \Re e\{izF^*\}, \tag{1.55}$$

where $\Re e$ stands for "*real part of*". So, the contribution ΔM to the total moment M of the force ΔF which acts at a surface element Δz located at position z, is given by:

$$\Delta M = \Re e\{iz\Delta F^*\}.$$

Using expression (1.53) for the force on the obstacle at a certain point z, we find for ΔM the relation:

$$\Delta M = \Re e\{iz\frac{i}{2}\rho\Delta z\left(\frac{d\Omega}{dz}\right)^2\}.$$

On integrating this over the whole surface of the obstacle, one obtains:

$$M = -\frac{\rho}{2}\Re e\{\oint_{\text{surface}} dz\ z\ \left(\frac{d\Omega}{dz}\right)^2\}. \tag{1.56}$$

Below we will discuss several examples:

1. A tennis ball without spin.

When a tennis ball without spin moves straigth through the air, then it has a flow pattern around it which resembles the fluid flow pattern of the streamfunction for the infinite circular cylinder defined in (1.17). So, we "approximate" the tennis ball by an infinite circular cylinder. In that case the complex potential is given by (compare formula 1.31)

$$\Omega(z) = v_0\left(ze^{-i\alpha} + \frac{R^2e^{i\alpha}}{z}\right),\ \text{for}\quad |\ z\ |\geq R, \tag{1.57}$$

where v_0 represents the speed of the tennis ball.

We observe here the following: The first term in the complex potential (1.57), which reads:

$$v_0ze^{-i\alpha},$$

represents an uniform fluid flow (compare 1.27 and 1.7). This is how the air "moves" at large distances in the coordinate frame attached to the center of the tennis ball and moving parallel to an inertial frame at rest. For small values of the angle α, the velocity of the fluid is pointing towards the right (see for example the figure of example 1.7). So, in that case the tennis ball moves towards the left. This we will assume here and in the following. When we consider moreover the x-axis to represent the horizontal surface of the Earth and the y-axis the vertical perpendicular to the Earth's surface, then the above term represents the tennis ball coming down towards the Earth at an angle equal to α in the negative direction of the x-axis.

The second term in the complex potential (1.57), *i.e.*:

$$v_0 \frac{R^2 e^{i\alpha}}{z},$$

becomes more important near the origin. It takes care of the deviation from an uniform motion of the fluid in the vicinity of the tennis ball. We also might notice that this term is singular at the origin, which represents the center of the obstacle. So, we may expect non-vanishing integrals when we integrate expression (1.53) on the contour $\mid z \mid= R$ in the complex plane, which represents the surface of the cylinder.

Next we must determine the integrand of (1.53), in order to find the force on the cylinder: First we calculate the derivative with respect to z of the complex potential defined in (1.57). This gives:

$$\frac{d\Omega}{dz} = v_0\{e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{z^2}\}. \tag{1.58}$$

Of this expression we need the square, *i.e.*:

$$\left(\frac{d\Omega}{dz}\right)^2 = (v_0)^2\{\frac{R^4 e^{2i\alpha}}{z^4} - \frac{2R^2}{z^2} + e^{-2i\alpha}\}. \tag{1.59}$$

We find here that the integrand for the contour integration of formula (1.53) has no residue at the singularity at $z = 0$, because there is no term linear in $1/z$ in expression (1.59). Consequently (see 1.48 and 1.50), the total force on the tennis ball due to the fluid flow around it equals zero, *i.e.*:

$$F = 0. \tag{1.60}$$

In order to determine the moment on the tennis ball, we must multiply the expression (1.59) with z. This gives:

$$\left(\frac{d\Omega}{dz}\right)^2 = (v_0)^2\{\frac{R^4 e^{2i\alpha}}{z^3} - \frac{2R^2}{z} + ze^{-2i\alpha}\}. \tag{1.61}$$

So, in this case the integrand of the contour integration of formula (1.56) has a residue at the singularity $z = 0$, which is equal to $-2v_0^2 R^2$. Consequently, we find for the integral of formula (1.56) the result:

$$\oint_{\text{surface}} dz\, z \left(\frac{d\Omega}{dz}\right)^2 = 2\pi i(-2v_0^2 R^2) = -4\pi i v_0^2 R^2.$$

The real part of this expression vanishes, which gives for the total moment on the tennis ball due to the fluid flow around it, the result:

$$M = 0. \tag{1.62}$$

The tennis ball does not feel any force (see 1.60), neither any moment in the fluid flow. This result might also have been obtained from a simple symmetry argument. In example (1.17) we have represented the flow pattern for this case. Now, we might observe that this pattern is completely symmetric for a rotation of 180° in the xy-plane. This implies that the speed (*i.e.* $\mid \vec{v}(x,y) \mid$) of the fluid flow near the surface of the cylinder is the same at two opposite points. As a consequence, according to Bernoulli's law, the pressure at one place of the surface of the cylinder is the same as the pressure at the opposite point, thus resulting in forces which compensate each other. Which gives as a result that the total force on the cylinder vanishes.

Problem 11:
Show that formally, using the relation (1.58) for the complex velocity, one has the following identity:

$$\mid \vec{v}(z = Re^{i(\varphi + 180^\circ)}) \mid = \mid \vec{v}(z = Re^{i\varphi}) \mid .$$

This shows in formula the above discussed symmetry.

---

2. A tennis ball with spin.

A tennis ball which rotates (spinning) when it moves through the air, has a different flow pattern around it than the tennis ball without spin (1.57). Here, again in the infinite cylinder approximation, we might use the following complex potential to represent its motion:

$$\Omega(z) = v_0 \left(ze^{-i\alpha} + \frac{R^2 e^{i\alpha}}{z} \right) + i\frac{\kappa}{2\pi}\log(z), \quad \text{for} \quad \mid z \mid \geq R, \tag{1.63}$$

where v_0 as in the previous case represents the speed of the center of mass of the tennis ball with respect to the air and where κ represents the amount of spin.

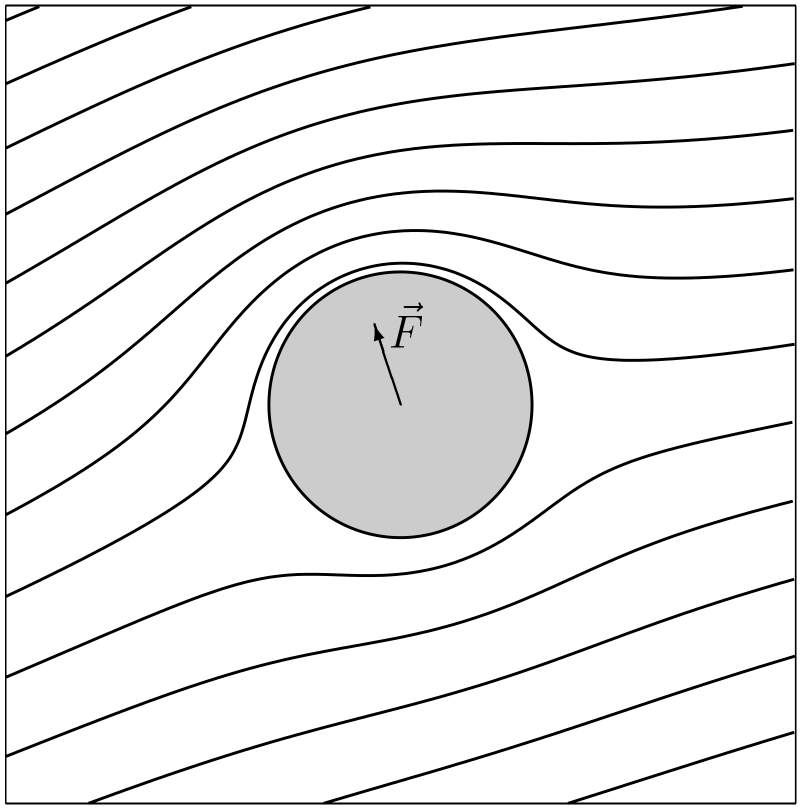
The first two terms of the complex potential (1.63) are the same as in the previous example of the tennis ball without spin. So the third term:

$$i\frac{\kappa}{2\pi}\log(z)$$

is supposed to describe the effect of rotation of the tennis ball in this case. By comparing this term to the formulas (1.30), (1.16) and (1.10), we see that it indeed corresponds to a rotating flow around the origin. Remember that the tennis ball is supposed to move from the right to the left. So, when its spin is counterclockwise, then the air moves relatively clockwise as in the figure related to formula (1.10). The latter motion is described by positive values of the vortex parameter κ.

Notice moreover that this logarithmic term is singular at the origin as is the term linear in $1/z$ in (1.63).

The flow pattern corresponding to the above complex potential (1.63) is shown in the figure below:



In order to determine the force and the moment acting on the tennis ball due to the fluid flow, we follow the same procedure as in the previous case: For the contour integral of formula (1.53) we find then:

$$\oint_{\text{surface}} dz \left(\frac{d\Omega}{dz}\right)^2 = 2\pi i\{i\frac{\kappa}{\pi}v_0 e^{-i\alpha}\} = -2\kappa v_0 e^{-i\alpha}.$$

Inserting this into formula (1.53) and taking the complex conjugate, we obtain for the force on the tennis ball due to the fluid flow around it, the following expression:

$$F = i\rho\kappa v_0 e^{i\alpha} = \rho\kappa v_0\{-\sin(\alpha) + i\cos(\alpha)\}.$$

The force $\vec{F}$ is then given by:

$$\vec{F} = \rho\kappa v_0\{-\sin(\alpha)\hat{x} + \cos(\alpha)\hat{y}\}. \tag{1.64}$$

What is the interpretation of this result? Let us consider that the x-axis is parallel to the surface of the Earth and the y-axis perpendicular to it and pointing towards higher altitudes. Then, when the tennis ball moves horizontally from the right to the left (which implies $\alpha = 0$), the direction of the force (1.64) depends here on the sign of κ: For $\kappa > 0$ the force acts upward and for $\kappa < 0$, downward. Now the direction of rotation can be studied from the flow pattern which belongs to the stream function given in (1.10). If we take that figure (which is for a positive value of κ) in front of us, then we notice that while the ball is moving in the negative x-direction, it appears like rolling on the air. So, the velocity of the air with respect to the surface of the ball is at the bottom side smaller than at the top side. This gives according to Bernoulli's theorem (1.40) a larger pressure at the bottom side than at the top side of the surface of the tennis ball. And consequently an upward resultant force.

In a tennis game, while the motion of a tennis ball without spin is more or less the motion of a projectile, in the case of a horizontal spin (which means that the axis of the related cylinder is horizontal), the ball deviates accelerated from a projectile's orbit in upward, or downward directions. The latter is of course the preferred spin in a tennis game, because it might result in having the ball drop right behind the net.

In order to determine the moment excerted by the fluid on the cylinder, we also follow the procedure outlined in the previous case for a tennis ball without spin. For the integrand of the contour integral in formula (1.56) we find:

$$\oint_{\text{surface}} dz\; z\, \left(\frac{d\Omega}{dz}\right)^2 = 2\pi i\left\{-\left(\frac{\kappa}{2\pi}\right)^2 - 2v_0^2R^2\right\}.$$

This expression has no real part and thus results in a vanishing moment, also in the case of a tennis ball with spin, *i.e.*:

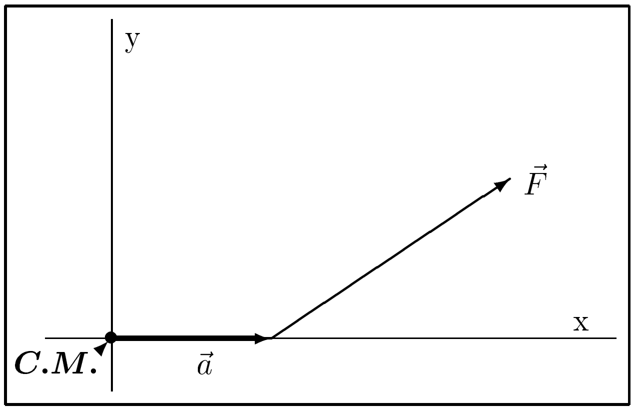
$$M = 0. \tag{1.65}$$

As a consequence, the angular momentum (spin) which is initially given to the ball will not change due to the forces described here.

The fact that the moment vanishes in this case can be understood from symmetry arguments, similar to the ones for the vanishing force in the previous example: From the flow pattern one notices that there is a mirror symmetry around the axis which makes an angle given by α with the vertical. To each point at the right of this symmetry axis corresponds a similar point at the left of it. Consequently, there is as much force acting at one side of the axis (which incidentally also passes through the C.M. of the body) as on the other side, which leads to compensating contributions to the total moment.

3. The lift arm.

In the examples (1.57) and (1.63), the center of mass of the obstacle coincides with the origin of the coordinate system. This we assume always to be the case. In such cases, indicates the moment of a force the lift arm with respect to the center of mass of the object. In general one has for a force $\vec{F}$ which does not act in the center of mass of a body, the following: Let the point of application of $\vec{F}$ be given by $\vec{a} = a\hat{x}$, as depicted in the figure below:



The moment of the force is then, according to (1.54), given by:

$$M = aF_y. \tag{1.66}$$

The quantity a in the above formula is called the *lift arm* of the force due to the fluid flow.

In the previous example of the tennis ball without spin, the moment vanishes and therefore so does the lift arm in that case. Consequently, the force is supposed to be applied in the center of the tennis ball.

4. A primitive air wing.

Let us consider for an air wing an infinitely long cylinder with an elipsoidal cross section. In order to find the complex potential for such case one must study the deformation of a circle into an ellipse. However, one must at the same time keep the analyticity property of the complex potential. Now, an ellipse can be obtained from a circle by the following analytic transformation:

$$z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right). \tag{1.67}$$

The study of the details of the above transformation, is left as an excercise for the reader.

Problem 12:
Show that the part of the complex ζ-plane exterior to the circle given by $\mid \zeta \mid = R$ with $R > 1$, is by (1.67) transformed into the part of the z-plane which is exterior to the ellipse given by:

$$\left[\frac{2\Re e(z)}{R + R^{-1}}\right]^2 + \left[\frac{2\Im m(z)}{R - R^{-1}}\right]^2 = 1, \tag{1.68}$$

where $\Re e$ stands for "*real part of*" and $\Im m$ stands for "*imaginary part of*".

The pricipal axes of the ellipse (1.68) are along the x- and y-directions and their lengths are given by $(R + 1/R)$ and $(R - 1/R)$ respectively. In the ζ-plane we take for the complex potential the following expression:

$$\omega(\zeta) = v_0\left(\zeta e^{-i\alpha} + \frac{R^2 e^{i\alpha}}{\zeta}\right) + i\frac{\kappa}{2\pi}\log(\zeta), \quad \text{for} \quad \mid \zeta \mid \geq R, \tag{1.69}$$

which is exactly the same expression as given in (1.63) for the tennis ball with spin, but where z is everywhere substituted by ζ. In the z-plane we will now study the complex potential which is defined by:

$$\Omega(z) = \omega(\zeta(z)), \tag{1.70}$$

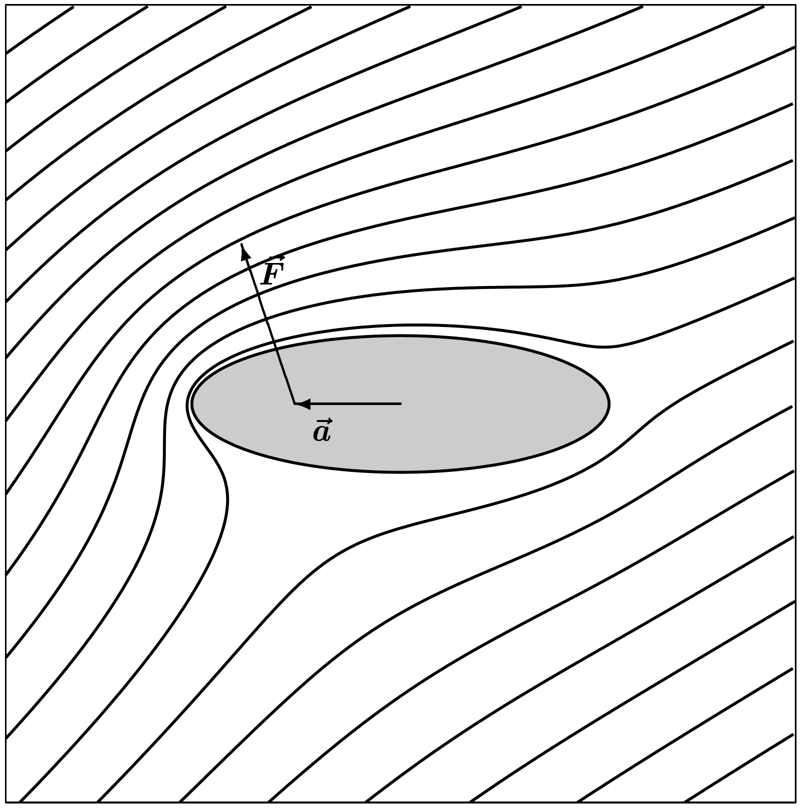
where $\zeta(z)$ represents the inverse of the transformation given in (1.67).
We notice that the fluid flow at large distances might still be given by an expression which represents a constant fluid flow, *i.e.*

$$2v_0 z e^{-i\alpha},$$

as in the case of the tennis ball. Large values of z correspond to large values of ζ (the small values of ζ are excluded in formula 1.69), according to relation (1.67).

So, except for a factor two, the long distance behaviour is the same as for the other cases. But at short distances the situation is different.

The flow pattern for this complex potential is shown in the figure below:



In order to establish the force and the moment of the force on the surface of the elipsoidal cylinder, we proceed as follows: First we determine dz in terms of $d\zeta$, *i.e.*:

$$dz = \frac{dz}{d\zeta}d\zeta = \frac{1}{2}\left(1 - \frac{1}{\zeta^2}\right)d\zeta.$$

Next, we find the expression for $d\Omega/dz$ in terms of $d\omega/d\zeta$:

$$\frac{d\Omega}{dz} = \frac{d\zeta}{dz}\frac{d\omega}{d\zeta} = \left(\frac{dz}{d\zeta}\right)^{-1}\frac{d\omega}{d\zeta} = \frac{2}{1-\zeta^{-2}}\frac{d\omega}{d\zeta}.$$

So, for the integrand of the contour integral of formula (1.53) we obtain here the following:

$$dz\left(\frac{d\Omega}{dz}\right)^2 = d\zeta\frac{2}{1-\zeta^{-2}}\left(\frac{d\omega}{d\zeta}\right)^2.$$

The contour in the z-plane is the ellipse defined in (1.68) and the corresponding contour in the ζ-plane is the circle $\mid \zeta \mid = R$. So, in going from one variable to the other, we find for the contour integrations the relation:

$$\oint_{\text{ellipse}} dz \left(\frac{d\Omega}{dz}\right)^2 = \oint_{\text{circle}} d\zeta \frac{2}{1-\zeta^{-2}} \left(\frac{d\omega}{d\zeta}\right)^2. \quad (1.71)$$

In order to proceed, we must find the residue of the integrand in the righthand side of the above formula (1.71) at the singularity at $\zeta = 0$. For this purpose we might introduce the following Laurent expansions (1.49):

$$\frac{1}{1-\zeta^{-2}} = 1 + \zeta^{-2} + \zeta^{-4} + \cdots,$$

and

$$\frac{d\omega}{d\zeta} = -\frac{v_0 R^2 e^{i\alpha}}{\zeta^2} + \frac{i\kappa/2\pi}{\zeta} + v_0 e^{-i\alpha}.$$

Combining these expansions one obtains for the full Laurent expansion of the relevant integrand in (1.71) the expression:

$$\frac{2}{1-\zeta^{-2}} \left(\frac{d\omega}{d\zeta}\right)^2 = \cdots + \frac{-2\left(\frac{\kappa}{2\pi}\right)^2 + 2v_0^2 e^{-2i\alpha} - 4v_0^2 R^2}{\zeta^2} + \frac{4i\left(\frac{\kappa}{2\pi}\right) v_0 e^{-i\alpha}}{\zeta} + v_0^2 e^{-i\alpha}. \quad (1.72)$$

The wanted residue at the singularity $\zeta = 0$ is the coefficient of the term which is linear in $1/\zeta$. After performing the contour integral of formula (1.53), taking the complex conjugate and identifying the components of the force, one finds for the force excerted on the air wing by the fluid flow, the following:

$$\vec{F} = 2\rho\kappa v_0\{-\sin(\alpha)\hat{x} + \cos(\alpha)\hat{y}\}. \quad (1.73)$$

In order to determine the moment of the above force, we must multiply the expression (1.72) with $z = (\zeta + 1/\zeta)/2$. This gives:

$$\frac{1}{2}(\zeta + \zeta^{-1}) \frac{2}{1-\zeta^{-2}} \left(\frac{d\omega}{d\zeta}\right)^2 = \cdots + \frac{-(\kappa/2\pi)^2 + 2v_0^2 e^{-2i\alpha} - v_0^2 R^2}{\zeta} + \cdots.$$

So, the residue at the singularity $\zeta = 0$ is thus given by:

$$-(\kappa/2\pi)^2 + 2v_0^2 e^{-2i\alpha} - v_0^2 R^2.$$

The real part of this expression does not contribute to the moment of the force on the air wing, but the imaginary part does. After performing the contour integration of formula (1.56) and taking the real part of the result, one finds for the moment:

$$\begin{aligned} M &= -\frac{\rho}{2}\Re e\{2\pi i(-2v_0^2 i \sin(2\alpha))\} \\ &= -2\pi\rho v_0^2 \sin(2\alpha). \quad (1.74) \end{aligned}$$

The lift arm of the rotation induced by this non-zero value for M follows from the expression (1.66), *i.e.*:

$$a = \frac{M}{F_y}.$$

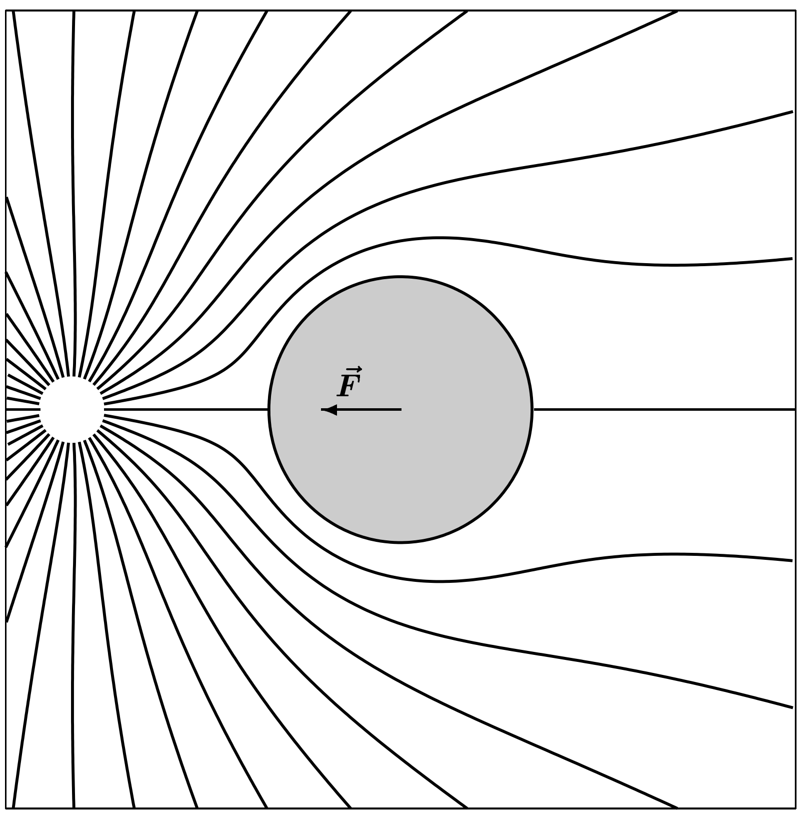
Using the results (1.73) for the force and (1.74) for the moment excerted on the air wing, one finds for the lift arm here:

$$a = -2\pi \left(\frac{v_0}{\kappa}\right) \sin(\alpha). \tag{1.75}$$

The circulation of the air around the air wing which comes from the logarithm in the complex potential (1.67), is in the case of real air-planes caused by a slightly more fancy construction of the air wing than an elipsoidal infinite cylinder, and sets in at high velocities. The moment is necessary for the lift of an air-plane.

5. The ping-pong ball in an air stream.

The situation of a light ping-pong ball which is lifted by an air stream directed towards the floor, is at first very surprising. Of course, when the airstream and gravity both work in the same direction, why would a ping-pong ball be going



upward? But the theorem of Bernoulli solves this miracle and shows that the force of the air stream is always directed towards the source and not depending on the weight of the ball. So, for a light ball the resultant might very well be upward. The flow pattern is shown in the above figure.

As in the examples of the tennis ball, we "approximate" the ping-pong ball by an infinite circular cylinder. The corresponding complex potential is given by:

$$\Omega(z) = \frac{\mu}{2\pi} \log\left\{ z + \frac{R^2}{z} - z_0 - \frac{R^2}{z_0} \right\}, \quad z_0 \text{ real and } | z_0 |> R. \tag{1.76}$$

In order to determine the force and the moment on the obstacle in this case, we first study the singularity pattern of the above complex potential. We notice that the argument of the logarithm can be factorized, which gives for the complex potential the following expression:

$$\Omega(z) = \frac{\mu}{2\pi} \log\left\{ \frac{1}{z}(z - \frac{R^2}{z_0})(z - z_0) \right\}.$$

In this form the singularities of the analytic function $\Omega(z)$ are more obvious, *i.e.* the points:

$$z = 0,\ z = \frac{R^2}{z_0}, \text{ and } z = z_0.$$

The first and the second singularity are inside the circle $| z |= R$ which represents the surface of the obstacle. The third singularity is outside the obstacle. So, according to the formulas (1.51) and (1.50), we must know the residues at the first two singularities of the integrands of the expressions (1.53) and (1.56).

Let us start with the integrand of formula (1.53) which gives the force at the obstacle. Its Laurent expansions at the poles $z = 0$ and $z = R^2/z_0$ are respectively given by:

$$\left(\frac{d\Omega}{dz}\right)^2 = \left(\frac{\mu}{2\pi}\right)^2 \left\{ \frac{1}{z^2} + \frac{2(\frac{1}{z_0} + \frac{z_0}{R^2})}{z} + \cdots \right\}, \tag{1.77}$$

and

$$\left(\frac{d\Omega}{dz}\right)^2 = \left(\frac{\mu}{2\pi}\right)^2 \left\{ \frac{1}{(z - \frac{R^2}{z_0})^2} + \frac{2\frac{z_0^3}{R^2(R^2 - z_0^2)}}{z - \frac{R^2}{z_0}} + \cdots \right\}. \tag{1.78}$$

Performing the integral (1.53) by use of the residues which can be extracted from the above Laurent expansions, and taking the complex conjugate of the result, gives then for the force the expression:

$$F = \rho \frac{\mu^2}{2\pi} \frac{R^2}{z_0(z_0^2 - R^2)}. \tag{1.79}$$

This force is along the x-direction because the position of the source z_0 is along the x-axis as indicated in the formula for the present complex potential (1.76). Moreover, its direction is determined by the sign of z_0, since all other factors in the expression for the force (1.79) are positive. For negative values of z_0 the direction is towards negative values of x and for positive values of z_0 towards positive values of x. From which we may conclude that the force is always pointing towards the source.

For the moment of the force we may use a similar symmetry argument as for the tennis ball with spin: The moments stemming from the forces on one side of the ping-pong ball are in equilibrium with the moments coming from the other side. The total moment must thus vanish, *i.e.*:

$$M = 0. \tag{1.80}$$

Alternatively one might use the above Laurent expansions (1.77) and (1.78) to calculate the moment of the force. This is left as an excercise to the reader.

So, in experiment one observes the following: The ping-pong ball is atracted by the source, but the ping-pong ball does not start rotating as a consequence of the forces.

A warning is in place here: The reader might think that z_0 in the formula (1.76) for the complex potential can be substituted by any complex number. That is however not true. It would lead to different conclusions about the direction of the force with respect to the position of the source than in the above case. But the choice of the radius parameter R determines the choice of z_0. A real value of R implies a real value of z_0.

Chapter 2

Transformations and invariant quantities.

At present one believes that the laws of physics are the same for any observer in the Universe. As a consequence it has to be assumed that the description of a phenomenon, which takes place either in a laboratory provoked in an experiment or uncontrolled somewhere in the Universe, can be formulated in such a way that for two different observers the related equations have the same appearance. For example, the total energy and the linear momentum of a point particle might differ from observer to observer, but its invariant mass (rest mass) is the same for all observers.

In mechanics an observer is characterized by a coordinate system. Each observer describes a point in space and time by a set of coordinates. However, two different observers might use different coordinates for the description of the same point. The mathematical relations between two different sets of coordinates are called *transformations*. Examples are Galilean transformations, Poincaré transformations and general coordinate transformations.

Physics is in fact the discovery of those transformations and the related invariant quantities. For simple mechanical systems, observers may select a different origin for their coordinate system (translation), rotated coordinate systems (rotations) and moving coordinate systems with constant relative velocities (boosts), without changing the formulation of the laws of physics.

Before the discovery of special relativity, it was thought that inertial systems are connected by Galilean transformations. After Einstein we know that Galilean transformations are only approximately obeyed by Nature for low relative velocities. In special relativity the allowed coordinate transformations are called Poincaré transformations, which include translations and Lorentz transformations. The latter include boosts and rotations. But also the Poincaré transformations of special relativity are only approximately valid in the case of weak gravitational fields. Einstein showed that in the presence of a gravitational source physics laws can be formulated in such a way that they remain invariant under general coordinate transformations.

In this chapter we will concentrate on rotations and study the invariant quantities related to those transformations.

2.1 Rotations in two dimensions.

Although our aim is the study of the properties of rotations in three dimensions, it is rather instructive to first limit ourselves to rotations in two dimensions. Therefore, let us consider a two dimensional orthogonal coordinate system $\mathcal{S}$ characterized by coordinates x_1 and x_2 and by the orthonormal basis vectors $\hat{e}_1$ and $\hat{e}_2$. A vector $\vec{x}$ is in this system represented by the expression:

$$\vec{x} = (x_1, x_2) = x_1\hat{e}_1 + x_2\hat{e}_2 = x_i\hat{e}_i. \tag{2.1}$$

The expression at the righthand side of the above formula (2.1) is sometimes referred to as the Einstein summation convention, or equivalently as a contraction of the index i. The orthonormality of the basis can be expressed by means of the Kronecker delta symbol as follows:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \quad , \quad i, j = 1, 2. \tag{2.2}$$

Furthermore we consider a different orthogonal coordinate system $\mathcal{S}'$ characterized by coordinates x_1' and x_2' and by the orthonormal basis vectors $\hat{e}_1'$ and $\hat{e}_2'$. The basis vectors of the coordinate system $\mathcal{S}'$ are related to the basis vectors of the coordinate system $\mathcal{S}$ by a rotation $R(\hat{z}, \alpha)$ around the origin over an angle indicated by α, *i.e.*:

$$\hat{e}_1' = \cos(\alpha)\hat{e}_1 - \sin(\alpha)\hat{e}_2 \quad \text{and} \quad \hat{e}_2' = \sin(\alpha)\hat{e}_1 + \cos(\alpha)\hat{e}_2.$$

The above basis transformation can be written in a more compact formulation, when we first define for the rotation $R(\hat{z}, \alpha)$ a matrix given by:

$$R(\hat{z}, \alpha) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{2.3}$$

In terms of the matrix elements of the above defined rotation matrix $R(\hat{z}, \alpha)$ one finds for the above basis transformation and its inverse respectively the expressions:

$$\hat{e}_i' = R_{ij}\hat{e}_j \quad \text{and} \quad \hat{e}_j = R_{ij}\hat{e}_i'. \tag{2.4}$$

The relations between the coordinates of the unprimed system $\mathcal{S}$ and the coordinates of the primed system $\mathcal{S}'$ can be found using the above expressions (2.4) and realizing that the same vector $\vec{x}$ can be represented in either system by an expression similar to (2.1), *i.e.*:

$$\begin{aligned} \vec{x} &= x_i'\hat{e}_i' \\ &= x_j\hat{e}_j = x_j\{R_{ij}\hat{e}_i'\} = \{R_{ij}x_j\}\hat{e}_i'. \end{aligned}$$

From this relation we deduce that for the coordinates x_i $(i = 1, 2)$ of $\vec{x}$ in $\mathcal{S}$ and the coordinates x_i' $(i = 1, 2)$ of $\vec{x}$ in $\mathcal{S}'$ yields the following transformation rule:

$$x_i' = R_{ij}x_j. \tag{2.5}$$

Problem 13:
Show that the reverse of the above relations (2.5) is given by:

$$x_i = \left(R^T\right)_{ij} x'_j = R_{ji} x'_j. \tag{2.6}$$

From the result of the above problem one can also infer that for rotations holds:

$$R^T = R^{-1}. \tag{2.7}$$

Matrices which have this property are said to be *orthogonal*. It implies that rows and columns of such matrix represent orthonormal vectors, as can be seen from:

$$R_{ik}R_{jk} = R_{ik}\left(R^T\right)_{kj} = \left(RR^T\right)_{ij} = \left(RR^{-1}\right)_{ij} = (\mathbf{1})_{ij} = \delta_{ij}. \tag{2.8}$$

Problem 14:
Show that also:

$$R_{ki}R_{kj} = \delta_{ij}. \tag{2.9}$$

Notice moreover that the determinant of the rotation matrix (2.3) equals one.

2.2 The rotation group $SO(2)$.

It can easily be shown that rotations around the origin in two dimensions form a group.

Problem 15:
Show that rotations $R(\hat{z},\alpha)$ as defined in (2.3) form a group, *i.e.* show that:

(i) The product of two rotations $R(\hat{z},\alpha)$ and $R(\hat{z},\beta)$ is again a rotation, $R(\hat{z},\alpha+\beta)$.
(ii) The product has the property of associativity, *i.e.*:

$$\{R(\hat{z},\alpha)R(\hat{z},\beta)\}\,R(\hat{z},\gamma) = R(\hat{z},\alpha)\,\{R(\hat{z},\beta)R(\hat{z},\gamma)\}\,.$$

(iii) There exists a unit operation, $R(\hat{z},0) = \mathbf{1}$
(iv) The inverse of a rotation exists and is also a rotation.

This group is called the *special orthogonal group* in two dimensions, abreviated by $SO(2)$. Orthogonal, because it consists of operations for which relation (2.7) holds and special, because the determinant of the defining matrices equals one (in general the determinant may be one or minus one for an orthogonal matrix).
The group elements can be characterized by one parameter, the rotation angle; that is that all possible rotations in two dimensions are given by:

$$R(\hat{z},\alpha)\quad,\quad -\pi < \alpha \leq +\pi. \tag{2.10}$$

In the neighbourhood of the unit operation ($\alpha = 0$), the above defined matrices form a one dimensional continuous matrix field. So, we might define the matrix A which is given by the derivative of the matrix field at $\alpha = 0$, as follows:

$$A = \left.\frac{d}{d\alpha}R(\hat{z},\alpha)\right|_{\alpha=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.11}$$

The matrix A is called the *generator* of rotations in two dimensions. In the following it will become clear why: For angles different from zero one has similarly:

$$\begin{aligned}\frac{d}{d\alpha}R(\hat{z},\alpha) &= \begin{pmatrix} -\sin(\alpha) & -\cos(\alpha) \\ \cos(\alpha) & -\sin(\alpha) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \\ &= A\;R(\hat{z},\alpha),\end{aligned}$$

which differential equation can be solved by:

$$R(\hat{z}, \alpha) = \exp\{\alpha A\}. \tag{2.12}$$

How do we interpret this exponential of a matrix? To answer this question, we first mention the following properties of the matrix A defined in (2.11):

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathbf{1}, \quad A^3 = -A, \quad A^4 = \mathbf{1}, \quad \cdots \tag{2.13}$$

Using the above properties of the generator A, we find for the exponential (2.12) the following interpretation:

$$\begin{aligned}
\exp\{\alpha A\} &= \mathbf{1} + \alpha A + \frac{1}{2!}(\alpha A)^2 + \frac{1}{3!}(\alpha A)^3 + \frac{1}{4!}(\alpha A)^4 + \cdots \\
&= \mathbf{1} + \alpha A + \frac{\alpha^2}{2!}A^2 + \frac{\alpha^3}{3!}A^3 + \frac{\alpha^4}{4!}A^4 + \cdots \\
&= \mathbf{1} + \alpha A - \frac{\alpha^2}{2!}\mathbf{1} - \frac{\alpha^3}{3!}A + \frac{\alpha^4}{4!}\mathbf{1} + \cdots \\
&= \{1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots\}\mathbf{1} + \{\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots\}A \\
&= \cos(\alpha)\mathbf{1} + \sin(\alpha)A \\
&= \begin{pmatrix} \cos(\alpha) & 0 \\ 0 & \cos(\alpha) \end{pmatrix} + \begin{pmatrix} 0 & -\sin(\alpha) \\ \sin(\alpha) & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \\
&= R(\hat{z}, \alpha).
\end{aligned}$$

Because of the above representation of a rotation in terms of a parameter, α, and the matrix A, it is that this matrix is called the generator of rotations in two dimensions.

2.3 The rotation group $SO(3)$.

Similar to the group of rotations around the origin in two dimensions, we have the group of rotations around the origin in three dimensions, called $SO(3)$. An important difference with rotations in two dimensions is that in three dimensions rotations do not commute. The three rotations around the principal axes of the orthogonal coordinate system (x, y, z) are given by:

$$R(\hat{x}, \alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad R(\hat{y}, \vartheta) = \begin{pmatrix} \cos(\vartheta) & 0 & \sin(\vartheta) \\ 0 & 1 & 0 \\ -\sin(\vartheta) & 0 & \cos(\vartheta) \end{pmatrix},$$

$$\text{and} \quad R(\hat{z}, \varphi) = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.14}$$

Problem 16:
As an example that rotations in general do not commute, show that for the rotations over angles of 90° around the x- and y-axis yields:

$$R(\hat{x}, 90^o)R(\hat{y}, 90^o) \neq R(\hat{y}, 90^o)R(\hat{x}, 90^o). \tag{2.15}$$

Rotations in three dimensions are characterized by three parameters. Below we will give two different possibilities. First we consider a rotation which rotates a point $\vec{a}$, defined by:

$$\vec{a} = (\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta)), \tag{2.16}$$

to the position $\vec{b}$, defined by

$$\vec{b} = (\sin(\vartheta')\cos(\varphi'), \sin(\vartheta')\sin(\varphi'), \cos(\vartheta')). \tag{2.17}$$

Notice that there exist an infinite number of rotations which may perform this operation. So, the above outlined procedure, which is left as an excercise for the reader, does not characterize unambiguously this rotation. Only when one also defines the rotation axis, then one is left with just one possibility.

Problem 17:
Show, using the definitions (2.14), (2.16) and (2.17), the following results:

$$\begin{aligned} R(\hat{y},-\vartheta)R(\hat{z},-\varphi)\vec{a} &= \hat{z}, \\ R(\hat{z},\varphi')R(\hat{y},\vartheta')\hat{z} &= \vec{b}, \quad \text{and} \\ R(\hat{y},\vartheta')R(\hat{y},-\vartheta) &= R(\hat{y},\vartheta'-\vartheta). \end{aligned}$$

From the results of the above problem, we may conclude that a rotation which transforms $\vec{a}$ (2.16) into $\vec{b}$ (2.17), is given by:

$$R(\varphi',\vartheta'-\vartheta,\varphi) = R(\hat{z},\varphi')R(\hat{y},\vartheta'-\vartheta)R(\hat{z},-\varphi). \tag{2.18}$$

This parametrization of an arbitrary rotation in three dimensions is due to Euler, the three independent angles φ', $\vartheta'-\vartheta$ and φ are called the *Euler angles*. The difficulty here is to also indicate the rotation axis which procedure, although straightforward, is a tedious calculation.

For that reason we consider a second parametrization which involves the *generators* of rotations in three dimensions, as, similar to the matrix A (2.11) for two dimensions, are called the following three matrices which result from the three basis rotations defined in (2.14):

$$A_1 = \frac{d}{d\alpha}R(\hat{x},\alpha)\bigg|_{\alpha=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \frac{d}{d\vartheta}R(\hat{y},\vartheta)\bigg|_{\vartheta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad A_3 = \frac{d}{d\varphi}R(\hat{z},\varphi)\bigg|_{\varphi=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.19}$$

For now and for later use, let us introduce the Levi-Civita tensor ϵ_{ijk}, given by:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for} \quad ijk = 123,\ 312 \ \text{and} \ 231. \\ -1 & \text{for} \quad ijk = 132,\ 213 \ \text{and} \ 321. \\ 0 & \text{for all other combinations.} \end{cases} \tag{2.20}$$

This tensor has the following properties:

(i) For symmetric permutations of the indices:

$$\epsilon_{jki} = \epsilon_{kij} = \epsilon_{ijk}. \tag{2.21}$$

(ii) For antisymmetric permutations of indices:

$$\epsilon_{ikj} = \epsilon_{jik} = \epsilon_{kji} = -\epsilon_{ijk}. \tag{2.22}$$

In terms of the Levi-Civita tensor, we can define the matrix representation (2.19) for the generators of $SO(3)$ by:

$$(A_i)_{jk} = -\epsilon_{ijk}. \tag{2.23}$$

The introduction of the Levi-Civita tensor is very useful for the various derivations in the following, since it allows a compact way of formulating matrix multiplications, as we will see. However, one more property of this tensor should be given here, *i.e.* the contraction of one index in the product of two Levi-Civita tensors:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{ilm} &= \epsilon_{1jk}\epsilon_{1lm} + \epsilon_{2jk}\epsilon_{2lm} + \epsilon_{3jk}\epsilon_{3lm} \\ &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \end{aligned} \tag{2.24}$$

As a demonstration of their use, let us determine the commutator of two generators (2.19), using the above properties (2.21), (2.22) and (2.24). First we concentrate on one matrix element (2.23) of the commutator:

$$\begin{aligned} \{[A_i, A_j]\}_{kl} &= (A_iA_j)_{kl} - (A_jA_i)_{kl} = (A_i)_{km}(A_j)_{ml} - (A_j)_{km}(A_i)_{ml} \\ &= \epsilon_{ikm}\epsilon_{jml} - \epsilon_{jkm}\epsilon_{iml} = \epsilon_{mik}\epsilon_{mlj} - \epsilon_{mjk}\epsilon_{mli} \\ &= \delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl} - (\delta_{jl}\delta_{ki} - \delta_{ji}\delta_{kl}) = \delta_{il}\delta_{kj} - \delta_{jl}\delta_{ki} \\ &= \epsilon_{mij}\epsilon_{mlk} = -\epsilon_{ijm}\epsilon_{mkl} = \epsilon_{ijm}(A_m)_{kl} \\ &= (\epsilon_{ijm}A_m)_{kl}. \end{aligned}$$

So, for the commutator of the generators (2.19) we find:

$$[A_i, A_j] = \epsilon_{ijm}A_m. \tag{2.25}$$

This establishes, moreover, their relation with the so-called angular momentum operators in Quantum Mechanics.

In order to determine a second parametrization of a rotation in three dimensions, we define an arbitrary vector $\vec{n}$ by:

$$\vec{n} = (n_1, n_2, n_3), \tag{2.26}$$

as well as its "innerproduct" with the three generators (2.19), given by the expression:

$$\vec{n} \cdot \vec{A} = n_iA_i = n_1A_1 + n_2A_2 + n_3A_3. \tag{2.27}$$

In the following we need the higher order powers of this "innerproduct". Actually, it is sufficient to determine the third power of (2.27), *i.e.*:

$$(\vec{n}\cdot\vec{A})^3 = (n_iA_i)(n_jA_j)(n_kA_k) = n_in_jn_kA_iA_jA_k.$$

We proceed by determining one matrix element of the resulting matrix. Using the above property (2.24) of the Levi-Civita tensor, we find:

$$\begin{aligned}
\{(\vec{n}\cdot\vec{A})^3\}_{ab} &= n_in_jn_k\{A_iA_jA_k\}_{ab} = n_in_jn_k(A_i)ac(A_j)_{cd}(A_k)_{db} \\
&= -n_in_jn_k\epsilon_{iac}\epsilon_{jcd}\epsilon_{kdb} = -n_in_jn_k\{\delta_{id}\delta_{aj} - \delta_{ij}\delta_{ad}\}\epsilon_{kdb} \\
&= -n_dn_an_k\epsilon_{kdb} + n^2n_k\epsilon_{kab} = 0 - n^2n_k(A_k)_{ab} \\
&= \{-n^2\vec{n}\cdot\vec{A}\}_{ab}.
\end{aligned}$$

The zero in the forelast step of the above derivation, comes from the deliberation that using the antisymmetry property (2.22) of the Levi-Civita tensor, we have the following result for the contraction of two indices with a symmetric expression:

$$\epsilon_{ijk}n_jn_k = -\epsilon_{ikj}n_jn_k = -\epsilon_{ikj}n_kn_j = -\epsilon_{ijk}n_jn_k, \tag{2.28}$$

where in the last step we used the fact that contracted indices are dummy and can consequently be represented by any symbol.

So, we have obtained for the third power of the "innerproduct" (2.27) the following:

$$(\vec{n}\cdot\vec{A})^3 = -n^2\vec{n}\cdot\vec{A}. \tag{2.29}$$

Using this relation repeatedly for the higher order powers of $\vec{n}\cdot\vec{A}$, we may also determine its exponential, *i.e.*

$$\begin{aligned}
\exp\{\vec{n}\cdot\vec{A}\} &= \mathbf{1} + \vec{n}\cdot\vec{A} + \frac{1}{2!}(\vec{n}\cdot\vec{A})^2 + \frac{1}{3!}(\vec{n}\cdot\vec{A})^3 + \frac{1}{4!}(\vec{n}\cdot\vec{A})^4 + \cdots \\
&= \mathbf{1} + \vec{n}\cdot\vec{A} + \frac{1}{2!}(\vec{n}\cdot\vec{A})^2 + \frac{1}{3!}(-n^2\vec{n}\cdot\vec{A}) + \frac{1}{4!}(-n^2(\vec{n}\cdot\vec{A})^2) + \cdots \\
&= \mathbf{1} + \{1 - \frac{n^2}{3!} + \frac{n^4}{5!} - \frac{n^6}{7!} + \cdots\}(\vec{n}\cdot\vec{A}) + \\
&\quad + \{\frac{1}{2!} - \frac{n^2}{4!} + \frac{n^4}{6!} - \frac{n^6}{8!} + \cdots\}(\vec{n}\cdot\vec{A})^2 \\
&= \mathbf{1} + \{n - \frac{n^3}{3!} + \frac{n^5}{5!} - \frac{n^7}{7!} + \cdots\}(\hat{n}\cdot\vec{A}) + \\
&\quad + \{\frac{n^2}{2!} - \frac{n^4}{4!} + \frac{n^6}{6!} - \frac{n^8}{8!} + \cdots\}(\hat{n}\cdot\vec{A})^2.
\end{aligned}$$

We recognize here the Taylor expansions for the cosine and sine functions. So, substituting these goniometric functions for their expansions, we obtain the following result:

$$\exp\{\vec{n}\cdot\vec{A}\} = \mathbf{1} + \sin(n)(\hat{n}\cdot\vec{A}) + (1-\cos(n))(\hat{n}\cdot\vec{A})^2. \tag{2.30}$$

Next, we will show that this exponential operator leaves the vector $\vec{n}$ invariant. For that purpose we proof, using formula (2.28), the following:

$$\{(\vec{n}\cdot\vec{A})\vec{n}\}_i = (\vec{n}\cdot\vec{A})_{ij}n_j = (n_kA_k)_{ij}n_j = n_k(A_k)_{ij}n_j = -n_k\epsilon_{kij}n_j = 0,$$

or equivalently:

$$(\vec{n}\cdot\vec{A})\vec{n} = 0. \tag{2.31}$$

Consequently, the exponential operator (2.30) acting at the vector $\vec{n}$, gives the following result:

$$\exp\{\vec{n}\cdot\vec{A}\}\vec{n} = \left[\mathbf{1} + \vec{n}\cdot\vec{A} + \cdots\right]\vec{n} = \mathbf{1}\vec{n} = \vec{n} \tag{2.32}$$

So, the exponential operator (2.30) leaves the vector $\vec{n}$ invariant and of course also the vectors $a\vec{n}$, where a represents an arbitrary real constant. Consequently, the axis through the vector $\vec{n}$ is invariant, which implies that it is the rotation axis when the exponential operator represents a rotation, *i.e.* when this operator represents an orthogonal transformation with determinant one. One way to demonstrate this, is to consider vectors perpendicular to $\vec{n}$ and study how they transform. When, as indeed one finds, the operator (2.30) acts as a rotation on those vectors, then, by writing an arbitrary vector as a linear combination of $\vec{n}$ and two vectors perpendicular to $\vec{n}$, one finds that the operator (2.30) acts as a rotation on all vectors of the three-dimensional space and hence represents an orthogonal transformation. As a result one finds moreover that the rotation angle equals the length of the vector $\vec{n}$. In fact there might be a minus sign which depends on the choice of the direction of $\hat{n}$.

Here we will however not follow this procedure, but just study one example in the following problem:

Problem 18:

Show, that for $\vec{n} = (0,0,n)$ where $n > 0$ and hence $\hat{n} = (0,0,1)$, we find that the exponential operator defined in (2.30) represents a rotation around the z-axis (2.14):

$$\exp\{\vec{n}\cdot\vec{A}\} = R(\hat{z}, n).$$

Concludingly, we may state that we found a second parametrization of a rotation around the origin in three dimensions, *i.e.*:

$$R(n_1, n_2, n_3) = \exp\{\vec{n} \cdot \vec{A}\}, \tag{2.33}$$

where the rotation angle is determined by:

$$n = \sqrt{n_1^2 + n_2^2 + n_3^2},$$

and where the rotation axis is indicated by the direction of $\vec{n}$.

A warning is here in place: For matrices B and C it is in general not true that the product of their exponentials equals the exponential of their sum, as is the case for real numbers, *i.e.*:

$$\exp(B)\exp(C) \left\{ \begin{array}{ll} = \exp(B + C) & \text{if } \; [B, C] = 0 \\ \neq \exp(B + C) & \text{in general.} \end{array} \right.$$

So since the generators (2.19) do not commute (see 2.25), in general the product of two rotations represented by expressions similar to (2.30), is not equal to the exponential of the sum of the two exponents, but only when the exponents commute.

For small rotation angles one might for (2.30) use to a certain accuracy, a first order approximation, *i.e.*:

$$\exp\{\vec{n} \cdot \vec{A}\} \approx \mathbf{1} + \vec{n} \cdot \vec{A} = \mathbf{1} + \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}. \tag{2.34}$$

This expression is often referred to as an *infinitesimal rotation*. Notice that the operator $\vec{n} \cdot \vec{A}$ is here represented by an antisymmetric matrix and hence traceless.

2.4 Tensors and invariants under rotations.

In section (2.1) we have discussed the transformation rules in two dimensions for vectors under an orthogonal basis transformation induced by a rotation. Here we will generalize the formalism to any dimension.

Let us consider an orthogonal coordinate system $\mathcal{S}$ in n dimensions characterized by coordinates x_i $(i = 1, \cdots, n)$ and by the orthonormal basis vectors $\hat{e}_i$ $(i = 1, \cdots, n)$. And let us furthermore consider in the same n-dimensional space a different orthogonal coordinate system $\mathcal{S}'$ characterized by coordinates x'_i $(i = 1, \cdots, n)$ and by the orthonormal basis vectors $\hat{e}'_i$ $(i = 1, \cdots, n)$. The orthonormality of the basis vectors is expressed by:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \quad \text{and} \quad \hat{e}'_i \cdot \hat{e}'_j = \delta_{ij} \tag{2.35}$$

The basis vectors of $\mathcal{S}$ and $\mathcal{S}'$ are related via linear transformation rules, given by:

$$\hat{e}'_i = R_{ij}\hat{e}_j \quad \text{and} \quad \hat{e}_i = (R^{-1})_{ij}\hat{e}'_j. \tag{2.36}$$

Using the orthonormality relations (2.35), we find for the matrix elements R_{ij} of the transformation matrix R, the following property:

$$\begin{aligned}\delta_{ij} &= \hat{e}'_i \cdot \hat{e}'_j = \left(R_{ik}\hat{e}_k\right) \cdot \left(R_{jl}\hat{e}_l\right) = R_{ik}R_{jl}\hat{e}_k \cdot \hat{e}_l \\ &= R_{ik}R_{jl}\delta_{kl} = R_{ik}R_{jk} = R_{ik}\left(R^T\right)_{kj} \\ &= \left(RR^T\right)_{ij},\end{aligned}$$

or equivalently:

$$RR^T = \mathbf{1} \ \text{ or } \ R^T = R^{-1}. \tag{2.37}$$

Linear transformations for which the transposed of the matrix equals the inverse of the matrix, are said to be *orthogonal*. The determinant of such matrices equals ± 1, as can be seen from:

$$\{det(R)\}^2 = det(R)det(R) = det(R)det(R^T) = det(RR^T) = det(\mathbf{1}) = 1. \tag{2.38}$$

Rotations have determinant $+1$, anti-orthogonal transformations have determinant -1. Here we will mainly concentrate on rotations, but some properties are as well valid for anti-orthogonal transformations.

Let the position vector $\vec{x}$ in $\mathcal{S}$ be given by:

$$\vec{x} = x_i\hat{e}_i \ , \quad (i = 1, \cdots n), \tag{2.39}$$

and in $\mathcal{S}'$ by:

$$\vec{x} = x'_i\hat{e}'_i \ , \quad (i = 1, \cdots n). \tag{2.40}$$

The relation between the coordinates of $\vec{x}$ in $\mathcal{S}$ and the coordinates of $\vec{x}$ in $\mathcal{S}'$, using the orthonormality property of the basis vectors (2.35), the linear relations between the two different bases (2.36) and the definitions of the coordinates (2.39) and (2.40), follows from:

$$\begin{aligned}x'_i &= x'_k\delta_{ki} = x'_k\hat{e}'_k \cdot \hat{e}'_i = \vec{x} \cdot \hat{e}'_i = \vec{x} \cdot \left(R_{ij}\hat{e}_j\right) \\ &= R_{ij}\vec{x} \cdot \hat{e}_j = R_{ij}\left(x_k\hat{e}_k\right) \cdot \hat{e}_j = R_{ij}x_k\delta_{kj},\end{aligned}$$

or equivalently:

$$x'_i = R_{ij}x_j. \tag{2.41}$$

And similarly follows for the inverse relations:

$$x_i = \left(R^{-1}\right)_{ij} x'_j = \left(R^T\right)_{ij} x'_j = R_{ji}x'_j. \tag{2.42}$$

Under orthogonal transformations various quantities remain invariant. One of them is the innerproduct of two vectors. Consider two vectors $\vec{x}$ and $\vec{y}$, the coordinates of which transform as given in formula (2.41). Their innerproduct is in $\mathcal{S}$ given by:

$$\vec{x} \cdot \vec{y} = x_i y_i.$$

In $\mathcal{S}'$ we find, using the transformation rules (2.41) and the orthogonality property (2.37) of the rotation matrices:

$$x'_i y'_i = R_{ik} x_k R_{il} y_l = R_{ik} R_{il} x_k y_l = \delta_{kl} x_k y_l = x_k y_k. \tag{2.43}$$

In particular this implies that the length of a vector is invariant under orthogonal transformations. Quantities which are invariant under arbitrary orthogonal transformations are called *scalars for orthogonal transformations*. Formally, one might use the name *tensor of rank zero under orthogonal transformations*, but usually these quantities are just referred to as *scalars*. The latter habit is misleading though common practice. Misleading, because a quantity might be a scalar for one group, but not for another group of transformations. However, when the context is clear then it is of course a bit cumbersome to every time refer to this context.

Any function of a scalar is also a scalar. More general, a function f of the coordinates is a scalar under orthogonal transformations when it satisfies:

$$f'(x'_1, \cdots, x'_n) = f(x_1, \cdots, x_n). \tag{2.44}$$

Quantities which transform similar to the components of the position vectors as in formula (2.41), are called vectors or tensors of rank one. The gradient of a scalar function $f(\vec{x})$ is an example of a vector, *i.e.*:

$$\frac{\partial f'}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{\partial f}{\partial x_j} = \frac{\partial R_{kj} x'_k}{\partial x'_i}\frac{\partial f}{\partial x_j} = R_{kj}\frac{\partial x'_k}{\partial x'_i}\frac{\partial f}{\partial x_j} = R_{kj}\delta_{ki}\frac{\partial f}{\partial x_j},$$

or equivalently:

$$\frac{\partial f'}{\partial x'_i} = R_{ij}\frac{\partial f}{\partial x_j}. \tag{2.45}$$

When we compare the resulting relation to the formula given in (2.41), then we see that the transformation rule for the gradient of a vector is exactly equal to the transformation rule for a position vector.

An n component function of position $\vec{v}(\vec{x})$ given by:

$$\vec{v}(\vec{x}) = \left(v_1(\vec{x}), \cdots, v_n(\vec{x})\right),$$

is called a *vector field* under orthogonal transformations when its components transform like a vector, *i.e.*

$$v'_i(x'_1, \cdots, x'_n) = R_{ij} v_j(x_1, \cdots, x_n). \tag{2.46}$$

The gradient of a scalar field is an example of a vector field.

The divergence of a vector field is a scalar field as follows:

$$\frac{\partial v_i'}{\partial x_i'} = \frac{\partial x_k}{\partial x_i'}\frac{\partial R_{ij}v_j}{\partial x_k} = R_{ik}R_{ij}\frac{\partial v_j}{\partial x_k} = \delta_{jk}\frac{\partial v_j}{\partial x_k},$$

or equivalently:

$$\frac{\partial v_i'}{\partial x_i'} = \frac{\partial v_j}{\partial x_j}. \tag{2.47}$$

An example of a tensor of rank two is the set of n^2 quantities defined by:

$$t_{ij}(x_1, \cdots, x_n) = \frac{\partial v_i(x_1, \cdots, x_n)}{\partial x_j}, \tag{2.48}$$

where $\vec{v}(\vec{x})$ represents a vector field. Using similar transformation rules as in (2.45), one finds that the quantities (2.48) transform under an orthogonal transformation, as follows :

$$t_{ij}' = \frac{\partial v_i'}{\partial x_j'} = R_{jl}\frac{\partial R_{ik}v_k}{\partial x_l} = R_{ik}R_{jl}t_{kl}. \tag{2.49}$$

Each index of the tensor field is in the above transformation rule, separately contracted like the index of a vector field. Notice that the relation (2.49) might also be written as:

$$\boldsymbol{t}' = R\boldsymbol{t}R^T. \tag{2.50}$$

For the particular case of three dimensions we will frequently make use of the Levi-Civita tensor ϵ_{ijk} introduced in (2.20). For example, the vector product of two vectors in three dimensions can be expressed in terms of this tensor as follows:

$$\begin{aligned}
\vec{c} &= \vec{a} \times \vec{b} \\
&= (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) \\
&= (\epsilon_{1jk}a_jb_k,\ \epsilon_{2jk}a_jb_k,\ \epsilon_{3jk}a_jb_k),
\end{aligned}$$

or equivalently:

$$c_i = \epsilon_{ijk}a_jb_k. \tag{2.51}$$

Also, the determinant of a 3×3 matrix B can be expressed by:

$$det(B) = \epsilon_{ijk}B_{1i}B_{2j}B_{3k} = \epsilon_{ijk}B_{i1}B_{j2}B_{k3}.$$

In particular, one finds for a rotation R in three dimensions the following expression for the determinant:

$$\epsilon_{ijk}R_{1i}R_{2j}R_{3k} = \epsilon_{ijk}R_{i1}R_{j2}R_{k3} = det(R) = 1. \tag{2.52}$$

This relation can be generalized, using the orthogonality properties of a 3×3 rotation matrix, to give:

$$\epsilon_{ijk}R_{ai}R_{bj}R_{ck} = \epsilon_{ijk}R_{ia}R_{jb}R_{kc} = \epsilon_{abc}. \tag{2.53}$$

These equations show also the transformation properties of the Levi-Civita tensor under rotations. So, under rotations this tensor transforms as a tensor of rank three. But, be careful, under orthogonal transformations in general there is an extra minus sign involved for anti-orthogonal transformations. The complete transformation rule for the Levi-Civita tensor can then for orthogonal transformations be expressed as above in (2.53) but multiplied with the determinant of the transformation matrix. Such quantities are called *pseudo-tensors* or *axial tensors*.

From the relations (2.53) one may also infer the following identities:

$$\epsilon_{ijk}R_{kl} = R_{im}R_{jn}\epsilon_{lmn} \quad \text{and} \quad \epsilon_{ijk}R_{lk} = R_{mi}R_{nj}\epsilon_{lmn} \tag{2.54}$$

Problem 19:

Proof the identities (2.53) and (2.54).

A possible strategy to proof these identities is to first define three vectors which are in the following way related to the rows of a rotation matrix R:

$$\vec{R}^{(i)} = (R_{i1}, R_{i2}, R_{i3}),$$

or in a more compact notation:

$$\left(\vec{R}^{(i)}\right)_j = R_{ij}.$$

For these vectors, using the orthogonality property of the rotation matrices (2.37), the definition of the vector product (2.51) and the fact that det(R)=1, it is easy to verify that:

$$\vec{R}^{(i)} \cdot \vec{R}^{(j)} = \delta_{ij} \quad \text{and} \quad \vec{R}^{(i)} \times \vec{R}^{(j)} = \epsilon_{ijk}\vec{R}^{(k)}.$$

The remaining details of the proof are then straightforward.

An example of an axial vector field is the vector product (2.51) of two vector fields. Using the relations (2.54) we find for their transformation under rotations the following rule:

$$\begin{aligned} c'_i &= (\vec{a}' \times \vec{b}')_i = \epsilon_{ijk}a'_j b'_k = \epsilon_{ijk}(R_{jm}a_m)(R_{kn}b_n) \\ &= \epsilon_{mnl}R_{il}a_m b_n = R_{il}(\vec{a} \times \vec{b})_l. \end{aligned} \tag{2.55}$$

We find, as expected, that the vector product transforms as a vector under rotations. But, for an anti-orthogonal transformation O, which has a negative determinant, *i.e.* $det(O) = -1$, one obtains a different result. For example, space inversion leads for the vector product (2.51) to:

$$a'_i = -a_i \text{ and } b'_j = -b_j, \text{ consequently } c'_k = c_k.$$

That is that the components of the vector product $\vec{c} = \vec{a} \times \vec{b}$ transform different than the component of the vectors $\vec{a}$ and $\vec{b}$. Consequently, the vector product is not a vector, but an axial vector under orthogonal transformations. One may show that the complete transformation of a vector product involves a determinant of the transformation matrix O, as follows:

$$c'_i = det(O) O_{ij} c_j. \tag{2.56}$$

One may similarly modify the transformation rules for the Levi-Civita tensor given in (2.53).

A second rank tensor $\boldsymbol{t}$ has two important quantities associated with it, which are invariant under rotations: its *determinant* and its *trace*. That these quantities are scalars under rotations can, using the transformation property of a second rank tensor (2.50), be seen as follows: For the determinant we use the general properties of matrix multiplication, *i.e.* $det(AB) = det(BA) = det(A)det(B)$, and the fact that for an orthogonal transformation the transposed equals the inverse:

$$det(\boldsymbol{t}') = det(R\boldsymbol{t}R^T) = det(\boldsymbol{t}R^T R) = det(\boldsymbol{t}), \tag{2.57}$$

For the trace we return to the transformation rule of the second rank tensor (2.49) and use the orthogonality property (2.37) of the rotation matrices, to find:

$$Tr(\boldsymbol{t}') = t'_{ii} = R_{ik} R_{il} t_{kl} = \delta_{kl} t_{kl} = Tr(\boldsymbol{t}). \tag{2.58}$$

A third invariant quantity associated with a second rank tensor is related to the fact that the product of two second rank tensors also forms a second rank tensor, *i.e.*:

$$\boldsymbol{A}'\boldsymbol{B}' = R\boldsymbol{A}R^T R\boldsymbol{B}R^T = R\boldsymbol{A}\boldsymbol{B}R^T = (\boldsymbol{A}\boldsymbol{B})'.$$

Consequently, also the trace of the product of two second rank tensors forms an invariant under rotations, *i.e.*:

$$Tr\{(\boldsymbol{A}\boldsymbol{B})'\} = Tr\{\boldsymbol{A}'\boldsymbol{B}'\} = Tr\{\boldsymbol{A}\boldsymbol{B}\}.$$

When we apply this latter property to the product of a second rank tensor with itself, then we find the following invariant quantity under rotations:

$$Tr\{\boldsymbol{t}^2\} = (\boldsymbol{t}^2)_{ii} = t_{ij} t_{ji}. \tag{2.59}$$

The above defined three invariant quantities of a second rank tensor will play an important role in the coming chapters.

A second rank tensor M which is symmetric, remains symmetric under a rotation of the coordinate system, *i.e.*

$$\begin{aligned} M'_{ji} &= (RMR^T)_{ji} = R_{jk}M_{kl}(R^T)_{li} \\ &= (R^T)_{kj}M_{lk}R_{il} = (RMR^T)_{ij} = M'_{ij}. \end{aligned} \tag{2.60}$$

Similarly, remains an anti-symmetric second rank tensor anti-symmetric under rotations.

Because of the above properties, the space of all possible second rank tensors can be organized in at least three subspaces which are invariant under rotations:

1 All second rank tensors which are proportional to the unit matrix. The orthogonality property (2.37) guarantees that such matrices transform into themselves under rotations and thus remain in the same subspace.
2 All traceless symmetric second rank tensors. Since the trace is an invariant under rotations (see formula 2.58) and the symmetry preserved (2.60), rotations transform these tensors amongst each other.
3 All anti-symmetric second rank tensors which are also transformed amongst themselves by rotations.

An arbitrary second rank tensor can always be written as the sum of three second rank tensors, each out of one of the three above defined subspaces, as follows:

$$M_{ij} = \frac{1}{n}Tr(M)\delta_{ij} + s_{ij} + t_{ij}, \tag{2.61}$$

where n represents the dimension of the coordinate system and where the matrices $\boldsymbol{s}$ and $\boldsymbol{t}$ are respectively defined according to:

$$s_{ij} = \frac{1}{2}\left(M_{ij} + M_{ji}\right) - \frac{1}{n}Tr(M)\delta_{ij}, \tag{2.62}$$

and

$$t_{ij} = \frac{1}{2}\left(M_{ij} - M_{ji}\right). \tag{2.63}$$

Problem 20:

Show, that when M represents a second rank tensor with respect to rotations, then for $\boldsymbol{s}$ and $\boldsymbol{t}$ defined as above in respectively the formulas (2.62) and (2.63) yields:

(i) $\boldsymbol{s}$ is traceless and symmetric,
(ii) $\boldsymbol{t}$ is anti-symmetric, and
(iii) $\boldsymbol{s}$ and $\boldsymbol{t}$ are second rank tensors under rotations.

2.5 Quadratic surfaces.

The main topic of this section consists of quadratic surfaces in two dimensions. However, most of the formalism can without difficulty be extended to three or more dimensions. In two dimensions quadratic surfaces (*i.e.* a quadratic line in the xy-plane) are circles, ellipses or hyperboles centered in the origin and can be represented by equations of the following type:

$$ax^2 + 2bxy + cy^2 = constant, \tag{2.64}$$

i.e. an expression quadratic in x and y (a, b and c represent here constants). We assume that we can always arrange things such that the constant at the right hand side of equation (2.64) equals 1. Furthermore, as before (see section 2.1), we prefer to refer to x and y as x_1 and x_2 respectively. This way we can compactify the notation of the above equation (2.64) as follows:

$$M_{ij}x_ix_j = 1, \tag{2.65}$$

where:

$$M_{11} = a \quad , \quad M_{12} + M_{21} = 2b \;\; and \;\; M_{22} = c.$$

In general, the off-diagonal matrix elements of the matrix M could be chosen to be different, as long as their sum equals $2b$. But one prefers to select a symmetric matrix M for the representation of a quadratic surface, *i.e.*:

$$M_{12} = M_{21} = b.$$

So, instead of defining a quadratic surface by the relation (2.64), one might just refer to the matrix M as to represent a quadratic surface. For example, the unit circle is represented by the unit matrix $\mathbf{1}$ whereas a circle of radius r is represented by the matrix $\mathbf{1}/r^2$. An ellipse which has its principal axes along the x-axis (of length $2a$) and the y-axis (of length $2b$), is represented here by a matrix given by:

$$M\{\text{ellipse } (a,b)\} = \begin{pmatrix} a^{-2} & 0 \\ 0 & b^{-2} \end{pmatrix}. \tag{2.66}$$

The same surface can also be described in the rotated coordinate system $\mathcal{S}'$ as defined by the relation (2.4). The related equation similar to (2.65) would in the primed system read:

$$M'_{ij}x'_ix'_j = 1. \tag{2.67}$$

Clearly, there must exist a relation between the above defined matrix M' which describes the quadratic surface in the coordinate system $\mathcal{S}'$ and the previously defined matrix M which describes the same surface in the unprimed coordinate system. This relation can most easily be deduced by substituting in the above equation (2.67) the expressions (2.5), which relate the coordinates of a vector in the unprimed coordinate system with its primed coordinates in the system $\mathcal{S}'$, leading to:

$$M'_{ij} R_{ik} x_k R_{jl} x_l = 1.$$

Comparing the above result with the equation (2.65), one finds for the elements of the two matrices:

$$M'_{ij} R_{ik} R_{jl} = M_{kl},$$

which implies in terms of the matrices the following transformation rules:

$$R^T M' R = M \quad , \quad or \quad M' = RMR^T. \tag{2.68}$$

From the above transformation rule, in comparison with formula (2.50), one concludes that the matrix M transforms as a second rank tensor under rotations.

As an example, let us determine the matrix which describes the above defined ellipse (2.66) in the rotated system $\mathcal{S}'$ given by the rotation $R(\hat{z}, \alpha)$ defined in (2.3). Using the above transformation rule (2.68), we find:

$$M'\{\text{ellipse } (a,b)\} = \begin{pmatrix} \dfrac{\cos^2(\alpha)}{a^2} + \dfrac{\sin^2(\alpha)}{b^2} & \cos(\alpha)\sin(\alpha)\{\frac{1}{a^2} - \frac{1}{b^2}\} \\ & \\ \cos(\alpha)\sin(\alpha)\{\frac{1}{a^2} - \frac{1}{b^2}\} & \dfrac{\sin^2(\alpha)}{a^2} + \dfrac{\cos^2(\alpha)}{b^2} \end{pmatrix}. \tag{2.69}$$

Notice that the above matrix is again symmetric.

Matrices, like the above symmetric matrix (2.69), might also be considered as the most general parametrization of all possible ellipses in the xy-plane which have the center in the origin. The constants a and b indicate the lengths of the principal axes (*i.e.* $2a$ and $2b$ respectively), whereas the angle α refers to the orientation of those axes with respect to the x-axis, *i.e.*:

$$M\{\text{ellipse } (a,b,\alpha)\} = \begin{pmatrix} \dfrac{\cos^2(\alpha)}{a^2} + \dfrac{\sin^2(\alpha)}{b^2} & \cos(\alpha)\sin(\alpha)\{\frac{1}{a^2} - \frac{1}{b^2}\} \\ & \\ \cos(\alpha)\sin(\alpha)\{\frac{1}{a^2} - \frac{1}{b^2}\} & \dfrac{\sin^2(\alpha)}{a^2} + \dfrac{\cos^2(\alpha)}{b^2} \end{pmatrix}. \tag{2.70}$$

Reversely, we may also conclude from the above example (2.69) that any symmetric matrix describes a circle or an ellipse centered in the origin, when its eigenvalues are positive.

In section (2.4) we discussed some general properties of the transformation of matrix operators like the above matrix M, under orthogonal coordinate transformations, below we will verify the relevant results of that section to the above matrix M defined in formula (2.66):

First we observe that the matrix (2.69) is again symmetric. This is a general result for a symmetric tensor operator (*i.e.* a matrix for which $M_{lk} = M_{kl}$) under an orthogonal basis transformation as is shown in equation (2.60). Next, we notice that for the *traces* of the matrices (2.66) and (2.69) holds the following:

$$Tr(M') = \frac{\cos^2(\alpha)}{a^2} + \frac{\sin^2(\alpha)}{b^2} + \frac{\sin^2(\alpha)}{a^2} + \frac{\cos^2(\alpha)}{b^2} = \frac{1}{a^2} + \frac{1}{b^2} = Tr(M).$$

This is also generally true for orthogonal basis transformations, as has been shown in (2.58).

One also might compare the *determinants* of the matrices (2.66) and (2.69), in order to find:

$$\begin{aligned} det(M') &= \left(\frac{\cos^2(\alpha)}{a^2} + \frac{\sin^2(\alpha)}{b^2}\right)\left(\frac{\sin^2(\alpha)}{a^2} + \frac{\cos^2(\alpha)}{b^2}\right) \\ &\quad - \cos^2(\alpha)\sin^2(\alpha)\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{1}{a^2}\,\frac{1}{b^2} = det(M). \end{aligned}$$

This general property of orthogonal basis transformations is shown in formula (2.57).

Let us return to the expression (2.69) in order to discuss one more property of orthogonal basis transformations. In the system $\mathcal{S}$, the matrix (2.66) which represents the ellipse, has the two eigenvectors $\vec{v} = (1, 0)$ and $\vec{w} = (0, 1)$, respectively corresponding to the eigenvalues a^{-2} and b^{-2}. The directions indicated by $\vec{v}$ and $\vec{w}$ are called the *principal axes* of the ellipse. The eigenvalue relations are in this case given by:

$$M\{\text{ellipse } (a, b)\}\vec{v} = a^{-2}\vec{v} \;\text{ and }\; M\{\text{ellipse } (a, b)\}\vec{w} = b^{-2}\vec{w}. \tag{2.71}$$

In the rotated coordinate system $\mathcal{S}'$ given by the rotation (2.3), the vectors $\vec{v}$ and $\vec{w}$ are respectively given by the expressions:

$$\vec{v} = \cos(\alpha)\hat{e}'_1 + \sin(\alpha)\hat{e}'_2 \;\text{ and }\; \vec{w} = -\sin(\alpha)\hat{e}'_1 + \cos(\alpha)\hat{e}'_2.$$

One may easily verify that expressed in the above components these vectors are the eigenvectors of the transformed matrix M' (2.69), *i.e.*:

$$M'\{\text{ellipse } (a, b)\}\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} = a^{-2}\begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$$

and

$$M'\{\text{ellipse } (a, b)\}\begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix} = b^{-2}\begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}.$$

From this example we might conclude that in general the principal axes of a quadratic surface are indicated by the eigenvectors of the corresponding symmetric matrix in any orthogonal coordinate system.

Consequently, there exists always an orthogonal coordinate system for which the symmetric matrix which describes a quadratic surface, is diagonal. One finds such system by first determining the eigenvectors of the matrix and then selecting the unit vectors of the new coordinate system in the direction of those eigenvectors. Whether or not the related basis transformation R represents a rotation depends on the order of the unit vectors.

In three dimensions, quadratic surfaces are spheres, ellipsoids and hyperboloids centered in the origin. A circle of radius r is given by the relation:

$$x^2 + y^2 + z^2 = r^2,$$

or equivalently by an expression similar to (2.65) but now with the implicit summations running from 1 to 3 and where:

$$M\{\text{sphere (radius } r)\} = \begin{pmatrix} r^{-2} & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}. \tag{2.72}$$

An elipsoid which has its principal axes along the x-axis (length $2a$), along the y-axis (length $2b$) and the z-axis (length $2c$), is represented by:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1,$$

or equivalently by the matrix:

$$M\{ellipsoid(a, b, c)\} = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix}. \tag{2.73}$$

Hyperboloids which have their principal axes along the x-, y- and z-axis are obtained by replacing one or two of the diagonal elements of the matrix defined in (2.73) by minus that diagonal element.

The properties of tensor operators under orthogonal coordinate transformations which are shown in this section for two dimensional quadratic surfaces, are also valid in three dimensions.

Chapter 3

The theory of small deformations.

For the study of deformable media, *i.e.* solids which are not considered as rigid bodies, but which can change their shape and volume, it is necessary to first set up a framework for the description of deformation itself. Here we will concentrate on small deformations of a solid. Large enough to reveal the properties of the material of which the solid body is made. But small enough to allow for reasonable approximations.

When a solid is stretched, twisted or compressed, the material points of the solid are displaced. For small deformations, one assumes that the local structure of the solid remains in tact. It means that neighboring material points do not move far away from each other and remain neighboring points throughout the deformation. An accurate definition of a small deformation is somewhat tedious to be formulated. For here, we rely on common sense. For instance an elastic spring can be stretched quite a lot before it looses its elastic properties. But a piece of ceramic should not be dealt with in a comparable way. A more practical definition might be to consider a small deformation, any change of form of a solid which can be described by methods to be studied below.

This chapter is organized as follows: First we set up a simple mathematical framework for the description of small deformations in two dimensions. Then, once we obtained a formalism which is suitable for generalization, we study the general theory of small deformations in three dimensions. At the end of this chapter a practical application of the formalism is shown, which is related to the consequences of crystal symmetries.

3.1 Deformations of the unit circle.

In section (2.5) we have seen that the unit circle is a quadratic surface which can be represented by the unit matrix $\mathbf{1}$, *i.e.*:

$$\delta_{ij}\; x_i x_j = 1. \tag{3.1}$$

In this section we will study deformations of the unit circle, such that the resulting "surface" is again quadratic.

Let us consider the unit circle to be drawn on a thin rubber sheet in the xy-plane. In fact we have here in the back of our mind an xy-plane which is depicted on top of our desk and a rubber sheet which can move freely over the same desk. A deformation of the rubber sheet is supposed to have as a consequence that the circle is deformed into an ellipse. In practice this will be only true for infinitesimally small deformations of a realistic rubber sheet, but from the mathematical point of view that does not bother us here. Later on in this chapter, we will understand why it is useful to study the above described idealized deformations.

We indicate the coordinates of material points in the undeformed rubber sheet by $\vec{x}$ and the coordinates of the same points on the sheet after a certain deformation by $\vec{X}$. Below we will study a few examples:

1. Stretching the rubber sheet in the direction of the x-axis.

As a first example we pull the rubber sheet on both sides such that it is stretched in the direction of the x-axis. When we stretch the rubber sheet in the direction of the x-axis, then the y-coordinates of the points on the sheet are supposed to remain the same, *i.e.*:

$$X_2 = x_2. \tag{3.2}$$

We might organize things in such a way that the points which are on the y-axis remain on the y-axis. In that case the new x-coordinates of material points on the sheet after the deformation (*i.e.* X_1), are supposed to be proportional to the old x-coordinates of the same points before the deformation (*i.e.* x_1). When we select for the proportionality constant the symbol a, then we find for the transformation which describes the above deformation, the following:

$$X_1 = a x_1 \quad , \quad a > 0. \tag{3.3}$$

In order to find the quadratic surface which follows from the deformation of the unit circle under the above transformations (3.2) and (3.3), we may just perform the following substitution of the inverse transformations in the relation (3.1):

$$1 = \delta_{ij}\; x_i x_j = (x_1)^2 + (x_2)^2 = \left(\frac{X_1}{a}\right)^2 + (X_2)^2. \tag{3.4}$$

This equation describes an ellipse which has its principal axes along the x-axis and the y-axis (compare the representation 2.66 for an ellips in two dimensions). The sizes of the principal axes of the ellipse are indicated by a in the x-direction and by 1 in the y-direction (which means that these sizes are equal to $2a$ and 2 respectively).

The transformations (3.2) and (3.3) which describe the deformation under consideration of the rubber sheet, could also have been represented by a matrix D, defined by:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = D \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad , \quad D = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

We will refer here to the matrix D as the *deformation matrix*.

The inverse of the transformations (3.2) and (3.3) are then represented by the expression:

$$x_i = (D^{-1})_{ij} X_j . \tag{3.6}$$

Inserting this into the relation for the unit circle (3.1) leads to the following relation:

$$1 = \delta_{ij} \ (D^{-1})_{ik} X_k (D^{-1})_{jl} X_l = \{(D^{-1})^T D^{-1}\}_{kl} X_k X_l . \tag{3.7}$$

When one substitutes next the definition (3.5) of the matrix D into the above relations, then one obtains again the equation (3.4) for the ellipse. The relations (3.7) are however more general and can be used for any linear deformation of the unit circle.

The area of the unit circle (3.1), equals π; the area of the ellipse (3.4), πa. So, we may conclude that the areas of the deformed figures are in this case all a factor a larger than the areas of their originals. For a small "volume" element $\Delta\sigma$ this might be seen as follows:

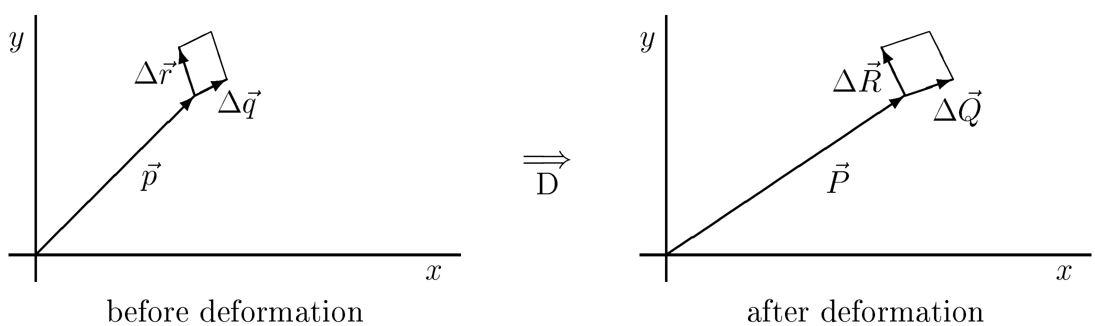
Let the "volume" element before the deformation be a "small" parallogram and let its position be indicated by the position of one of its corners (see figure below): *i.e.* $\vec{p} = (p_1, p_2)$ before the deformation and $\vec{P} = (ap_1, p_2)$ after the deformation. The positions of the other two nearby corners are indicated by:

$$\vec{p} + \Delta\vec{q} = (p_1 + \Delta q_1 \ , \ p_2 + \Delta q_2) \ \text{ and } \ \vec{p} + \Delta\vec{r} = (p_1 + \Delta r_1 \ , \ p_2 + \Delta r_2)$$

before the deformation, and by:

$$\vec{P} + \Delta\vec{Q} = (a\{p_1 + \Delta q_1\} \ , \ p_2 + \Delta q_2) \ \text{ and } \ \vec{P} + \Delta\vec{R} = (a\{p_1 + \Delta r_1\} \ , \ p_2 + \Delta r_2)$$

after the deformation. The above described situation is illustrated in the figure below for the case $a = 3/2$.



Before the deformation the area $\Delta\sigma$ of the paralellogram is given by:

$$\Delta\sigma = | \ det \begin{pmatrix} \Delta q_1 & \Delta r_1 \\ \Delta q_2 & \Delta r_2 \end{pmatrix} | \tag{3.8}$$

After the deformation the area $\Delta\Sigma$ of the deformed paralellogram is given by:

$$\Delta\Sigma = |\ det \begin{pmatrix} \Delta Q_1 & \Delta R_1 \\ \Delta Q_2 & \Delta R_2 \end{pmatrix} | = |\ det \begin{pmatrix} a\Delta q_1 & a\Delta r_1 \\ \Delta q_2 & \Delta r_2 \end{pmatrix} |$$

$$= |\ det \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta q_1 & \Delta r_1 \\ \Delta q_2 & \Delta r_2 \end{pmatrix} \right\} | = |\ det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |\ \Delta\sigma. \quad (3.9)$$

So we may conclude that the size of figures increase here by a factor which is equal to the determinant of the deformation matrix (3.5).

2. Stretching the rubber sheet in both the x- and the y-direction.

As a second example we pull the rubber sheet as well in the x-direction as in the y-direction. Let the corresponding deformation matrix be given by:

$$D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad , \quad a, b > 0. \quad (3.10)$$

Using the same techniques as in the previous example (3.7), we find for the "deformed" unit circle in this case the relation:

$$1 = \left(\frac{X_1}{a}\right)^2 + \left(\frac{X_2}{b}\right)^2 . \quad (3.11)$$

This represents an ellipse which is centered in the origin and has its principal axes in the directions of the x-axis (length $2a$) and the y-axis (length $2b$).

One might consider the above deformation as being the result of two consecutive deformations of the rubber sheet: First we stretch the sheet in the direction of the x-axis, such that the new x-coordinates of all points of the sheet are a factor a larger than their old x-coordinates. This can be given by (compare 3.5):

$$D_x = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} . \quad (3.12)$$

Then we perform a similar deformation in the y-direction, for which the proportionality constant equals b, represented by:

$$D_y = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} . \quad (3.13)$$

The complete deformation of this example is then simply represented by the matrix product of D_x, representing the deformation in the x-direction (3.12), and D_y, representing the deformation in the y-direction (3.13), *i.e.*:

$$D = D_y D_x .$$

It is left to the reader to verify that the result of this matrix multiplication is indeed equal to the matrix given in (3.10).

The area of the ellipse (3.11) described by the deformation matrix (3.10), is here a factor ab larger than the area of the unit circle, which factor is, as in the previous case, equal to the determinant of the deformation matrix.

3. Stretching and rotating the rubber sheet.

We leave the rubber sheet in the deformed situation of the previous case (3.10) and then rotate the sheet over an angle given by α. Let the corresponding rotation matrix be given by:

$$R(\hat{z},\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{3.14}$$

The resulting new coordinates of points on the sheet are related to their original positions before any deformation, by the matrix product:

$$\begin{aligned} D &= R(\hat{z},\alpha) D_y D_x \\ &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.15}$$

The matrix which describes the quadratic "surface" of the deformed unit circle (see 3.7) is here consequently given by:

$$\left(D^{-1}\right)^T D^{-1} = \begin{pmatrix} \dfrac{\cos^2(\alpha)}{a^2} + \dfrac{\sin^2(\alpha)}{b^2} & \cos(\alpha)\sin(\alpha)\{\frac{1}{a^2} - \frac{1}{b^2}\} \\ \cos(\alpha)\sin(\alpha)\{\frac{1}{a^2} - \frac{1}{b^2}\} & \dfrac{\sin^2(\alpha)}{a^2} + \dfrac{\cos^2(\alpha)}{b^2} \end{pmatrix}.$$

This is exactly the result we found earlier for a rotated ellipse (see 2.70), as of course had to be expected.

One should notice that the form of the ellipse has not changed here with respect to the ellipse of the previous example (3.11). A rotation does not involve any deformation, but merely changes the orientation of circles and ellipses. Consequently, the area of the ellipse is here as in the previous case without a rotation, a factor ab larger than the area of the unit circle. A factor which might also have been concluded from the determinant of the deformation matrix D (3.15) as follows:

$$det(D) = det(R(\hat{z},\alpha)) det(D_y D_x) = det(D_y D_x),$$

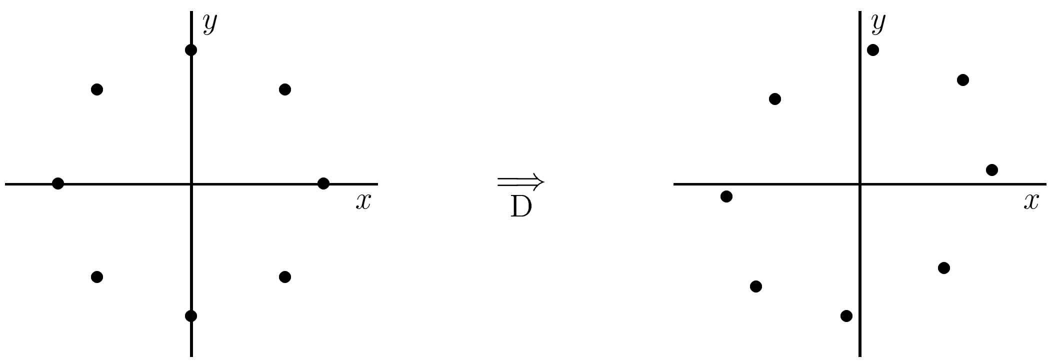
where we have used the fact that the determinant of a rotation matrix equals one, which too demonstrates that rotations do not involve any deformations.

4. Shear.

As a final example we will study here a deformation of the rubber sheet which is known in the literature as *shear* and which is given by the following deformation matrix:

$$D = \begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix} \quad , \quad |e| < 1. \tag{3.16}$$

Below is graphically represented (for the case $e = 0.1$), the effect of the deformation defined by the above matrix D on various points of the unit circle:



before deformation (•) after deformation (•)

The unit circle seems to transform under the deformation defined by the matrix D given in (3.16) into an ellipse which has its principal axes into directions which make angles of about 45^o with the x-axis. But let us follow the procedure outlined in the first example (3.7) of this section, in order to see which quadratic surface one obtains here. We start by determining the product of the transposed of the inverse of the above matrix D and the inverse itself, to obtain:

$$(D^{-1})^T D^{-1} = \frac{1}{(1-e^2)^2} \begin{pmatrix} 1+e^2 & -2e \\ -2e & 1+e^2 \end{pmatrix} \tag{3.17}$$

The relation for the quadratic surface which represents the deformed circle on the rubber sheet, follows by substituting the above result (3.17) for the product of the transposed of the inverse of D and the inverse of D itself, into the expression (3.7). However, besides ending up with a quadratic expression in X_1 and X_2, not much information would be gained from the resulting expression.

A more instructive method is to first inspect the expression (3.17) for its eigenvectors and eigenvalues. There is in fact no difficulty to find that the eigenvectors are given by $(X_1, X_2)_a = (1, 1)$ and $(X_1, X_2)_b = (-1, 1)$, *i.e.*:

$$\frac{1}{(1-e^2)^2} \begin{pmatrix} 1+e^2 & -2e \\ -2e & 1+e^2 \end{pmatrix} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix} = \frac{1}{(1 \pm e)^2} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}. \tag{3.18}$$

Consequently, the principal axes of the related quadratic surface are in those directions which thus should be chosen as the axes of an orthogonal coordinate system for which the matrix (3.17) is diagonal. The related basis transformation might be given by the following rotation (in general there are four different possibilities):

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

This matrix corresponds to a rotation angle of -45^o, in agreement with the guess we made before about the angle of the principal axes with respect to the x-axis. More explicitly, this matrix corresponds to the choice $\hat{e}_1' = \hat{e}_1/\sqrt{2} + \hat{e}_2/\sqrt{2}$ and $\hat{e}_2' = -\hat{e}_1/\sqrt{2} + \hat{e}_2/\sqrt{2}$ for the basis in the primed coordinate system. The diagonalization of the matrix which describes the quadratic surface of the deformed unit circle, is just performing this rotation on the matrix obtained in (3.17) using the transformation rule (2.50), which gives the following result:

$$R\{(D^{-1})^T D^{-1}\}R^T = \begin{pmatrix} \dfrac{1}{(1+e)^2} & 0 \\ 0 & \dfrac{1}{(1-e)^2} \end{pmatrix} , \tag{3.19}$$

One obtains the matrix corresponding to an ellipse, the principal axes of which have the respective lengths $2(1+e)$ and $2(1-e)$.

Consequently, we may conclude that the above deformation D is equal to stretching the rubber sheet in a direction which makes an angle of 45^o with the x-axis, because in that direction half the axis of the ellipse is longer than the radius of the unit circle by a fraction e, and squeezing the sheet in the perpendicular direction in which direction half the axis of the ellipse is shorter than the radius of the unit circle by the same fraction.

From the expression (3.19) it follows that the area of the ellipse is here $(1+e)(1-e)$ times smaller than the area of the unit circle. Notice that, as in the previous examples, this factor follows directly from the determinant of the deformation matrix (3.16). One can proof this relation with the procedure outlined in the first example.

3.2 The general form of the deformation matrix.

At this point we might discuss the forms which the deformation matrix D can take for a rubber sheet. Stretching and compressing the sheet is limited here to a linear relation, like the one in formula (3.5), between the initial and final positions of the material points of the sheet. The simplest general form is the deformation matrix (3.10) discussed in example 2 of the previous section. The deformation is there represented by a symmetric matrix with positive eigenvalues a and b. The constants a and b of (3.10) cannot be negative independently. For instance, a deformation described by a matrix similar to (3.10) but with $a > 0$ and $b < 0$ involves a mirror transformation around the x-axis. Such deformation implies that the material points of the rubber sheet have to pass through each other; a process which we judge not to occur for a small deformation. However, both $a < 0$ and $b < 0$ in (3.10) represents a possible deformation. But in that case we can decompose the deformation matrix into a deformation of the type (3.10) followed by a rotation over π radians, *i.e.*:

$$
\begin{aligned}
D &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \\
&= \mathcal{R}(\hat{z}, 180^o)\mathcal{D}_S^+(-a,-b) \quad , \quad a, b < 0,
\end{aligned}
$$

where:

$$
\mathcal{D}_S^+(p,q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \quad , \quad p, q > 0. \tag{3.20}
$$

The matrix $\mathcal{D}_S^+$ is symmetric and has positive eigenvalues.

So, in (3.20) we have represented a deformation matrix D by the product of a symmetric deformation with positive eigenvalues $\mathcal{D}_S^+$ and a rotation $\mathcal{R}$. We will assume in the following that this is the most general form for a small deformation. That is that we assume that any reasonable deformation, also in three dimensions, can always be represented by the following product:

$$
D = \mathcal{R}\mathcal{D}_S^+. \tag{3.21}
$$

In three dimensions, three negative eigenvalues for D represents a deformation which involves space inversion, which is rejected by us as a possible realistic deformation of a solid. Also one negative and two positive eigenvalues for D implies space inversion and is thus rejected for the same reason. Only a deformation matrix with two negative and one positive eigenvalues may represent a realistic deformation as in those cases the deformation can be seen as the result of an allowed deformation which is represented by a matrix with only positive eigenvalues and a rotation.

In the above discussion, we have implicitly understood that the deformation matrix has only real eigenvalues. But that is in fact not the case. For a symmetric matrix, however, one can prove that all its eigenvalues are real. Consequently, only in the case that the rotation angle of $\mathcal{R}$ in formula (3.21) equals zero or π, the deformation matrix has real eigenvalues. So, what about the other cases?

Let us assume for $\mathcal{R}$ in (3.21) an arbitrary rotation angle. The form of the matrix in (3.7) which describes the "deformed" quadratic surface, is in the general case (3.21) given by:

$$
\left(D^{-1}\right)^T D^{-1} = \left[(\mathcal{R}\mathcal{D}_S^+)^{-1}\right]^T (\mathcal{R}\mathcal{D}_S^+)^{-1} = \mathcal{R}\left(\mathcal{D}_S^+\right)^{-2}\mathcal{R}^{-1}. \tag{3.22}
$$

Here we have used the fact that for a rotation holds $\mathcal{R}^T = \mathcal{R}^{-1}$ and for a symmetric matrix $\mathcal{D}_S^T = \mathcal{D}_S$.

Next, let us assume that $\vec{x}$ is an eigenvector of $\mathcal{D}_S^+$ with eigenvalue λ, *i.e.*:

$$
\mathcal{D}_S^+\vec{x} = \lambda\vec{x} \quad , \quad \lambda > 0. \tag{3.23}
$$

We find then that for $\mathcal{R}\vec{x}$, which represents the rotated vector $\vec{x}$ after a rotation given by $\mathcal{R}$, the following holds:

$$
\left(D^{-1}\right)^T D^{-1}\mathcal{R}\vec{x} = \frac{1}{\lambda^2}\mathcal{R}\vec{x}. \tag{3.24}
$$

Apparently represents $\mathcal{R}\vec{x}$ the eigenvector of the matrix which describes the "deformed" quadratic surface. Consequently, $\mathcal{R}\vec{x}$ indicates a principal axis of the quadratic surface (see section 2.5, formula 2.71), whereas moreover two times its eigenvalue (*i.e.* 2λ) represents the length of that principal axis.

When, in the three dimensional case λ_1, λ_2 and λ_3 are the three positive eigenvalues of $\mathcal{D}_S^+$, then the volume of the "deformed" quadratic surface as given by formula (3.7) is

$$\lambda_1\lambda_2\lambda_3 = det(\mathcal{D}_S^+) \tag{3.25}$$

times larger than the volume of the original unit sphere.

3.3 Infinitesimal deformations of the unit circle.

In this section we consider deformations of the unit circle which are small enough to allow first order approximations, and which can be described by deformation matrices of the form:

$$D = 1 + \epsilon,$$

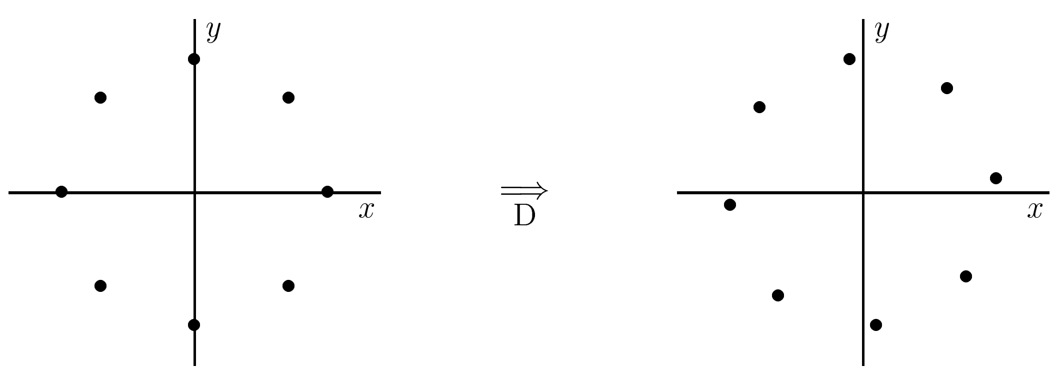
where ϵ represents a matrix for which all matrix elements are much smaller than unity.

1. Infinitesimal rotation.

Let us consider the deformation of the unit circle given by the following deformation matrix:

$$D = \begin{pmatrix} 1 & -\kappa \\ \kappa & 1 \end{pmatrix} \quad , \quad |\kappa| \ll 1. \tag{3.26}$$

Below is graphically represented (for the case $\kappa = 0.1$), the effect of the deformation defined by the above matrix D on various points of the unit circle:



before deformation (•) after deformation (•)

By eye we guess that the effect of D on the unit circle probably only results in a rotation of the unit circle. This is to first order in κ confirmed by calculations, as we will see below. First let us determine the inverse of the matrix D to first order in κ, *i.e.*:

$$D^{-1} = \frac{1}{1+\kappa^2}\begin{pmatrix} 1 & \kappa \\ -\kappa & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & \kappa \\ -\kappa & 1 \end{pmatrix}. \tag{3.27}$$

The above equation implies that the inverse of D is to first order in κ equal to the transposed of D (compare the expressions 3.26 and 3.27) which is the defining property of an orthogonal matrix (see formula 2.37). In fact corresponds the deformation matrix here to a small rotation over an angle which is equal to κ, as can be seen from the following:

$$R(\hat{z},\kappa) = \begin{pmatrix} \cos(\kappa) & -\sin(\kappa) \\ \sin(\kappa) & \cos(\kappa) \end{pmatrix} \approx \begin{pmatrix} 1 & -\kappa \\ \kappa & 1 \end{pmatrix} \quad , \quad \text{for } |\kappa| \ll 1. \tag{3.28}$$

The relation for the "deformed" circle follows by substituting the above result (3.27) for the inverse of D into the expression (3.7) in order to find:

$$1 = \{D\, D^{-1}\}_{kl} X_k X_l = \delta_{kl}\; X_k X_l, \tag{3.29}$$

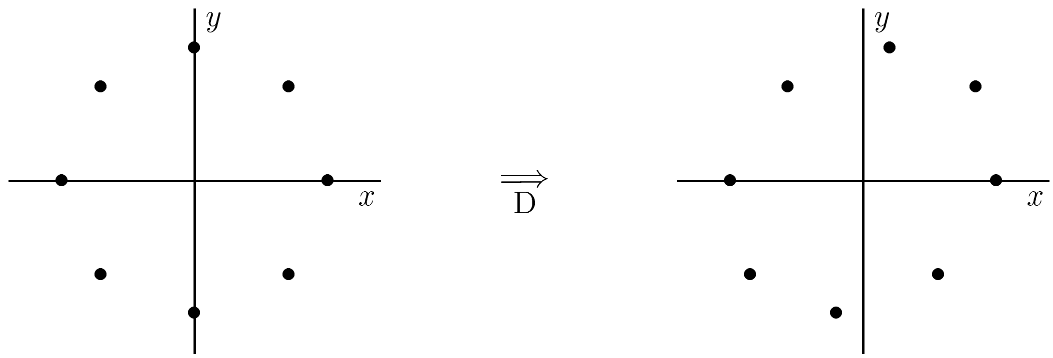
which is again the relation for the unit circle. Consequently, we may conclude that the above deformation D is to first order in κ equal to a rotation of the rubber sheet, which does not lead to any deformation.

2. Simple shear.

An infinitesimal deformation which is known in the literature as *simple shear* consists of the combination of an infinitesimal shear deformation (see example 3.16) and an infinitesimal rotation. The related deformation matrix is given by:

$$D = \begin{pmatrix} 1 & 2\epsilon \\ 0 & 1 \end{pmatrix} \quad , \quad |\epsilon| \ll 1. \tag{3.30}$$

Each point of the sheet suffers a deformation in the x-direction which depends linearly on its y-coordinate. So, points on the x-axis remain in their positions, but points at growing distances from the x-axis suffer larger and larger deformations. When we take ϵ positive, then the effect is in the direction of the positive x-axis for points in the upper half xy-plane and in the opposite direction for points in the lower half plane. Below this is graphically represented (for the case $\epsilon = 0.1$) for various points of the unit circle:



before deformation (•) after deformation (•)

As in the case of shear, the unit circle seems to transform under the deformation defined by the matrix D given in (3.30) into an ellipse which has its principal axes in directions which make angles of about 45^o with the x-axis. But, it is easy to show that in this case the angle is not exactly 45^o, but smaller by an amount ϵ. In order to do so, we first write equation (3.30) as the combination of two infinitesimal deformations, a rotation $R(\hat{z}, -\epsilon)$ and a shear deformation of an amount also given by ϵ. In the resulting matrix multiplication we allow only first order terms in ϵ, *i.e.*:

$$R(\hat{z}, -\epsilon)D(\epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} = \begin{pmatrix} 1+\epsilon^2 & 2\epsilon \\ 0 & 1-\epsilon^2 \end{pmatrix} \approx \begin{pmatrix} 1 & 2\epsilon \\ 0 & 1 \end{pmatrix},$$

or equivalently:

$$RD = \{1 + \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}\}\{1 + \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}\} \approx 1 + \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}. \tag{3.31}$$

The result is to first order in ϵ equal to the deformation matrix (3.30) of this example.

A rotation does not imply any deformation of a material object, but changes the orientation of the principal axes of quadratic surfaces like the deformed circle of this example. Consequently, the only real deformation comes in this example from the last matrix on the righthandside of equation (3.31), which is the symmetric part of the deformation matrix.

The eigenvectors of the symmetric part of the deformation matrix have been discussed previously (see formula 3.18). Here we find that, according to relation (3.24), the eigenvectors are found by performing moreover the above rotation to the eigenvectors of formula (3.18), as follows:

$$R(\hat{z}, -\epsilon)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\epsilon \\ 1-\epsilon \end{pmatrix} = \begin{pmatrix} \cos(45^\circ - \epsilon) \\ \sin(45^\circ - \epsilon) \end{pmatrix}$$

and

$$R(\hat{z}, -\epsilon)\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1+\epsilon \\ 1+\epsilon \end{pmatrix} = \begin{pmatrix} -\sin(45^\circ - \epsilon) \\ \cos(45^\circ - \epsilon) \end{pmatrix}.$$

Notice that the determinant of the symmetric part of expression (3.31) is to first order in ϵ equal to one. Using formula (3.25), this means that to first order approximation no changes in volume are involved in a shear deformation.

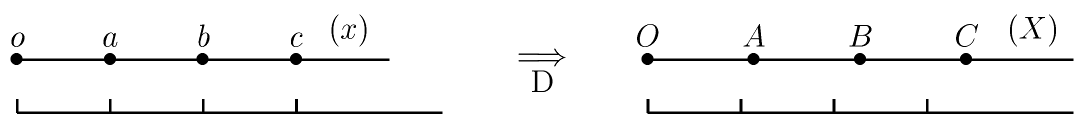
As a conclusion for this section, we might state that any infinitesimal deformation consists in general of three parts (compare equation 3.31), the unity, an antisymmetric matrix which represents a rotation, and a symmetric matrix which represents a real deformation.

3.4 The displacement vector and the strain tensor.

In this section we study small deformations of elastic materials and its consequences for infinitesimal volume elements. As a preparation we first consider here some examples and then we go in the next section into the formal theory.

1. Stretching an elastic string.

In the figure below we show the deformation of an elastic string when it is stretched. We assume that the string is along the x-axis and stretched in the positive direction of the x-axis. Moreover, we have fixed one extremum of the elastic string in the origin.



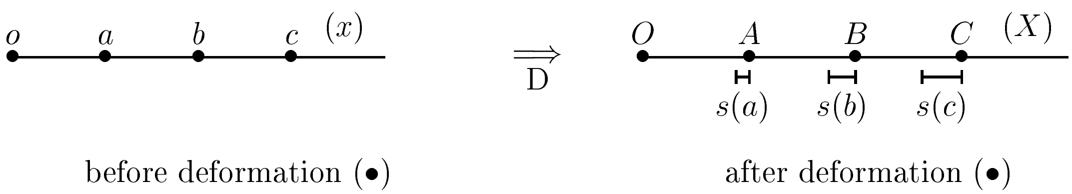
before deformation (•) after deformation (•)

The material points (point particles) of the string which before the deformation are in the positions indicated by $x = o$, a, b and c, are after the deformation found in the respective positions $x = O$, A, B and C. The points indicated by $x = o$ and $x = O$ come here both in the origin of the x-axis, because they represent the fixed extremum of the string. For an ideal elastic string the deformation is linear, which means that the ratio's OA/oa, OB/ob and OC/oc are equal. A deformation which has that property, is called a *homogeneous deformation*.

For a homogeneous deformation we might represent the above ratio by $(1 + \epsilon)$, where ϵ is here supposed to be a small number. Consequently, if the initial position of a material point of the string is indicated by x and its final position after the deformation by $X(x)$, then one has for a homogeneous deformation:

$$X(x) = (1 + \epsilon)x. \tag{3.32}$$

As a consequence, the displacement of a material point depends on its initial position. In the figure below this fact is demonstrated.



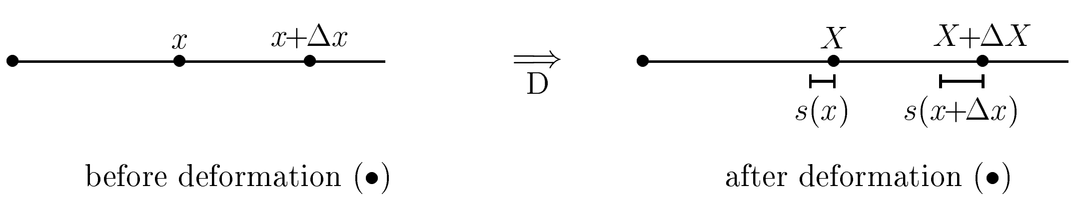
The displacement of the material point which initially is in the position $x = a$ is in the above figure indicated by $s(a)$ and similarly are indicated the displacements of b by $s(b)$ and c by $s(c)$. So, instead of the expression (3.32) for the relation between initial positions x and final positions X, one might also describe the above deformation by the displacement of a material point which has the initial position x, according to:

$$s(x) = X(x) - x. \tag{3.33}$$

For a homogeneous deformation the displacement of material points depends linearly on its position with respect to the fixed extremum of the string. In that case one obtains for the displacement function the expression:

$$s(x) = X(x) - x = (1 + \epsilon)x - x = \epsilon x. \tag{3.34}$$

In the following, we are interrested in the deformation of a small region around a certain point with initial position x. Thereto, we study the effect of a deformation on neighboring points. In the figure below we have two neighboring points which are indicated by x and $x + \Delta x$.



After the deformation the initial distance Δx in between the two neighboring points x and $x + \Delta x$ changes to ΔX which is the distance of the new positions X and $X + \Delta X$ of those points after the deformation. This can be expressed in terms of the displacement function as follows:

$$\begin{aligned}\Delta X &= \{X + \Delta X\} - X \\ &= \{x + \Delta x + s(x + \Delta x)\} - \{x + s(x)\} = \Delta x + s(x + \Delta x) - s(x).\end{aligned} \tag{3.35}$$

The above expression for the new distance of two neighboring points after a small deformation, can be further developed by making a Taylor expansion of $s(x + \Delta x)$ around the point x, *i.e.*:

$$s(x + \Delta x) = s(x) + \left(\left. \frac{ds}{dx} \right|_x \right) \Delta x + \frac{1}{2!} \left(\left. \frac{d^2 s}{dx^2} \right|_x \right) (\Delta x)^2 + \cdots . \tag{3.36}$$

In the specific example of a homogeneous deformation, the second and higher order derivatives in the Taylor series do not contribute. In general those higher order derivatives will of course not disappear.

When we insert the Taylor series expansion of $s(x + \Delta x)$ around the point x in the expression (3.35) for the distance in between two neighboring points after a deformation, then we find for a homogeneous deformation the result:

$$\Delta X = \Delta x + \left(\left. \frac{ds}{dx} \right|_x \right) \Delta x = \Delta x + \epsilon \Delta x. \tag{3.37}$$

The constant ϵ which represents the fractional deformation of any line element of the string in the case of a homogeneous deformation, is called the *strain* of the deformation.

For a strain which is not homogeneous, one can still always define a displacement function $s(x)$, which describes the displacement of an arbitrary material point x, which after the deformation comes at the position $X(x)$, *i.e.*:

$$X(x) = x + s(x). \tag{3.38}$$

The above relation (3.37) remains then valid for the differential dX, which is the limit for $\Delta x \to 0$ of the expression (3.35). However, in the general case the fractional change (*strain*) differs from point to point in the elastic string and thus becomes a function of the initial position x. The resulting function is called the strain function and represented by ϵ, as is shown below:

$$dX = dx + \left(\left. \frac{ds}{dx} \right|_x \right) dx = dx + \epsilon(x) dx. \tag{3.39}$$

For deformations in two and three dimensions, the displacement function becomes a vector, the *displacement vector*, and the strain function a tensor, the *strain tensor*. In the following we will study the examples of shear (see example 4 of section 3.1) and simple shear (see example 2 of section 3.3) in two dimensions.

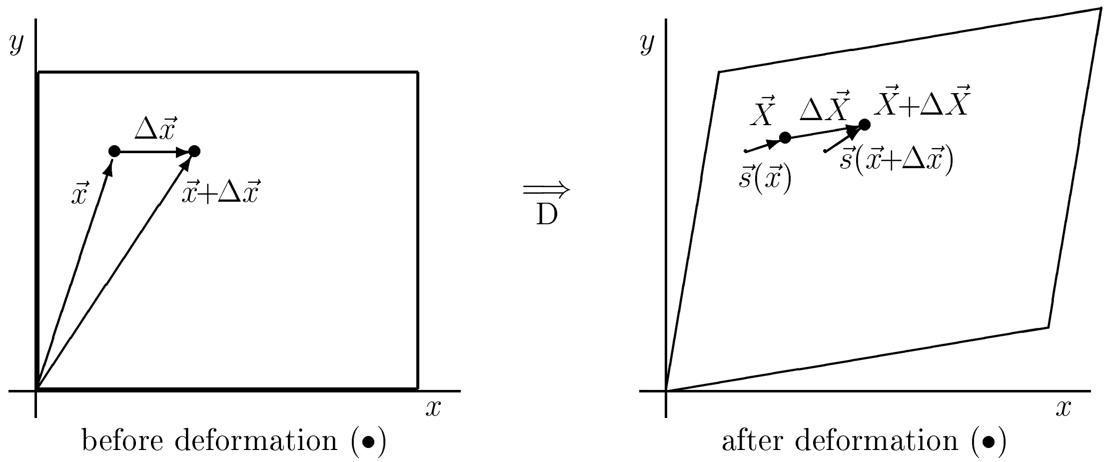
2. Shear in two dimensions.

The displacement vector $\vec{s}(\vec{x})$ for shear in two dimensions is given by (compare example 4 of section 3.1, formula 3.16):

$$\vec{s}(\vec{x}) = (s_1(x_1,x_2),\ s_2(x_1,x_2))$$

$$= (\epsilon x_2,\ \epsilon x_1)\,. \tag{3.40}$$

In the figure below the effect of the above displacement vector is shown.



The relation between the final and initial position of a material point, equivalent to formula (3.38) for the elastic string in one dimension, becomes here a vector equation. For each of the two components we have:

$$X_i(x_1,x_2) = x_i + s_i(x_1,x_2). \tag{3.41}$$

The differential $d\vec{X}$ of $\vec{X}(\vec{x})$ which is important for the study of the deformation of an infinitesimal region around the point $\vec{x}$, has consequently also two components, *i.e.*:

$$dX_1(\vec{x}) = dx_1 + \left(\left.\frac{\partial s_1}{\partial x_1}\right|_{\vec{x}}\right) dx_1 + \left(\left.\frac{\partial s_1}{\partial x_2}\right|_{\vec{x}}\right) dx_2$$

and

$$dX_2(\vec{x}) = dx_2 + \left(\left.\frac{\partial s_2}{\partial x_1}\right|_{\vec{x}}\right) dx_1 + \left(\left.\frac{\partial s_2}{\partial x_2}\right|_{\vec{x}}\right) dx_2.$$

These equations can in general be compactified as follows:

$$dX_i(\vec{x}) = dx_i + \left(\left.\frac{\partial s_i}{\partial x_j}\right|_{\vec{x}}\right) dx_j. \tag{3.42}$$

In the case of the above displacement vector (3.40) one obtains for the differential of $\vec{X}$ the expression:

$$\begin{pmatrix} dX_1 \\ \\ dX_2 \end{pmatrix} = \begin{pmatrix} dx_1 + \epsilon dx_2 \\ \\ dx_2 + \epsilon dx_1 \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ \\ dx_2 \end{pmatrix} . \tag{3.43}$$

The matrix at the righthandside of this matrix equation for the deformation of infinitesimal distances around the point $\vec{x}$, might be recognized as the deformation matrix (3.16) which was studied in example 4 of section (3.1). Consequently, we know that small circles around any point of the xy-plane are deformed into small ellipses, the principal axes of which are in directions which make angles of 45^o with the x-axis.

The above equation (3.43) can also be written in the following form:

$$\begin{pmatrix} dX_1 \\ \\ dX_2 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ \\ \epsilon & 0 \end{pmatrix} \right\} \begin{pmatrix} dx_1 \\ \\ dx_2 \end{pmatrix} . \tag{3.44}$$

In this form one might compare the equation with the relation (3.39) in order to find that the second matrix on the righthandside of the above equation corresponds to the function $\epsilon(x)$ in the one dimensional case. Indeed, the strain tensor is in the case of shear given by:

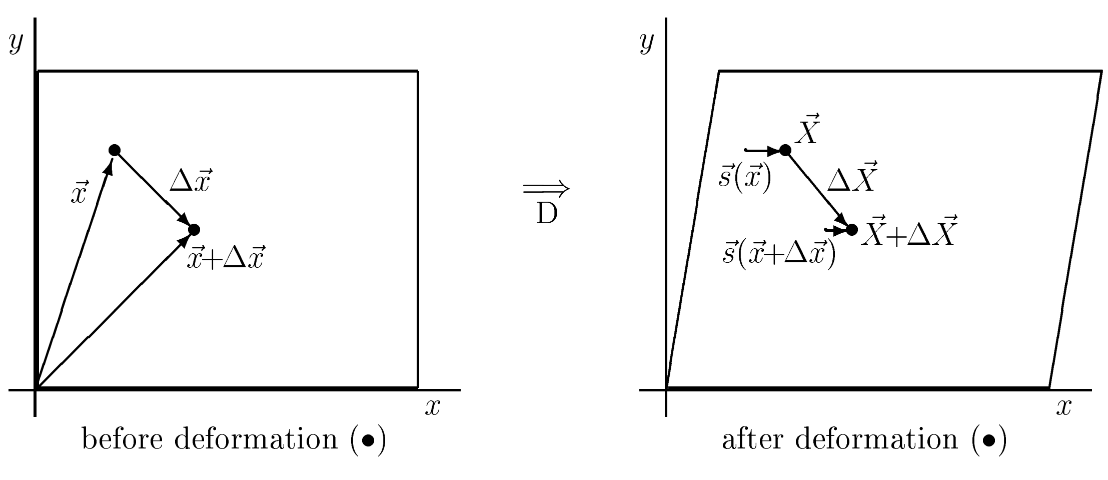
$$\epsilon(\vec{x}) = \begin{pmatrix} 0 & \epsilon \\ \\ \epsilon & 0 \end{pmatrix} . \tag{3.45}$$

3. Simple shear in two dimensions.

The displacement vector $\vec{s}(\vec{x})$ for simple shear in two dimensions is given by (compare example 2 of section 3.3, formula 3.30):

$$\vec{s}(\vec{x}) = (2\epsilon x_2,\ 0) . \tag{3.46}$$

In the figure below the effect of the above displacement vector is shown.



The strain tensor follows from an equation similar to the one in the previous case. Using expression (3.42) we find:

$$\begin{pmatrix} dX_1 \\ \\ dX_2 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2\epsilon \\ \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} dx_1 \\ \\ dx_2 \end{pmatrix}. \tag{3.47}$$

But now we remember that from the study we made in section (3.3) it followed that in this case there is an infinitesimal rotation involved and that in fact one should split the matrix as follows:

$$\begin{pmatrix} 1 & 2\epsilon \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} + \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}. \tag{3.48}$$

The unit matrix together with the antisymmetric tensor define the infinitesimal rotation. The strain tensor defines the deformation of a small region of the xy-plane. Consequently, this tensor relates to the deformation of distances, or equivalently of quadratic surfaces. From section (3.1) we know that only the third, *symmetric*, tensor is responsible for the deformation of quadratic surfaces and so, is the strain tensor for this case.

3.5 Deformations in three dimensions.

After all the preparation of the previous pages, the formal theory of deformation is now straightforward. Let us denote, as before, the position of a material point before the deformation by $\vec{x}$, and after deformation by $\vec{X}(\vec{x})$. And let the displacement vector field which defines the deformation be given by $\vec{s}(\vec{x})$. Then, is the basic relation for the displacement of material points in a body due to deformation, given by:

$$\vec{X}(\vec{x}) = \vec{x} + \vec{s}(\vec{x}). \tag{3.49}$$

For the study of the behavior of small domains in the body under the a deformation, it for us is sufficient to consider infinitesimal deformations. We thereto determine the differential of $\vec{X}$ which follows from the equation (3.49) and which is an expression similar to the one we obtained in the two-dimensional case in equation (3.42), the only difference being that the summation over the indices runs here from 1 to 3, *i.e.*:

$$dX_i(\vec{x}) = dx_i + \left(\frac{\partial s_i}{\partial x_j}\bigg|_{\vec{x}} \right) dx_j. \tag{3.50}$$

The partial derivatives of the components of the displacement vector field form a tensor as has been shown in section (2.4) in the formulas (2.48), (2.49) and (2.50). The above formula can be rewritten as follows:

$$dX_i(\vec{x}) = dx_i + \frac{1}{2}\left\{ \left(\frac{\partial s_i}{\partial x_j}\bigg|_{\vec{x}} \right) - \left(\frac{\partial s_j}{\partial x_i}\bigg|_{\vec{x}} \right) \right\} dx_j \; + \; \frac{1}{2}\left\{ \left(\frac{\partial s_i}{\partial x_j}\bigg|_{\vec{x}} \right) + \left(\frac{\partial s_j}{\partial x_i}\bigg|_{\vec{x}} \right) \right\} dx_j.$$

This way one obtains two tensors: the antisymmetric tensor field $r_{ij}(\vec{x})$ given by:

$$r_{ij}(\vec{x}) = \frac{1}{2}\{\left(\left.\frac{\partial s_i}{\partial x_j}\right|_{\vec{x}}\right) - \left(\left.\frac{\partial s_j}{\partial x_i}\right|_{\vec{x}}\right)\} \tag{3.51}$$

and the symmetric strain tensor field $\epsilon_{ij}(\vec{x})$ defined by:

$$\epsilon_{ij}(\vec{x}) = \frac{1}{2}\{\left(\left.\frac{\partial s_i}{\partial x_j}\right|_{\vec{x}}\right) + \left(\left.\frac{\partial s_j}{\partial x_i}\right|_{\vec{x}}\right)\} \tag{3.52}$$

Using the definitions (2.19) of the generators of the rotation group, the relation (2.24) for the product of Levi-Civita tensors and the expression (2.51) for the components of the curl of a vector field, the antisymmetric tensor (3.51) can be rewritten according to:

$$\begin{aligned} r_{ij}(\vec{x}) &= \frac{1}{2}\{\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il}\}\left(\left.\frac{\partial s_k}{\partial x_l}\right|_{\vec{x}}\right) = \frac{1}{2}\epsilon_{mij}\epsilon_{mkl}\left(\left.\frac{\partial s_k}{\partial x_l}\right|_{\vec{x}}\right) \\ &= \frac{1}{2}(-A_m)_{ij}\left(-\nabla\times\vec{s}\,|_{\vec{x}}\right)_m = \frac{1}{2}\left\{\left(\nabla\times\vec{s}\,|_{\vec{x}}\right)\cdot\vec{A}\right\}_{ij} \end{aligned}$$

Inserting the above into equation (3.50) one obtains the following expression for the differential of $\vec{X}$:

$$dX_i(\vec{x}) = \left\{\delta_{ij} + \frac{1}{2}\left\{\left(\nabla\times\vec{s}\,|_{\vec{x}}\right)\cdot\vec{A}\right\}_{ij} + \epsilon_{ij}(\vec{x})\right\}\ dx_j\ ,$$

Or in terms of the matrices:

$$d\vec{X}(\vec{x}) = \{\mathbf{1} + \frac{1}{2}\left(\nabla\times\vec{s}\,|_{\vec{x}}\right)\cdot\vec{\boldsymbol{A}} + \boldsymbol{\epsilon}(\vec{x})\}d\vec{x}. \tag{3.53}$$

Provided that the components of the vector $curl(\vec{s})$ are small, one may approximate:

$$\mathbf{1} + \frac{1}{2}\left(\nabla\times\vec{s}\,|_{\vec{x}}\right)\cdot\vec{\boldsymbol{A}} \approx \exp\{\frac{1}{2}\left(\nabla\times\vec{s}\,|_{\vec{x}}\right)\cdot\vec{\boldsymbol{A}}\},$$

which represents a rotation around the vector $curl(\vec{s})$, with a rotation angle indicated by half the absolute value of that vector (see formula 2.30). Let us introduce for this rotation the notation:

$$\mathcal{R}(\vec{s},\vec{x}) = \exp\{\frac{1}{2}\left(\nabla\times\vec{s}\,|_{\vec{x}}\right)\cdot\vec{\boldsymbol{A}}\}. \tag{3.54}$$

Now, also provided that the matrix elements of the strain tensor $\boldsymbol{\epsilon}$ are small, one might approximate the matrix in equation (3.53) by:

$$\mathbf{1} + \frac{1}{2}\left(\nabla\times\vec{s}\,|_{\vec{x}}\right)\cdot\vec{\boldsymbol{A}} + \boldsymbol{\epsilon}(\vec{x}) \quad\approx\quad \mathcal{R}(\vec{s},\vec{x})\{\mathbf{1} + \boldsymbol{\epsilon}(\vec{x})\}, \tag{3.55}$$

which represents the product of a rotation field $\mathcal{R}$ and a symmetric deformation field (see also formula 3.21) which describes how volumes (e.g. spheres) deform (into e.g. ellipsoides).

For infinitesimal deformations, *i.e.* $\vec{s}(\vec{x}) \to 0$, the above approximations are of course exact. But in experiment one deals with finite deformations. Whether or not the above approximations hold in a realistic situation, depends on the experimental accuracy one wants to obtain. In the ultimate case one has to return to the Taylor expansions, like the one in (3.36) and take into account higher order terms.

3.6 The dilatation.

In the case of a small deformation for which the approximation (3.55) holds, the deformation of a small sphere centered in the point $\vec{x}$ of a solid, is described by the strain tensor according to:

$$D(\vec{x}) = \mathbf{1} + \boldsymbol{\epsilon}(\vec{x}). \tag{3.56}$$

Following the procedure described in the previous sections (see for example formula 3.9), the volume $\Delta\sigma(\vec{x})$ of a domain around the point $\vec{x}$ changes under the deformation which is represented by the above deformation matrix (3.56), to a volume $\Delta\Sigma(\vec{X}(\vec{x}))$, given by:

$$\Delta\Sigma(\vec{X}(\vec{x})) = det\{D(\vec{x})\}\Delta\sigma(\vec{x}). \tag{3.57}$$

When the matrix elements of the strain tensor are small, then a first order approximation of this expression might be usefull. In such case, one can easily verify that the only terms which are linear in the matrix elements of the strain tensor in the determinant $D(\vec{x})$ (3.56) come from the diagonal elements of D, *i.e.*

$$\begin{aligned} det\{D(\vec{x})\} &\approx \{1+\epsilon_{11}(\vec{x})\}\{1+\epsilon_{22}(\vec{x})\}\{1+\epsilon_{33}(\vec{x})\} \\ &\approx 1+\epsilon_{11}(\vec{x})+\epsilon_{22}(\vec{x})+\epsilon_{33}(\vec{x}), \end{aligned}$$

or equivalently:

$$det\{D(\vec{x})\} \approx 1 + Tr\{\boldsymbol{\epsilon}(\vec{x})\}. \tag{3.58}$$

The latter quantity is moreover invariant under rotations of the coordinate system, because the strain tensor transforms as a tensor under rotations (see formulas 3.52 and 2.49) and consequently (see formula 2.58), its trace is an invariant under rotations. This is not so much surprising, since the the deformation matrix $D(\vec{x})$ also is a tensor under rotations and thus its determinant is an invariant quantity, but for a first order approximation this property might have been lost.

The expression (3.58) can be further simplified using the definition (3.52) of the strain tensor, to obtain:

$$det\{D(\vec{x})\} \approx 1 + \left(\left.\frac{\partial s_i}{\partial x_i}\right|_{\vec{x}} \right) = 1 + \left(\nabla \cdot \vec{s}|_{\vec{x}} \right). \tag{3.59}$$

The fractional change in volume $\Delta(\vec{x})$ around a point $\vec{x}$ in a solid is called the *dilatation* of a deformation and can now for small deformations, using the above expressions (3.57) and (3.59), be given by:

$$\Delta(\vec{x}) = \frac{\Delta\Sigma(\vec{X}(\vec{x})) - \Delta\sigma(\vec{x})}{\Delta\sigma(\vec{x})} \approx \left(\nabla \cdot \vec{s}|_{\vec{x}}\right). \tag{3.60}$$

For homogenuous isotropic deformations, the strain tensor is constant and proportional to the unit matrix. In that case the above formula for the fractional change in volume has a very simple consequence for a similar quantity for the change in length. By comparing the first line in formula (3.58) with the above equation (3.60), one finds that the fractional change in length equals one third of the fractional change in volume for isotropic deformations to first order in the strain tensor diagonal elements. If the deformations are moreover homogenuous, then this number is a constant for the whole solid. In practice this is used for isotropic deformations of materials under pressure or temperature changes: One just has to measure the fractional volume change of such materials in order to find the change in any direction, or vice versa if that experimentally is more easy.

3.7 Thermal expansion coefficients.

A piece of material deforms when we excert pressure on it. The amount of deformation is described here by the strain tensor. Consequently, the strain tensor of a given body is a function of pressure p. Also temperature influences the volume of a body. Many materials expand for increasing temperature. So, the strain tensor of a body is also a function of temperature T. The measurable quantities for such materials, to be studied in this section, are the so-called *thermal expansion coefficients*, which are defined by:

$$\alpha_{ij} = \left(\frac{\partial\epsilon_{ij}(p,T)}{\partial T}\right)_p . \tag{3.61}$$

The suffix p indicates here that the measurement must be done under constant pressure or tension. This is a necessary condition for the experimental determination of the thermal expansion coefficients, but not of much relevance for the subject of this section.

Like the matrix $\boldsymbol{\epsilon}$, the matrix $\boldsymbol{\alpha}$ is a symmetric matrix which behaves as a tensor under rotations. Moreover, from the definition (3.61) it might be clear that any symmetry of $\boldsymbol{\epsilon}$ is also a symmetry of $\boldsymbol{\alpha}$. Such symmetries exist for crystaline structures as we will discuss below.

1. Monoclinic structures.

In a monoclinic material exists a crystal axis for which the atomic arrangement has the same appearance after a rotation of 180^o around that axis. Such crystal axis is called a *twofold axis*. It implies for such materials that $\boldsymbol{\alpha}$ is invariant under a rotation of 180^o around the twofold symmetry crystal axis. When we take this axis in the z-direction, then a rotation of 180^o around that axis is represented by the following matrix:

$$R(\hat{z},180^o) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.62}$$

From formula (2.50) in section (2.4) we know that under the above rotation of the coordinate system, the tensor $\boldsymbol{\alpha}$ on one hand transforms as follows:

$$\boldsymbol{\alpha}' = R\boldsymbol{\alpha}R^T = \begin{pmatrix} \alpha_{11} & \alpha_{12} & -\alpha_{13} \\ \alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{3.63}$$

However, since on the other hand the thermal expansion coefficients should in this case be invariant under the transformation (3.62), (*i.e.* the above matrix (3.63) must be equal to the matrix $\boldsymbol{\alpha}$), it follows that:

$$\alpha_{13} = \alpha_{31} = 0 \quad \text{and} \quad \alpha_{23} = \alpha_{32} = 0.$$

Consequently, the matrix $\boldsymbol{\alpha}$ for monoclinic materials has the following form:

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix},$$

which implies that there exist only four non-vanishing thermal expansion coefficients for monoclinic materials, *i.e.* α_{11}, $\alpha_{12} = \alpha_{21}$, α_{22} and α_{33}.

2. Ortho-rhombic structures.

In a material which has an ortho-rhombic structure, exist three mutually perpendicular twofold axes. When we take these axes along the x-, the y- and the z-directions, then the rotations for which $\boldsymbol{\alpha}$ is invariant are respectively $R(\hat{x},180^o)$, $R(\hat{y},180^o)$ and $R(\hat{z},180^o)$. One finds in this case for transformations similar to equation (3.63):

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & \alpha_{23} \\ -\alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_{11} & -\alpha_{12} & \alpha_{13} \\ -\alpha_{21} & \alpha_{22} & -\alpha_{23} \\ \alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_{11} & \alpha_{12} & -\alpha_{13} \\ \alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix}.$$

From these equations one deduces that:

$$\alpha_{12} = \alpha_{21} = 0 \ , \quad \alpha_{13} = \alpha_{31} = 0 \ , \quad \alpha_{23} = \alpha_{32} = 0.$$

This results for ortho-rhombic materials in a matrix for the thermal expansion coefficients of the form:

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix} .$$

Only three non-vanishing coefficients.

3. Cubic structures.

For a material with a cubic lattice all rotations which transform the cube into itself leave the appearance of the atomic structure of that material invariant. When we take the axes of the unit lattice cube along the x-, the y- and the z-directions, then the rotations for which $\boldsymbol{\alpha}$ is invariant are rotations of 90^o, 180^o and -90^o around those axes. Such crystal axes are called *fourfold axes.*

The matrix $\boldsymbol{\alpha}$ of the thermal expansion coefficients for such materials is proportional to the unit matrix, *i.e.* $\boldsymbol{\alpha} = \alpha_{11}\,\mathbf{1}$. There is only one independent coefficient in this case.

Problem 21:
Verify the above affirmation for cubic structures.

---

4. Hexagonal structures.

A material which has a hexagonal crystal structure, contains a sixfold crystal axis. The invariance rotations around that axis have rotation angles of 60^o, 120^o, 180^o, 240^o and 300^o. When this axis is in the z-direction, the matrix $\boldsymbol{\alpha}$ takes the form:

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix} .$$

There are only two different coefficients.

Problem 22:
Perform the various symmetry rotations for hexagonal structures on the matrix $\boldsymbol{\alpha}$ as given in (3.61), in order to verify the above indicated form for $\boldsymbol{\alpha}$.

Notice, that in experiment the measured matrix of thermal expansion coefficients might come out very different. This is because the x-, y- and z-directions of our laboratory do not necessarily coincide with the symmetry axes of the crystal.

Let us assume, for example, that the symmetry axis in the case of the hexagonal structure, is in our laboratory in the direction (ϑ, φ), then this means that there is a rotation of coordinate systems involved (see section 2.3), given by:

$$R(\vartheta, \varphi) = R(\hat{z}, \varphi) R(\hat{y}, \vartheta). \tag{3.64}$$

So, the matrix of thermal expansion coefficients which we would measure in our laboratory equals:

$$\begin{aligned} \boldsymbol{\alpha}' &= R(\vartheta, \varphi) \boldsymbol{\alpha} R^T(\vartheta, \varphi) \\ &= \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix} + \end{aligned}$$

$$+(\alpha_{33}-\alpha_{11}) \begin{pmatrix} \sin^2(\vartheta)\cos^2(\varphi) & \sin^2(\vartheta)\cos(\varphi)\sin(\varphi) & \cos(\vartheta)\sin(\vartheta)\cos(\varphi) \\ \sin^2(\vartheta)\cos(\varphi)\sin(\varphi) & \sin^2(\vartheta)\sin^2(\varphi) & \cos(\vartheta)\sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta)\sin(\vartheta)\cos(\varphi) & \cos(\vartheta)\sin(\vartheta)\sin(\varphi) & -\sin^2(\vartheta) \end{pmatrix}.$$

Besides being symmetric, the above tensor $\boldsymbol{\alpha}'$ shows six different matrix elements. Nevertheless, if we know that the crystal has a hexagonal structure, then we can use the invariants of a tensor under rotation (see section 2.4) in order to determine α_{11} and α_{33}, *i.e.*:

$$det(\boldsymbol{\alpha}') = det(\boldsymbol{\alpha}) = (\alpha_{11})^2 \alpha_{33} \quad \text{and} \quad Tr(\boldsymbol{\alpha}') = Tr(\boldsymbol{\alpha}) = 2\alpha_{11} + \alpha_{33}.$$

Chapter 4

Forces due to stress.

A solid under pressure or stress develops internal reaction forces due to the attractive and repulsive forces amongst its atoms and molecules. At interatomic distances such forces might fluctuate enormously from place to place. However, at distances which are still microscopic, but large with respect to the size of an atom, the average forces due to stress are here assumed to vary smoothly from one place to the other in a solid.

Let us no longer refer to the atomic structure of a solid, but instead consider a small domain $\mathcal{A}$ inside an elastic body. At its surface, the domain $\mathcal{A}$ is in contact with neighbouring domains. Consequently, when on the elastic body forces are excerted which give rise to deformation, like pressure, stretching or frictional forces, then the domain $\mathcal{A}$ is subject to stress forces through its contact at its surface with the surrounding domains. The basic assumption is here that the resulting force field which describes these forces at the surface of the domain $\mathcal{A}$, varies smoothly as a function of position.

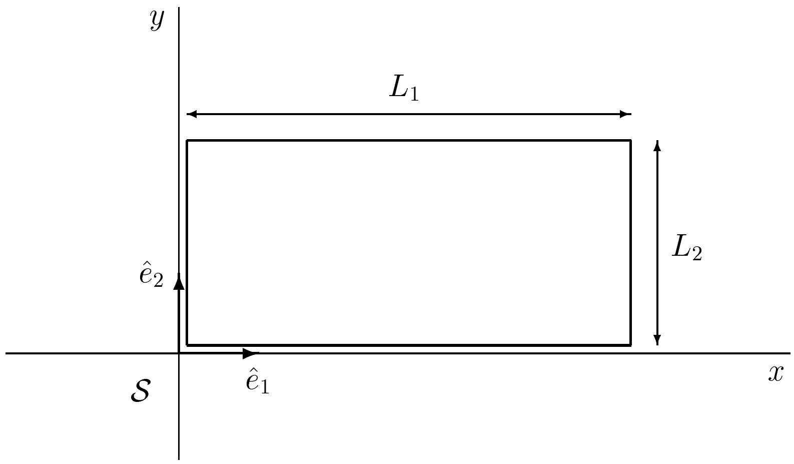
Next to forces due to stress, a solid might be subject to external forces, like gravitational forces, or even internal forces which have a different origin then the applied stress. Such forces will be called *body forces*. Their behavior is assumed to be essentially different from the stress forces. Whereas the stress force fields represent the contact forces between the material points in a solid due to tension and consequently depend in strength and direction on the size and the orientation of the contact surface, the body force field represents the effects of a distant field source (like the Earth in the case of gravitation) and has therefore a definite direction in each point of the solid. We assume moreover here, that body forces, like the gravitational force, are proportional to the density of the solid.

4.1 Hooke's law.

When we apply pressure at the surface of a solid, then its dimensions change. For small deformations, the changes in the sizes of an elastic body, like the increase or decrease of its length, are proportional to the applied force. This observation has been formulated for the first time in the seventeenth century by R.Hooke.

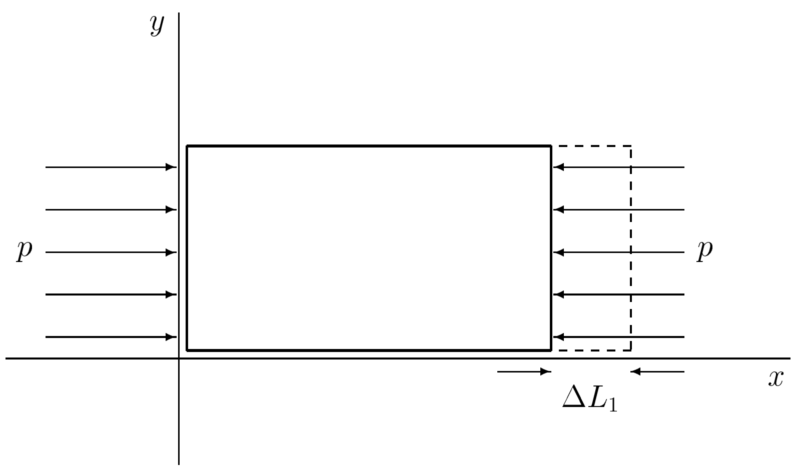
In the previous chapter we have studied the description of deformation in terms of the strain tensor, the matrix elements of which have been related to the fractional changes in dimensions of a solid. It is therefore logic that a generalization of Hooke's law involves the matrix elements of the strain tensor. Here we will follow an intuitive approach in order to understand in which way Hooke's law might be generalized. In the next sections we will follow a more rigorous method for the same purpose.

Let us consider a bar of some elastic material, which has the form of a rectangular parallelopiped. And let us take its edges along the axes of our coordinate system (see figure below):



Let us for convenience only consider two dimensions and a coordinate system $\mathcal{S}$ as indicated in the figure above. The lengths of the sides of the bar are L_1 in the x-direction and L_2 in the y-direction.

When we apply pressure at the two opposite sides of the elastic bar, then its length in the direction of the applied force diminuishes. This situation is shown in the figure below:



A pressure of magnitude p is applied at the left- and righthand surface of the bar, because of which its length in the x-direction changes by the amount ΔL_1. Hooke's observation for the fractional change in length can here be formulated as follows:

$$\frac{\Delta L_1}{L_1} = -ap, \tag{4.1}$$

where the constant of proportionality a has to be determined experimentally.

The compressed bar contains elastic energy. This can be determined following the standard procedure: An infinitesimal change $d(\Delta L_1)$ in ΔL_1 contributes to the total work W an amount given by:

$$dW = \text{force} \times d(\Delta L_1).$$

The strength of the force is here equal to the area L_2 times the pressure p, and using Hooke's relation (4.1), one finds:

$$dW = \frac{L_2}{aL_1}(\Delta L_1)\, d(\Delta L_1).$$

Integration of this expression leads to the well-known relation of the elastic potential energy and the change in size of an elastic body in one dimension, *i.e.*:

$$W = \frac{1}{2}\left(\frac{L_2}{aL_1}\right)(\Delta L_1)^2. \tag{4.2}$$

The constant of proportionality (L_2/aL_1) is called the elasticity constant of the bar. The constant $(1/a)$ represents an elasticity constant of the material of which the bar has been made. For a constant, the result (4.2) is quite reasonable: A long elastic string can more easily be stretched than a small string. So, the work might well be proportional to the inverse of the length L_1. Also, a thicker elastic string is less easily stretched than a thin string. So, the work might perfectly be proportional to the thickness L_2.

The forces at the left- and righthand side of the bar are opposite and equal in strength. They are moreover perpendicular to the surface of the bar and directed towards its interior. We represent these forces by a vector $\vec{t}_1$, which is called the *tension* of the applied force. The strength of $\vec{t}_1$ is equal to the applied force per unit area. In the above case we have thus:

$$\mid \vec{t}_1 \mid = p. \tag{4.3}$$

The direction of $\vec{t}_1$ is chosen in the negative x-direction, because the force is directed towards the interior of the bar. Consequently:

$$\vec{t}_1 = -p\hat{e}_1. \tag{4.4}$$

The index 1 of $\vec{t}_1$ comes from the face where the force acts: The face which is perpendicular to the x-direction has the index 1 in this context.

The tension related to pressure is always perpendicular to the surface where it is acting. But, in general the tension $\vec{t}_1$ might not be perpendicular to the left- and righthand faces. This is possible because the surface of a solid body allows frictions, and frictional forces are parallel to the surface. In two dimensions, the tension $\vec{t}_1$ may in general have two components, *i.e.*:

$$\vec{t}_1 = t_{11}\hat{e}_1 + t_{12}\hat{e}_2. \tag{4.5}$$

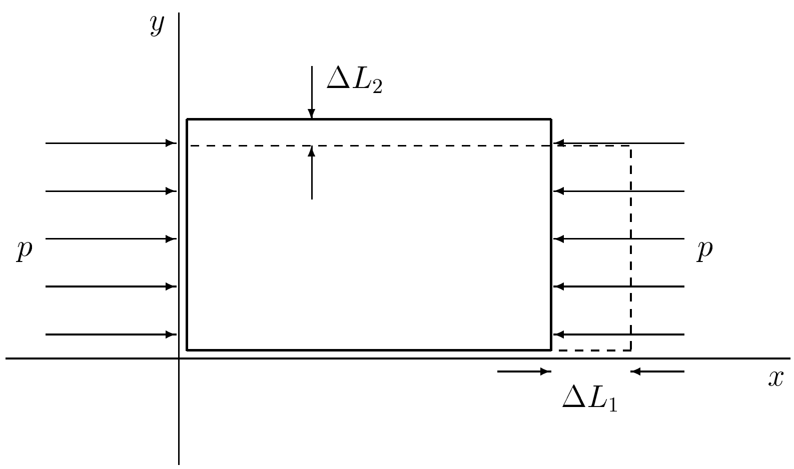
For the above example (4.4) the two components of $\vec{t}_1$ are given by:

$$t_{11} = -p \text{ and } t_{12} = 0.$$

Inserting this in formula (4.1) one obtains:

$$\frac{\Delta L_1}{L_1} = a t_{11}. \tag{4.6}$$

In example (4.1) it is suggested that the dimensions of the bar change only in the x-direction, when a pressure is excerted on its left- and righthand faces. However, more generally, one might consider the situation that the bar does not only suffer a deformation in the x-direction, but also in the y-direction when a force is excerted on the left- and rigthhand extremes of the bar. This is depicted below:



In this case Hooke's law takes the form:

$$\frac{\Delta L_1}{L_1} = -ap \text{ and } \frac{\Delta L_2}{L_2} = bp,$$

where both proportionality constants a and b depend on the material of which the bar has been made. Or, using the components of the tension defined in (4.5), one equivalently writes:

$$\frac{\Delta L_1}{L_1} = a t_{11} \text{ and } \frac{\Delta L_2}{L_2} = -b t_{11}. \tag{4.7}$$

The deformation matrix might in the above example have the form:

$$D = \begin{pmatrix} 1 + \frac{\Delta L_1}{L_1} & 0 \\ 0 & 1 + \frac{\Delta L_2}{L_2} \end{pmatrix}.$$

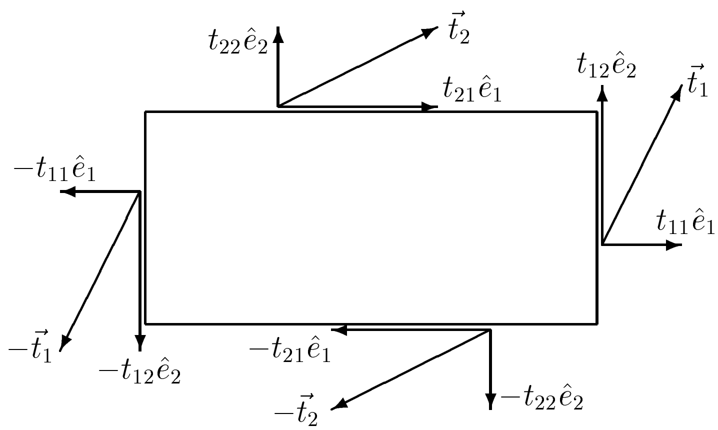
So, for small deformations, the strain tensor, which has been defined in (3.52), takes in this example the following form:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \frac{\Delta L_1}{L_1} & 0 \\ & \\ 0 & \frac{\Delta L_2}{L_2} \end{pmatrix}. \tag{4.8}$$

So we may identify the diagonal matrix elements of the strain tensor and the fractional changes in size of the elastic bar. This way, we obtain for relation (4.7) the expression:

$$\epsilon_{11} = at_{11} \text{ and } \epsilon_{22} = -bt_{11}. \tag{4.9}$$

Next, let us consider the most general case of pressure and frictional forces at the four faces of the two-dimensional bar, *i.e.*:



The first observation one might make from the figure is that the forces are in equilibrium, but for the moments of the forces this is not obvious. Consequently, in a situation of static equilibrium there must exist a condition which limits the freedom of all possible choices for the two forces. The moments of the two sets of opposite forces are given by:

$$\vec{M}_1 = L_1 F_{12}\hat{z} \text{ and } \vec{M}_2 = -L_2 F_{21}\hat{z}. \tag{4.10}$$

In (4.10) the forces F_{12} and F_{21} are related to the components t_{12} and t_{21} of respectively $\vec{t}_1$ and $\vec{t}_2$ via the following expression (remember that $\vec{t}$ represents the tension force per unit area):

$$F_{12} = L_2 t_{12} \text{ and } F_{21} = L_1 t_{21}.$$

Substituting this into (4.10) we find that the moments are proportional to the "volume" $(L_1 L_2)$ of the bar and that the related components of the tension vectors represent something like the moment per unit volume, *i.e.*:

$$\vec{M}_1 = L_1 L_2 t_{12}\hat{z} \text{ and } \vec{M}_2 = -L_1 L_2 t_{21}\hat{z}. \tag{4.11}$$

As a consequence we notice that for static equilibrium we must impose on the components of the tension vectors the condition that:

$$t_{12} = t_{21}. \tag{4.12}$$

In fact one might join the components of the tension vectors into a symmetric matrix, the stress tensor, given by:

$$\boldsymbol{t} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \quad , \quad t_{12} = t_{21}. \tag{4.13}$$

In general, this matrix describes all possible forces on the elastic bar in two dimensions. Its content for a specific situation must therefore be related to the matrix which describes the effects of those forces in terms of the deformation of infinitesimal domains of the bar. Small deformations can be represented by a symmetric strain tensor of the form:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} \quad , \quad \epsilon_{12} = \epsilon_{21}. \tag{4.14}$$

According to Hooke (4.1) for small deformations are the fractional changes in size of a solid body linearly proportional to the tension forces. Both, the tension forces and the deformations are here represented by matrices. So we expect that Hooke's law (4.9) can be generalized to the matrix elements of those matrices, and be formulated as follows:

$$t_{ij} = c_{ijkl}\epsilon_{kl}. \tag{4.15}$$

The sixteen material *elasticity constants* c_{ijkl} (for $i, j, k, l = 1, 2$) are not independent. The symmetry of as well the stress tensor (4.13) as the strain tensor (4.14) gives the following identities:

$$c_{jikl} = c_{ijkl} \text{ and } c_{ijlk} = c_{ijkl}. \tag{4.16}$$

These symmetry relations leave us with only nine independent elasticity constants in two dimensions.

A further reduction of the number of independent coefficients c_{ijkl} can be obtained from the following considerations: In (4.2) we found an expression for the elastic potential energy of the bar in the most simple case. From this formula and also using the relations (4.6) and (4.9) we can infer that:

$$\frac{\partial W}{\partial \epsilon_{11}} = L_1 \frac{\partial W}{\partial(\Delta L_1)} = \frac{L_2 L_1}{a}\frac{\Delta L_1}{L_1} = \frac{L_1 L_2}{a}\epsilon_{11} = L_1 L_2 t_{11}.$$

Here, $L_1 L_2$ represents the "volume" of the bar, so if we concentrate on the elastic potential energy per unit volume $\omega = W/V$, then we find for the above expression:

$$\frac{\partial \omega}{\partial \epsilon_{11}} = t_{11}.$$

In the following sections we will see that this can be generalized according to:

$$\frac{\partial \omega}{\partial \epsilon_{ij}} = t_{ij}. \tag{4.17}$$

An extra derivative with respect to a matrix element of the strain tensor gives then, also using the the generalized Hooke's law as shown in (4.15), the following symmetry relations for the elasticity coefficients:

$$c_{klij} = \frac{\partial t_{kl}}{\partial \epsilon_{ij}} = \frac{\partial^2 \omega}{\partial \epsilon_{ij} \partial \epsilon_{kl}} = \frac{\partial^2 \omega}{\partial \epsilon_{kl} \partial \epsilon_{ij}} = \frac{\partial t_{ij}}{\partial \epsilon_{kl}} = c_{ijkl} \tag{4.18}$$

These symmetry relations reduce the amount of independent elasticity constants further from nine to six. So, concludingly we may remark that the reaction of a two-dimensional solid to tension can for small deformations be completely described by six independent constants:

2 constants which describe the reaction of the bar in the x- and y-directions due to an applied force in the x-direction on the right and lefthand faces,

2 similar constants for an applied force in the y-direction on the upper and lower faces of the bar,

1 constant for shear due to a frictional force in the x-direction at the upper and lower faces of the bar,

1 similar constant for frictional forces in the left- and righthand faces of the bar.

For isotropic materials, the number of independent elasticity constants reduces further to two.

4.2 Stress forces and the stress tensor

Let us represent the internal forces due to stress in a solid by a second rank tensor field, *the stress tensor*, given by:

$$t_{ij}(\vec{x}) \quad , \quad i,j = 1,2,3. \tag{4.19}$$

In order to study its relation to the forces due to stress, we consider a volume element ΔV inside the solid. We furthermore consider a small element of its surface $\Delta A(\vec{x})$ located in the point $\vec{x}$ of the surface of the volume. This surface element should be small enough for the stress tensor (4.19) to be considered constant in all of its points. Such surface element is represented by a vector $\Delta \vec{A}(\vec{x})$, which has the following properties:

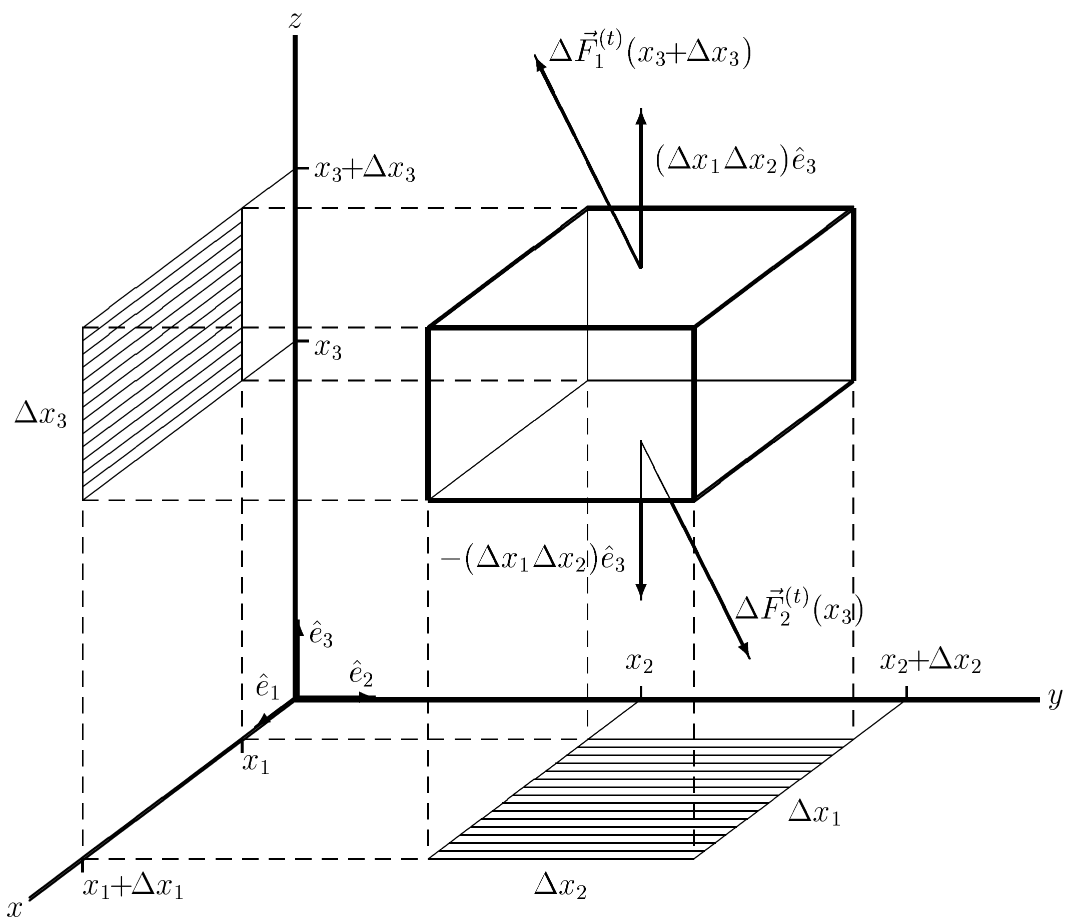
1. $\Delta \vec{A}(\vec{x})$ is perpendicular to the surface element $\Delta A(\vec{x})$,
2. $\mid \Delta \vec{A}(\vec{x}) \mid$ equals the area of the surface element $\Delta A(\vec{x})$, and
3. $\Delta \vec{A}(\vec{x})$ points outward with respect to the volume ΔV.

(4.20)

The force $\Delta\vec{F}^{(t)}$, which acts on the surface element $\Delta A(\vec{x})$ due to the stress forces inside the solid, is defined by:

$$\left(\Delta\vec{F}^{(t)}\right)_i = \left(\Delta\vec{A}(\vec{x})\right)_j t_{ji}(\vec{x}). \tag{4.21}$$

Let us first inspect whether the above definition makes any sense. For this purpose we might choose the stress tensor field to be constant in the domain around a certain the volume element of the solid. We select for a volume element the parallelopiped which is indicated in the figure below.



Its lower left corner in the back is situated at the point (x_1, x_2, x_3) and its three neighbour corners in the x-, y- and z-direction are respectively situated at $(x_1 + \Delta x_1, x_2, x_3)$, $(x_1, x_2 + \Delta x_2, x_3)$ and $(x_1, x_2, x_3 + \Delta x_3)$. The sides of the parallelopiped in those three directions have thus sizes Δx_1, Δx_2 and Δx_3 respectively.

Let us consider the force due to stress which acts at the top surface of the above volume element. In the figure this force is indicated by $\Delta\vec{F}_1^{(t)}(x_3{+}\Delta x_3)$. The top surface, as also indicated in this figure, is represented by a vector, given by:

$$\Delta\vec{A}(\text{top surface}) = (\Delta x_1\ \Delta x_2)\hat{e}_3. \tag{4.22}$$

So, from the definition (4.21) we obtain

$$\begin{aligned}\left(\Delta\vec{F}_1^{(t)}\right)_i &= (\Delta x_1 \Delta x_2 \hat{e}_3)_j t_{ji} \\ &= \Delta x_1 \Delta x_2 \delta_{3j} t_{ji} = \Delta x_1 \Delta x_2 t_{3i},\end{aligned}$$

or in vector notation:

$$\Delta\vec{F}_1^{(t)} = \Delta x_1 \Delta x_2 (t_{31}, t_{32}, t_{33}). \tag{4.23}$$

When we compare this result with forces acting on a surface element in a perfect fluid, then we observe one main difference: In a perfect fluid the force is always perpendicular to the surface element, whereas here the force due to stress on a surface element in a solid is not. The reason is that in solids and viscuous fluids there exist frictional forces on an interior volume due to stress parallel to the surface of the surface element, which forces are absent in a perfect fluid.

For the bottom surface of the volume element, the associated vector which represents that surface element, is pointing in the $-\hat{e}_3$ direction. For the rest everything is the same. So, we obtain for the force at the bottom surface, indicated by $\Delta\vec{F}_2^{(t)}(x_3)$ in the above figure, the following expression:

$$\Delta\vec{F}_2^{(t)} = -\Delta x_1 \Delta x_2 (t_{31}, t_{32}, t_{33}). \tag{4.24}$$

This is just opposite to the expression (4.23) for $\Delta\vec{F}_1^{(t)}$ and so the two forces cancel each other in the case of a constant stress tensor field.

Similarly, at the right- and lefthand surfaces of the volume element of the above figure, one obtains for the forces due to stress the expressions:

$$\pm\Delta x_1 \Delta x_3 (t_{21}, t_{22}, t_{23}). \tag{4.25}$$

And at the front- and backside surfaces act forces due to stress, given by:

$$\pm\Delta x_2 \Delta x_3 (t_{11}, t_{12}, t_{13}). \tag{4.26}$$

Three different expressions (4.23 and 4.24), (4.25) and (4.26) for the three principal directions in a solid, seems quite sufficient to describe the internal forces due to stress in a solid. Moreover, are the six forces at the various surfaces in perfect equilibrium for a constant stress tensor field, which means that the net force at the volume element vanishes.

When the stress tensor field is not constant, then the forces at opposite faces do not exactly cancel each other. A residual force due to stress remains. In order to calculate this force for an arbitrary stress tensor field, one must integrate the expression (4.21) over the whole surface of the volume element. Let us perform this integration for an arbitrary volume inside a solid under stress. The surface integral for the sum of all forces acting due to stress on all surface elements of the volume, becomes then:

$$\left[\vec{F}^{(t)}(\text{volume})\right]_i = \int_{\text{surface}} \left(d\vec{A}(\vec{x})\right)_j t_{ji}(\vec{x}). \tag{4.27}$$

Using the divergence theorem, this integration can be converted to a volume integration over the whole volume, *i.e.*:

$$\left[\vec{F}^{(t)}(\text{volume})\right]_i = \int_{\text{volume}} dV(\vec{x})\{\left(\frac{\partial t_{1i}}{\partial x_1}\bigg|_{\vec{x}}\right) + \left(\frac{\partial t_{2i}}{\partial x_2}\bigg|_{\vec{x}}\right) + \left(\frac{\partial t_{3i}}{\partial x_3}\bigg|_{\vec{x}}\right)\},$$

or in a more compact notation, using the Einstein summation convention for the terms in the integrand, one obtains:

$$\left[\vec{F}^{(t)}(\text{volume})\right]_i = \int_{\text{volume}} dV(\vec{x}) \left(\frac{\partial t_{ji}}{\partial x_j}\bigg|_{\vec{x}}\right). \tag{4.28}$$

This expression shows that in the case of a constant stress tensor field one finds also for an arbitrary volume in a solid that the total force due to stress vanishes, because then the integrand is zero everywhere.

In general represents the expression in the integrand of the above formula (4.28), the total force due to stress at an infinitesimal volume element in the solid per unit of volume.

Another important quantity related to rotations due to stress, is the moment (*i.e.* $M^{(t)}(\vec{x}) = \vec{x} \times F^{(t)}(\vec{x})$) which the stress forces excert on a volume element of a solid. For this we return to the surface of an arbitrary volume element and consider the contribution $\Delta M^{(t)}$ to the total moment of one surface element $\Delta A(\vec{x})$ with respect to the origin of the coordinate system. Using the component expression (2.51) for the *curl* of a vector and formula (4.21) for the definition of the force due to stress at the above surface element, we find:

$$\Delta\vec{M}^{(t)}(\vec{x}) = \vec{x} \times \Delta\vec{F}^{(t)}(\vec{x}),$$

or in components:

$$\begin{aligned}
\left[\Delta\vec{M}^{(t)}(\vec{x})\right]_i &= \epsilon_{ijk} x_j \left(\Delta\vec{F}^{(t)}(\vec{x})\right)_k \\
&= \epsilon_{ijk} x_j \left(\Delta\vec{A}(\vec{x})\right)_l t_{lk}(\vec{x}).
\end{aligned}$$

The total moment due to stress at the volume element is then obtained by integrating the above expression over the whole volume, *i.e.*:

$$\left[\vec{M}^{(t)}(\text{volume})\right]_i = \int_{\text{surface}} \left(d\vec{A}(\vec{x})\right)_l t_{lk}(\vec{x})\epsilon_{ijk} x_j.$$

Again using the divergence theorem, one finds the following volume integration:

$$\begin{aligned}
\left[\vec{M}^{(t)}(\text{volume})\right]_i &= \int_{\text{volume}} dV(\vec{x}) \left[\frac{\partial}{\partial x_l}\left(\epsilon_{ijk} x_j t_{lk}\right)\Big|_{\vec{x}}\right] \\
&= \int_{\text{volume}} dV(\vec{x})\epsilon_{ijk}\left\{\delta_{lj} t_{lk}(\vec{x}) + x_j \left(\frac{\partial t_{lk}}{\partial x_l}\bigg|_{\vec{x}}\right)\right\}.
\end{aligned}$$

After performing the summation over the first index of the Kronecker delta, which is the result of partial differentiating x_j with respect to x_l, we finally end up with:

$$\left[\vec{M}^{(t)}(\text{volume})\right]_i = \int_{\text{volume}} dV(\vec{x})\epsilon_{ijk}\left\{t_{jk}(\vec{x}) + x_j\left(\left.\frac{\partial t_{lk}}{\partial x_l}\right|_{\vec{x}}\right)\right\}. \tag{4.29}$$

The integrand of the above expression represents the i-th component of the moment due to stress which acts on an infinitesimal volume element of the solid per unit of volume, with respect to the origin of the coordinate system.

4.3 Equilibrium conditions.

When a solid is hold under stress in a fixed position, all volume elements must be in equilibrium, which implies that the forces acting at each volume element of the solid must be in equilibrium. Also when a deformation takes place in a quasi-static process, which means that at each stage of the process the system is in a perfect equilibrium, the sum of the internal forces must vanish. In the previous section we have determined the total force due to stress at an arbitrary volume element of the solid (see formula 4.28). However, next to the contact forces due to tension there might be body forces acting on the material points of the solid. These forces are supposed to be volumetric, which means that they can be represented by a force $\vec{f}^{(b)}(\vec{x})$ per unit of volume, and proportional to the density $\rho(\vec{x})$ of the medium. So, at a volume element of the solid acts the force $\vec{F}^{(b)}$ due to body forces which is given by:

$$\left[F^{(b)}(\text{volume})\right]_i = \int_{\text{volume}} dV(\vec{x})\rho(\vec{x})f_i^{(b)}(\vec{x}). \tag{4.30}$$

The total force on a volume element of the solid is then the sum of the contribution of the stress forces given in (4.28) and the above contribution (4.30) of the body forces. Internal equilibrium requires then the following:

$$\left[F^{(t)}(\text{volume})\right]_i + \left[F^{(b)}(\text{volume})\right]_i = 0,$$

or equivalently:

$$\int_{\text{volume}} dV(\vec{x})\left\{\left(\left.\frac{\partial t_{ji}}{\partial x_j}\right|_{\vec{x}}\right) + \rho(\vec{x})f_i^{(b)}(\vec{x})\right\} = 0.$$

This must be valid for an arbitrary volume element of the solid, hence the integrand must vanish, *i.e.*:

$$\left(\left.\frac{\partial t_{ji}}{\partial x_j}\right|_{\vec{x}}\right) + \rho(\vec{x})f_i^{(b)}(\vec{x}) = 0. \tag{4.31}$$

Also the internal moments must be in equilibrium in the situation that the deformation does not change in time.

The contribution of the body forces on a volume element of a deformed solid with respect to the origin of the coordinate system, is given by:

$$\left[\vec{M}^{(b)}(\text{volume})\right]_i = \int_{\text{volume}} dV(\vec{x})\epsilon_{ijk}x_j\rho(\vec{x})f_k^{(b)}(\vec{x}). \tag{4.32}$$

Equilibrium of the moments at a volume element of the solid requires then that the sum of this contribution and the contribution of the stress forces (4.29) vanish, also for an arbitrary volume element. This leads for the sum of the integrands:

$$\epsilon_{ijk}\left[t_{jk}(\vec{x}) + x_j\left\{\left(\left.\frac{\partial t_{lk}}{\partial x_l}\right|_{\vec{x}}\right) + \rho(\vec{x})f_k^{(b)}(\vec{x})\right\}\right] = 0.$$

Using also the equilibrium condition (4.31) for the internal forces, we find that the second term on the lefthand side vanishes, so we are left with the condition:

$$\epsilon_{ijk}t_{jk} = 0.$$

This is satisfied for a symmetric stress tensor (see formula 2.28), *i.e.*:

$$t_{ji}(\vec{x}) = t_{ij}(\vec{x}). \tag{4.33}$$

Consequently, static equilibrium requires that the stress tensor is symmetric. In the section on Hooke's law we obtained a similar result in two dimensions for a homogeneous deformation (4.12), the above result is a generalization in three dimensions for non-homogeneous deformations.

4.4 The work done by deformation.

It is well-known that the system of a stretched elastic string contains elastic potential energy. This energy has been stored into the system in the process of deformation. Below we will study the variation δW of the total stored energy W in the deformed solid, caused by a variation of the deformation. A variation of the deformation can be represented by a variation $\delta\vec{s}(\vec{x})$ of the displacement vector field, which has been introduced in section (3.4). We assume here for convenience that the solid has already suffered a certain amount of deformation, such that a given volume element of the solid is subject to stress forces, represented by a non-zero stress tensor $\boldsymbol{t}(\vec{x})$.

For a small (infinitesimal) variation $\delta\vec{s}(\vec{x})$ of the displacement vector field, we assume that the stress forces at the volume element remain constant. The total force at a volume element is given by the sum of the forces excerted at all surface elements due to the contact forces. Consequently, the amount of work, δW done by the stress forces on the volume element by a variation $\delta\vec{s}(\vec{x})$ of the displacement vector field, is equal to the sum of the work done at each of the surface elements by the stress forces, *i.e.*:

$$\delta W^{(t)}(\text{volume}) = \int_{\text{surface}} \left(d\vec{A}(\vec{x})\right)_j t_{ji}(\vec{x})\delta s_i(\vec{x}).$$

Using the divergence theorem, this converts into:

$$\delta W^{(t)}(\text{volume}) = \int_{\text{volume}} dV(\vec{x}) \left\{ \frac{\partial}{\partial x_j} \left(t_{ji} \delta s_i \right)\Big|_{\vec{x}} \right\}$$

$$= \int_{\text{volume}} dV(\vec{x}) \left\{ \left(\frac{\partial t_{ji}}{\partial x_j}\Big|_{\vec{x}} \right) \delta s_i(\vec{x}) + t_{ji}(\vec{x}) \left(\frac{\partial \delta s_i}{\partial x_j}\Big|_{\vec{x}} \right) \right\}.$$

The first term represents the work done by the stress forces on displacing an infinitesimal volume element $dV(\vec{x})$ over a distance $\delta\vec{s}(\vec{x})$. However, here we should also include the contribution to the total work of the body forces, to obtain:

$$\delta W(\text{volume}) = \delta W^{(t)}(\text{volume}) + \delta W^{(b)}(\text{volume})$$

or equivalently:

$$= \int_{\text{volume}} dV(\vec{x}) \left\{ \left[\left(\frac{\partial t_{ji}}{\partial x_j}\Big|_{\vec{x}} \right) + \rho(\vec{x}) f_i^{(b)}(\vec{x}) \right] \delta s_i(\vec{x}) + t_{ji}(\vec{x}) \left(\frac{\partial \delta s_i}{\partial x_j}\Big|_{\vec{x}} \right) \right\}.$$

Now, if one assumes that for the variation in the displacement vector field the system is in equilibrium (*i.e.* a deformation for which the system at each instant of time is in equilibrium, or a quasi-static process), then the first term vanishes according to (4.31). For the second term we use the property that for reasonably smooth functions the derivative of the variation of that function equals the variation of the derivative, to obtain:

$$\delta W(\text{volume}) = \int_{\text{volume}} dV(\vec{x}) t_{ji}(\vec{x}) \delta \left(\frac{\partial s_i}{\partial x_j}\Big|_{\vec{x}} \right).$$

Using furthermore the definition of the strain tensor (3.52) in terms of the derivatives of the displacement vector field, and the fact that the stress tensor is symmetric, then one finally finds the wanted expression:

$$\delta W(\text{volume}) = \int_{\text{volume}} dV(\vec{x}) t_{ij}(\vec{x}) \delta\epsilon_{ij}(\vec{x}). \tag{4.34}$$

The variation of the energy stored in the solid comes from the changes in the strain tensor, which indeed describes the the variation of the deformation in terms of fractional changes in size. The integrand in (4.34) can be interpreted as the variation in work, $\delta\omega(\vec{x})$, per unit of volume caused by a variation of the strain tensor:

$$\delta\omega(\vec{x}) = t_{ij}(\vec{x}) \delta\epsilon_{ij}(\vec{x}). \tag{4.35}$$

This is the relation we have anticipated on in (4.17).

4.5 The elasticity constants.

As discussed in the section (4.1) in general one may expect the following generalization of Hooke's law (see also formula 4.15 for the two-dimensional case):

$$t_{ij} = c_{ijkl}\epsilon_{kl}. \tag{4.36}$$

When we insert this linear relation between the tension and the strain into the expression for the variation of the work done by the stress forces due to a variation in the strain tensor, then we find:

$$\delta\omega(\vec{x}) = c_{ijkl}\epsilon_{kl}\delta\epsilon_{ij}. \tag{4.37}$$

On integrating this equation, keeping in mind that the work stored in the solid due to deformation must be zero when there is no deformation, one obtains an expression quadratic in the matrix elements of the strain tensor for the elastic potential energy ω per unit of volume of the solid. This means that ω as a function of the strain matrix elements contains terms like:

$$\frac{1}{2}c_{1111}\left(\epsilon_{11}\right)^2 \quad \text{and} \quad c_{1112}\epsilon_{11}\epsilon_{12}.$$

This agrees with what we expected to find in comparison to the simple relation (4.2) for the one-dimensional harmonic oscillator.

The generalized Hooke's law (4.36) suggests that there are $3^4 = 81$ elasticity constants c_{ijkl} (for $i, j, k, l = 1, 2, 3$). There are however several symmetry relations for those material constants, equivalent to the relations (4.16) and (4.18) but now for three dimensions. Those relations reduce the number of independent elasticity constants for an arbitrary material to only 21. This number can also be understood as follows:

$3 \times 3 = 9$	At each of the three faces of an elementary rectangular parallelopiped one can excert pressure. The strain tensor contains in each of the three cases the information on the deformation of the small spheres around any material point in the medium into small elipsoids. This information can be given by means of the lengths of the three principal axes.
$3 \times 2 \times 2 = 12$	One can apply frictional forces at each of the faces of the parallelopiped. There are two independent directions at each face. For shear forces we assumed that to first order the volume of the deformed spheres are the same as the volumes of the original spheres. In that case is the strain tensor traceless (see formula 3.45), and can be characterized by two independent constants.

The inverse of the generalized Hooke's law (4.36) allows to determine the deformation of a body as a function of the excerted forces on the body, *i.e.*:

$$\epsilon_{ij}(\vec{x}) = \sigma_{ijkl}t_{kl}(\vec{x}). \tag{4.38}$$

The proportionality constants σ_{ijkl} are called the *elasticity compliance constants.*

Chapter 5

Elasticity constants and symmetries.

According to the generalized Hooke's law (4.36) there exists a linear relation between the six independent matrix elements of the stress tensor $\boldsymbol{t}$ and the six independent matrix elements of the strain tensor $\boldsymbol{\epsilon}$. This would in general amount to 36 proportionality constants. But, we have seen in section (4.5) that because of the symmetry relations (4.16) and (4.18) only 21 are independent. However, when matter has certain symmetries, like cristaline structure, then the number of independent elasticity coefficients is less than 21.

5.1 The elasticity constants of isotropic media.

An approximation which holds quite reasonable for most materials, is the assumption that the internal structure of the substance out of which the solid is made, is isotropic. In the context of the theory of deformations, isotropy means that the relation between the deformation and the induced stress tensor takes the same form in any orthogonal coordinate system. Or equivalently, that the elasticity constants are *invariant* under rotations.

If we want to know, either in experiment or in theory, whether or not a certain elasticity constant vanishes, then we might consider the related deformation and determine the matrix elements of the stress tensor caused by that deformation (or vice-versa). Here and in the following we are interested in the elasticity constants for isotropic matter. Thereto, we start by studying the set of coefficients given by the relation:

$$t_{ij} = c_{ij11}\epsilon_{11}, \tag{5.1}$$

which represents the response of the medium to a deformation in the x-direction and which can be obtained from the generalized Hooke's relation (4.36) by putting all ϵ_{ij} equal to zero, but ϵ_{11}. So, we are dealing with a deformation represented by the following strain tensor:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

The above strain tensor (5.2) is invariant under rotations around the x-axis, *i.e.*:

$$R(\hat{x},\alpha)\boldsymbol{\epsilon}\{R(\hat{x},\alpha)\}^T = \boldsymbol{\epsilon}. \tag{5.3}$$

Problem 23:

Show that the strain tensor (5.2) is invariant under rotations around the x-axis, $R(\hat{x},\alpha)$, defined in formula (2.14).

Consequently, since in an isotropic medium one coordinate system is as good as any other, the stress tensor $\boldsymbol{t}$ must satisfy the same symmetry (5.3).

Let us consider a general symmetric stress tensor $\boldsymbol{t}$, given by the matrix:

$$\boldsymbol{t} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}. \tag{5.4}$$

Now, since the above matrix must be invariant under any of the transformations given by (5.3), we might first study some special cases and then check whether or not the resulting conditions on the matrix elements of (5.4) are sufficient to satisfy (5.3). Under a rotation over 180° around the x-axis, using the definition in (2.14) for the rotation matrix $R(\hat{x},\alpha=180°)$, the above stress tensor $\boldsymbol{t}$ turns into:

$$R(\hat{x},180°)\boldsymbol{t}\{R(\hat{x},180°)\}^T = \begin{pmatrix} t_{11} & -t_{12} & -t_{13} \\ -t_{12} & t_{22} & t_{23} \\ -t_{13} & t_{23} & t_{33} \end{pmatrix}. \tag{5.5}$$

In comparing the above result (5.5) with the matrix shown in (5.4), we find that the stress tensor is invariant under the above transformation when its matrix elements satisfy:

$$t_{12} = t_{13} = 0.$$

These conditions are not enough to make $\boldsymbol{t}$ invariant under all rotations around the x-axis. For example a rotation over 90° around the x-axis gives the result, already including the fact that four off-diagonal matrix elements must vanish:

$$R(\hat{x},90°)\boldsymbol{t}\{R(\hat{x},90°)\}^T = \begin{pmatrix} t_{11} & 0 & 0 \\ 0 & t_{33} & -t_{23} \\ 0 & -t_{23} & t_{22} \end{pmatrix}. \tag{5.6}$$

So, we find for the two basis transformations (5.5) and (5.6) that we must impose on $\boldsymbol{t}$ the following conditions in order to be invariant under those transformations:

$$t_{12} = t_{13} = t_{23} = 0 \ \text{ and } \ t_{22} = t_{33}.$$

It is an easy task to verify that the resulting matrix:

$$\boldsymbol{t} = \begin{pmatrix} t_{11} & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{22} \end{pmatrix}, \qquad (5.7)$$

is invariant under all rotations around the x-axis, *i.e.*:

$$R(\hat{x}, \alpha)\boldsymbol{t}\{R(\hat{x}, \alpha)\}^T = \boldsymbol{t}.$$

Problem 24:

Show that the above stress tensor (5.7) is invariant under rotations around the x-axis, $R(\hat{x}, \alpha)$, defined in formula (2.14).

So, for the deformation (5.2) which is fully determined by the 11-element of the strain tensor, we find that isotropic media will respond with a stress tensor of the form (5.7). We may then conclude that the off-diagonal matrix elements of the stress tensor t_{12}, t_{13} and t_{23} do not depend on the matrix element ϵ_{11} of the strain tensor, and consequently that the corresponding elasticity constants vanish for isotropic media. We may also conclude that the diagonal matrix elements t_{22} and t_{33} respond in the same way to a deformation given by ϵ_{11}, which implies that the related elasticity constants must be the same for isotropic media. Moreover, is the response of the matrix element t_{11} independent of the response of the other two diagonal matrix elements, which means that the corresponding elasticity constant may in general differ from the other two elasticity constants. In the table below we have collected the above observations on the elasticity constants.

c_{ij11}	$i = 1$	2	3
$j = 1$	a	0	0
2	0	b	0
3	0	0	b

Table 5.1: The elasticity constants for a deformation in the x-direction in an isotropic medium.

The constants a and b in table (5.1) are to be determined experimentally. The theory only predicts that in the case of isotropy no more than two different exist.

What about the other elasticity constants?

Let us next study the effect on the stress tensor of a deformation given by the following strain tensor:

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.8}$$

which corresponds to studying the elasticity constants defined by the relation:

$$t_{ij} = c_{ij22}\epsilon_{22}. \tag{5.9}$$

The above strain tensor (5.8) describes a deformation in the y-direction. But, it might be obvious that the response of an isotropic medium does not depend on any direction. So, one could predict without performing any calculations, which stress tensor is related to the deformation defined in (5.8).

However, formally, one might observe that a strain tensor of the form (5.8) can be obtained from the strain tensor given in (5.2) by a rotation over 90° around the z-axis, *i.e.*:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} = R(\hat{z}, 90^\circ) \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \{R(\hat{z}, 90^\circ)\}^T. \tag{5.10}$$

Now, since any coordinate system is equivalent for isotropic matter, we may just use the above basis transformation in order to study the case (5.8). For that purpose we also transform the stress tensor (5.7) to the new coordinate system. Using furthermore the results given in table (5.1), one finds:

$$R(\hat{z}, 90^\circ) \begin{pmatrix} a\epsilon_{11} & 0 & 0 \\ 0 & b\epsilon_{11} & 0 \\ 0 & 0 & b\epsilon_{11} \end{pmatrix} \{R(\hat{z}, 90^\circ)\}^T = \begin{pmatrix} b\epsilon_{11} & 0 & 0 \\ 0 & a\epsilon_{11} & 0 \\ 0 & 0 & b\epsilon_{11} \end{pmatrix}. \tag{5.11}$$

So, we deduce from (5.10) and (5.11) the following elasticity constants in the new coordinate system, which because of isotropy are also the elasticity constants in the original coordinate system, as are shown in the table below:

c_{ij22}	$i = 1$	2	3
$j = 1$	b	0	0
2	0	a	0
3	0	0	b

Table 5.2: The elasticity constants for a deformation in the y-direction in an isotropic medium.

At this point it might be perfectly clear to the reader how to proceed for the third direction. We will now concentrate on the off-diagonal matrix elements of the strain tensor and the stress tensor(s) induced by them, or equivalently the elasticity constants defined by for example:

$$t_{ij} = c_{ij12}\epsilon_{12}. \tag{5.12}$$

However, before attacking this problem, we first study the deformation defined by the following strain tensor:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & -\epsilon_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.13}$$

Since the relation between the stress tensor and the two diagonal matrix elements of the strain tensor (5.13) are established in the foregoing and collected in the tables (5.1) and (5.2), we can without difficulties construct the induced stress tensor, to obtain:

$$\boldsymbol{t} = \begin{pmatrix} (a-b)\epsilon_{11} & 0 & 0 \\ 0 & (b-a)\epsilon_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.14}$$

In itself the above deformation does not give any new information. But, by introducing at this stage a new coordinate system which is obtained from the old one by a rotation over 45° around the z-axis, we find for the rotated strain tensor (5.13) the following:

$$R(\hat{z},45^\circ)\boldsymbol{\epsilon}\{R(\hat{z},45^\circ)\}^T = \begin{pmatrix} 0 & \epsilon_{11} & 0 \\ \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

This represents exactly the type of deformation of our interest and related to the elasticity constants defined in (5.12). The stress tensor which is induced by this deformation, is consequently obtained by applying the same transformation as in (5.15) to the stress tensor (5.14), resulting in:

$$R(\hat{z},45^\circ)\boldsymbol{t}\{R(\hat{z},45^\circ)\}^T = \begin{pmatrix} 0 & (a-b)\epsilon_{11} & 0 \\ (a-b)\epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.16}$$

Again using the argument that in an isotropic medium one orthogonal basis is as good as any other, we find the elasticity constants collected in the table below:

c_{ij12}	$i=1$	2	3
$j=1$	0	$a-b$	0
2	$a-b$	0	0
3	0	0	0

Table 5.3: The elasticity constants for a shear deformation in the xy-plane in an isotropic medium.

The remaining undetermined elasticity constants follow either by similar arguments as in the above discussed cases, or by using the symmetry relations (4.16) and (4.18), from the above tables (5.1), (5.2) and (5.3).

We obtain the result that for isotropic media there are only two independent elasticity constants,

$$\text{for instance} \quad : \quad c_{1111} \ \text{ and } \ c_{1122}. \tag{5.17}$$

All the others can be expressed in terms of those two. In particular, using the tables (5.1), (5.2) and (5.3), one has the following identity:

$$c_{1212} = c_{1111} - c_{1122}. \tag{5.18}$$

Summarizing, one might express the above results in the following way:

$$\begin{aligned}
t_{11} &= c_{1111}\epsilon_{11} + c_{1122}\epsilon_{22} + c_{1122}\epsilon_{33}, \\
t_{22} &= c_{1122}\epsilon_{11} + c_{1111}\epsilon_{22} + c_{1122}\epsilon_{33}, \\
t_{33} &= c_{1122}\epsilon_{11} + c_{1122}\epsilon_{22} + c_{1111}\epsilon_{33}, \\
t_{23} &= (c_{1111} - c_{1122})\epsilon_{23}, \\
t_{13} &= (c_{1111} - c_{1122})\epsilon_{13}, \quad \text{and} \\
t_{12} &= (c_{1111} - c_{1122})\epsilon_{12}.
\end{aligned}$$

These relations are the generalization of (4.1) for isotropic matter. They can be written in a more compact form as follows:

$$\begin{aligned}
t_{ij} &= (c_{1111} - c_{1122})\epsilon_{ij} + c_{1122}(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})\delta_{ij} \\
&= (c_{1111} - c_{1122})\epsilon_{ij} + c_{1122}Tr(\boldsymbol{\epsilon})\delta_{ij}.
\end{aligned} \tag{5.19}$$

The constants defined by:

$$\lambda = c_{1122}, \quad \text{and} \quad 2\mu = c_{1111} - c_{1122}, \tag{5.20}$$

are known as the Lame elasticity constants. In terms of these constants we may express the generalized Hooke's relations for isotropic media (5.19) by:

$$t_{ij} = 2\mu\epsilon_{ij} + \lambda Tr(\boldsymbol{\epsilon})\delta_{ij}. \tag{5.21}$$

In experiment, one has to verify whether or not a substance may be considered isotropic. In that case one defines a constant as follows:

$$\gamma = \frac{c_{1212}}{c_{111} - c_{1122}}. \tag{5.22}$$

It measures the degree of anisotropy and is therefore called the *anisotropic factor*. Using the identity (5.18) one observes that for perfect isotropic media this factor is equal to one.

5.2 Elasticity compliance constants for isotropic media.

In formula (4.38) we have defined the elasticity compliance constants for the deformation of a body as a function of the excerted forces on that body, which is the inverse relation of the generalized Hooke's law (4.36). The inverse of the relations (5.21) for isotropic media can easily be determined. One might for instance write these equations as a 6×6 matrix relation, as follows:

$$\begin{pmatrix} t_{11} \\ t_{22} \\ t_{33} \\ t_{23} \\ t_{13} \\ t_{12} \end{pmatrix} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{pmatrix} .$$

Then in order to find the elsticity compliance constants, one has to determine the inverse of a 6×6 matrix, which because of its simple form reduces to calculating the inverse of two 3×3 matrices, one of which is proportional to the unit matrix. The resulting relations can compactly be denoted by:

$$\epsilon_{ij} = \frac{1}{2\mu(2\mu+3\lambda)}\{(2\mu+3\lambda)t_{ij} - \lambda Tr(\boldsymbol{t})\delta_{ij}\}. \tag{5.23}$$

From this expression one might extract the elasticity compliance constants σ_{ijkl} as defined in formula (4.38), for isotropic media.

Problem 25:

(i) Show, by means of substitution, that formula (5.23) is the inverse of formula (5.21).

(ii) Determine the elasticity compliance constants σ_{1111}, σ_{1122} and σ_{1212} for isotropic media in terms of the Lame constants λ and μ.

The elasticity compliance constants play an important role in engineering. Knowledge of these constants make it possible to determine the changes in dimension of the various components of a construction due to pressure and temperature changes.

Chapter 6

Fluids.

The differences between fluids and solids are well known and do not have to be discussed here. The similarities in their reactions to pressure are the subject of this chapter.

6.1 Hydrostatics.

In section (4.3) we discussed the equilibrium conditions for a solid which is subject to stress forces. The related formula is given in equation (4.31) for the force per unit volume in terms of the stress forces t_{ij} and the body forces $f_i^{(b)}$, *i.e.*:

$$\left(\left. \frac{\partial t_{ij}}{\partial x_j} \right|_{(\vec{x},t)} \right) + \rho(\vec{x},t) f_i^{(b)}(\vec{x}) = 0. \tag{6.1}$$

Now, in a fluid at rest the stress tensor can only have diagonal components, since a fluid at rest cannot sustain shear forces and consequently the forces on any surface element within the fluid are always acting normal to the surface. Moreover, because of the isotropy of a fluid, have these forces in all directions the same modulus. The stress tensor takes thus an extremely simple form for a fluid at rest:

$$t_{ij}(\vec{x},t) = \begin{pmatrix} -p(\vec{x},t) & 0 & 0 \\ 0 & -p(\vec{x},t) & 0 \\ 0 & 0 & -p(\vec{x},t) \end{pmatrix} ,$$

or equivalently in a more compact notation:

$$t_{ij}(\vec{x},t) = -p(\vec{x},t)\delta_{ij} .$$

Inserting the above stress tensor in the equilibrium equation (6.1), we obtain:

$$-\left\{ \nabla p \right\}_{(\vec{x},t)} + \rho(\vec{x},t) \vec{f}^{(b)}(\vec{x}) = 0. \tag{6.2}$$

For most applications the body force $\vec{f}^{(b)}$ just represents the gravitational field near the Earth's surface. If one takes the $\hat{e}_3$-axis along the vertical, then the gravitational field is near the surface of the Earth well approximated by:

$$\vec{f}^{(b)}(\vec{x}) = -g\hat{e}_3. \tag{6.3}$$

In the case of a gas one needs a further information in order to solve equation (6.2) for the density ρ and the pressure p. For an *ideal* gas at a constant temperature one has for example the relation:

$$\text{pressure} \times \text{volume} = \text{constant},$$

which gives the following relation for the density and the pressure of an ideal gas at constant temperature:

$$p(\vec{x},t) = \gamma\, \rho(\vec{x},t). \tag{6.4}$$

The proportionality constant γ depends on the molecular weight and the temperature of the gas.

Problem 26:

Assuming formula (6.3) for the gravitational field, (6.4) for an ideal gas, then, using the equilibrium equation (6.2), determine the pressure of air at altitude z above sea level, assuming moreover that the temperature of air is the same at all altitudes.

Determine a reasonable value for the proportionality constant γ in (6.4) in order to find the pressure and the density at 10 km above the Earth's surface.

The density of a liquid is for most practical purposes constant. This is called the *incompressibility* property of liquids.

Problem 27:

Let the atmospheric pressure at sea level be represented by p_{atm} and the density of sea water by ρ.

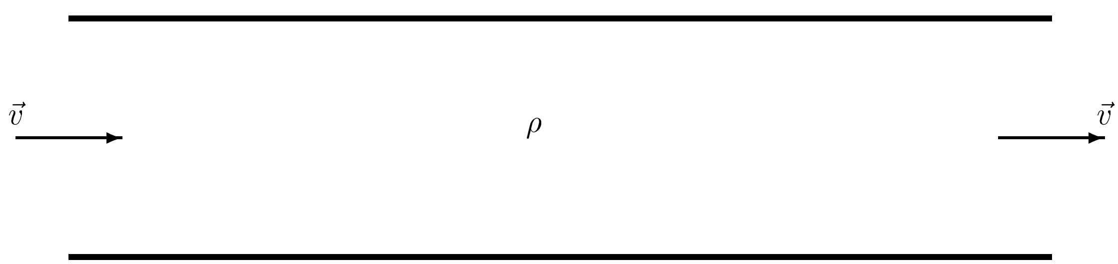
Find an expression for the pressure in the sea at a depth d below its surface. Estimate the numerical value of this pressure in terms of the atmospheric pressure at ten meters below the surface of the sea.

6.2 The current vector field.

The amount of fluid which passes at a certain location $\vec{x}$ at a certain instant of time t per unit of time and per unit area, is called the *current* of the fluid flow and is defined as follows:

$$\vec{j}(\vec{x},t) = \rho(\vec{x},t)\vec{v}(\vec{x},t). \tag{6.5}$$

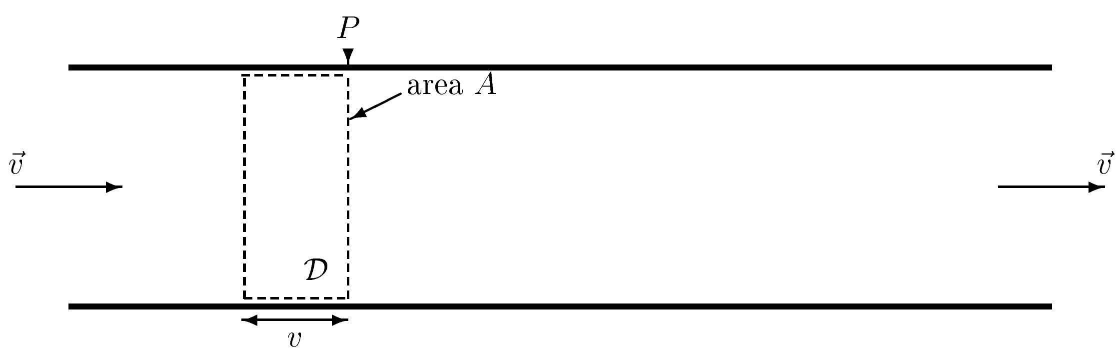
The precise meaning of this expression will be made clear in the following: Let us first consider a simple tube and a fluid of constant density $\rho(\vec{x},t) = \rho$. The fluid flows with a constant velocity through the tube, as indicated in the figure below:



Let the cross section of the tube be the same everywhere and its area be indicated by A. At a certain location of the tube, indicated by P in the figure below, passes per unit of time an amount of fluid which has a volume given by:

$$\text{fluid which passes in } P \ = Av \text{ per unit of time,} \tag{6.6}$$

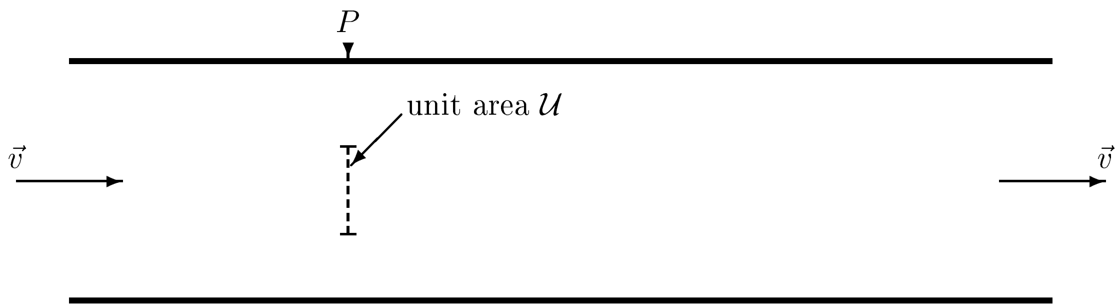
where v, which equals the modulus of $\vec{v}$, represents the speed of the material points of the fluid flow. The derivation of formula (6.6) for the fluid flow in the tube under consideration, is shown in the figure below:



The volume Av of fluid inside the domain $\mathcal{D}$ passes in one unit of time at the location P. The reason for this is, that the length of the domain $\mathcal{D}$ has the same value as the speed v of the fluid. Consequently, the amount of matter (*i.e.* the mass in the domain $\mathcal{D}$) which passes per unit of time at location P, is given by:

$$\text{mass which passes in } P \ = Av\rho \text{ per unit of time.} \tag{6.7}$$

When one next considers an unit area $\mathcal{U}$ at location P, as is shown in the following figure,



then, using the definition (6.5) for the current of the fluid flow and the relation (6.7) for the amount of matter which passes at location P per unit of time, one obtains for the amount of matter which passes through the unit area $\mathcal{U}$ per unit of time, the following:

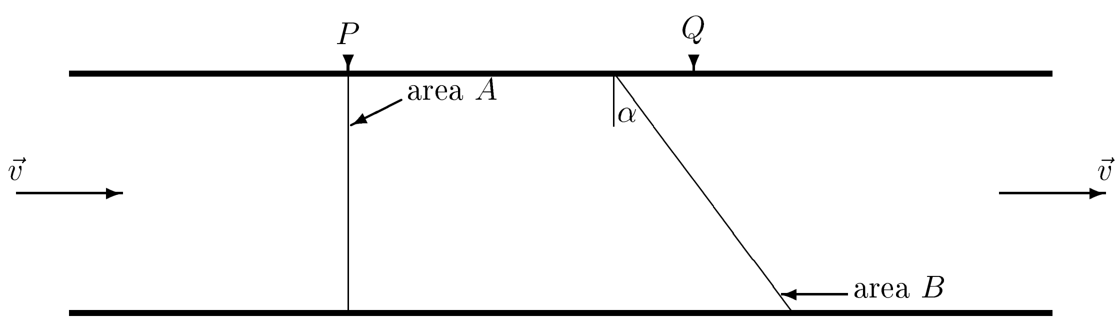
$$\text{mass which passes per unit of time through } \mathcal{U} \ = v\rho = \mid \vec{j} \mid .$$

The unit area $\mathcal{U}$ in the above figure is perpendicular to the velocity vector $\vec{v}$, and hence to the current vector $\vec{j}$, which leads us to the following interpretation of the current vector field:

1 The vector $\vec{j}(\vec{x},t)$ is pointing in the same direction as the velocity vector $\vec{v}(\vec{x},t)$ at the position $\vec{x}$ and at the instant of time t.

2 The modulus $\mid \vec{j}(\vec{x},t) \mid$ represents the amount of matter which passes through an area perpendicular to $\vec{j}(\vec{x},t)$ per unit area and per unit of time.

For a surface element which is not perpendicular to the current vector one might consider the following: In the same tube of the above example let us consider two different surface elements, one perpendicular to the current and with an area A at location P, another which makes an angle α with the previous surface element and has an area B at location Q. This situation is shown below:

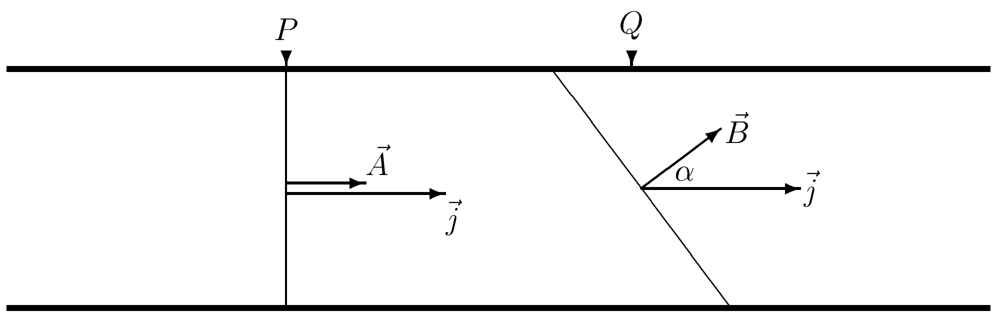


From the figure one might conclude that the relation between the areas A and B is given by the formula:

$$A = B\cos(\alpha). \tag{6.8}$$

The amount of matter passing through the surface element A at location P per unit of time must be equal to the amount of matter passing through the surface element B at location Q per unit of time. This observation is based on the principle of conservation of matter, and only valid if there are, as we assume to be the case here, no sources or sinks in between the locations P and Q.

Now, let us represent the two surface elements A and B by a vector (as defined in formula 4.20). This is depicted below:



In the above figure the area A of the surface element at location P is represented by the modulus of $\vec{A}$ and similar for B. We find then for the amount of matter passing per unit of time through the surface elements at P and Q respectively, using the principle of conservation of matter and the formulas (6.7), (6.5) and (6.8), the following:

$$\begin{aligned} \rho v A &= jA = \vec{j} \cdot \vec{A} \\ &= \rho v B \cos(\alpha) = jB\cos(\alpha) = \vec{j} \cdot \vec{B}. \end{aligned}$$

This way we obtain the following interpretation for the current field:

1 The vector $\vec{j}(\vec{x}, t)$ is pointing in the same direction as the velocity vector $\vec{v}(\vec{x}, t)$ at the position $\vec{x}$ and at the instant of time t.

2 Through an arbitrary surface element $\Delta\vec{A}(\vec{x})$ at position $\vec{x}$ passes per unit of time at instant t the amount of matter given by:

$$\vec{j}(\vec{x}, t) \cdot \Delta\vec{A}(\vec{x}). \tag{6.9}$$

6.3 The continuity equation.

In section (1.1) we mentioned that the density $\rho(\vec{x}, t)$ and the velocity $\vec{v}(\vec{x}, t)$ at a certain position $\vec{x}$ in a fluid flow at a certain instant of time t, are not independent. The related equation is the *equation of continuity* (1.3). In the following we discuss a derivation of this equation based on the conservation of matter.

Let us consider a fixed volume element $\mathcal{V}$ inside a fluid flow. Fluid enters and leaves this volume, passing through its fixed surface $\mathcal{A}$. As a consequence, the amount of fluid which is inside the volume element might vary with time. Once the density field $\rho(\vec{x}, t)$ is known as a function of coordinates and time, we can determine the total mass inside the volume element as a function of time, as follows:

$$M_{\mathcal{V}}(t) = \int_{\text{volume } \mathcal{V}} dV(\vec{x}) \rho(\vec{x}, t). \tag{6.10}$$

How this quantity varies with time, is given by its derivative with respect to t, according to:

$$\begin{aligned} \frac{d}{dt} M_{\mathcal{V}}(t) &= \lim_{\Delta t \to 0} \frac{M_{\mathcal{V}}(t + \Delta t) - M_{\mathcal{V}}(t)}{\Delta t} \\ &= \int_{\text{volume } \mathcal{V}} dV(\vec{x}) \left\{ \lim_{\Delta t \to 0} \frac{\rho(\vec{x}, t + \Delta t) - \rho(\vec{x}, t)}{\Delta t} \right\}, \end{aligned}$$

or consequently:

$$\frac{d}{dt} M_{\mathcal{V}}(t) = \int_{\text{volume } \mathcal{V}} dV(\vec{x}) \left(\left. \frac{\partial \rho}{\partial t} \right|_{(\vec{x}, t)} \right). \tag{6.11}$$

The partial derivative of the density field comes in this expression, rather than the total derivative, because the density is a function of the four independent variables x_1, x_2, x_3 and t whereas the derivative of the total mass is only in the fourth variable t since the other three are integrated.

Through a surface element $\Delta\vec{A}(\vec{x})$ of the surface $\mathcal{A}$ of the volume element $\mathcal{V}$ flows per unit of time the amount of matter given by the expression (6.9). The vector $\Delta\vec{A}(\vec{x})$ is supposed to point outward with respect to the interior of the volume element $\mathcal{V}$. So, more accurately, formula (6.9) represents the amount of matter which *leaves* the volume element $\mathcal{V}$ through the surface element represented by $\Delta\vec{A}(\vec{x})$. Consequently, the total amount of matter which leaves $\mathcal{V}$ per unit of time, is given by the integral of contributions (6.9) over the whole surface $\mathcal{A}$ of the volume element $\mathcal{V}$, *i.e.*:

$$J_{\mathcal{V}}(t) = \int_{\text{surface } \mathcal{A}} d\vec{A}(\vec{x}) \cdot \vec{j}(\vec{x}, t),$$

which, by virtue of the divergence theorem, alternatively can be expressed by:

$$J_{\mathcal{V}}(t) = \int_{\text{volume } \mathcal{V}} dV(\vec{x}) \left(\left. \frac{\partial j_i}{\partial x_i} \right|_{(\vec{x}, t)} \right). \tag{6.12}$$

The amount of matter which *enters* the volume element in a short time interval given by $(t, t + \Delta t)$, is then equal to:

$$-\Delta t \; J_{\mathcal{V}}(t).$$

This must be equal to the increase in total mass of the volume element $\mathcal{V}$ within the same time interval, which is expressed by:

$$M_{\mathcal{V}}(t + \Delta t) - M_{\mathcal{V}}(t) = -\Delta t \; J_{\mathcal{V}}(t).$$

In the limit of $\Delta t \to 0$, one finds then:

$$\frac{dM_{\mathcal{V}}(t)}{dt} = \lim_{\Delta t \to 0} \frac{M_{\mathcal{V}}(t+\Delta t) - M_{\mathcal{V}}(t)}{\Delta t} = -J_{\mathcal{V}}(t). \tag{6.13}$$

This formula expresses the conservation of matter, *i.e.* the increase of mass inside the volume element $\mathcal{V}$ is equal to the amount of matter which enters through its surface $\mathcal{A}$. Using the equations (6.11) for the derivative with respect to time of $M_{\mathcal{V}}(t)$ and (6.12) for $J_{\mathcal{V}}(t)$, we arrive at:

$$\int_{\text{volume } \mathcal{V}} dV(\vec{x}) \left\{ \left(\left.\frac{\partial \rho}{\partial t}\right|_{(\vec{x},t)} \right) + \left(\left.\frac{\partial j_i}{\partial x_i}\right|_{(\vec{x},t)} \right) \right\} = 0.$$

And, since the result has been obtained for an arbitrary volume element $\mathcal{V}$, the integrand must vanish, *i.e.*:

$$\left(\left.\frac{\partial \rho}{\partial t}\right|_{(\vec{x},t)} \right) + \left(\left.\frac{\partial j_i}{\partial x_i}\right|_{(\vec{x},t)} \right) = 0. \tag{6.14}$$

The above relation is known as the *continuity equation* for fluids.

Problem 28:

Consider for a collapsing system, the following density distribution as a function of position $\vec{x}$ and time t:

$$\rho(\vec{x},t) = \frac{M_0}{4\pi}\beta t \frac{e^{-\beta t \mid \vec{x} \mid}}{\mid \vec{x} \mid^2} \quad \text{, for } \ t > 0. \tag{6.15}$$

The parameter M_0 represents the total mass of the collapsing system and β, which has the units "per unit of length and per unit of time", indicates the collapse velocity.

(i) Determine the total mass $M(R,t)$ inside a sphere of radius R at instant $t > 0$.

(ii) Determine for a fixed instant $t > 0$ the limit $M(R \to \infty, t)$.

(iii) Determine for a fixed radius $R > 0$ the limit $M(R, t \to \infty)$.

Notice that all mass M_0, which for fixed time t is distributed over the whole space, *i.e.* $R \to \infty$, is inside a sphere of arbitrary radius $R > 0$ for $t \to \infty$. This means that all mass of the collapsing system is concentrating in the origin.

The expression (6.15) has the same value at all points of the surface of a sphere around the origin. So, we may expect that the modulus of the current shows the same phenomenon.

(iv) Using formula (6.13) determine the amount of matter which passes through the surface of a spere of radius R per unit of time.

(v) Determine the current of the fluid flow under consideration.

(vi) Verify that the current obtained in (v) and the density given in (6.15) satisfy the continuity equation (6.14).

(vii) Show, using the definition (6.5) for the current of a fluid flow, that the velocity field is given by:

$$\vec{v}(\vec{x}, t) = -\frac{\vec{x}}{t}.$$

The speed of material points exceeds here the velocity of light at distances which are given by $\mid \vec{x} \mid > ct$. Consequently, the density field (6.15) is not a realistic representation of a collapsing system.

6.4 Euler's equations for a fluid in motion.

In section (4.3) the equilibrium conditions for a solid which is subject to stress forces are discussed. Formula (6.1) shows the total force per unit volume in terms of the stress tensor and the body forces, which in case of equilibrium vanishes. For a fluid in motion we assume a similar expression for the force per unit volume with the following interpretation: At a small amount of matter ΔM which at instant t occupies a small domain ΔV of the fluid around the position $\vec{x}$, acts a force given by:

$$\Delta V \left\{ \left(\left. \frac{\partial t_{ij}}{\partial x_j} \right|_{(\vec{x},\, t)} \right) + \rho(\vec{x}, t) f_i^{(b)}(\vec{x}) \right\},$$

and consequently this amount of matter is accellerated with an accelleration given by:

$$\Delta M \vec{a}(\vec{x}, t) = \Delta V \left\{ \left(\left. \frac{\partial t_{ij}}{\partial x_j} \right|_{(\vec{x},\, t)} \right) + \rho(\vec{x}, t) f_i^{(b)}(\vec{x}) \right\}. \tag{6.16}$$

However, the position vector of the small amount of matter ΔM is not a priori known to us as a function of time, neither its velocity. So, how can we know its accelleration?

What we do know is the velocity field $\vec{v}(\vec{x}, t)$ of the fluid flow. The question is then: "How is the accelleration of ΔM related to the velocity field?".

Let the position of the small amount of matter under consideration at a later instant $t + \Delta t$ be given by $\vec{x} + \Delta\vec{x}$. Then its velocity at that later instant will be given by:

$$\vec{v}(\vec{x}+\Delta\vec{x}, t+\Delta t),$$

which by a Taylor expansion can be related to the velocity at the instant t, according to:

$$v_i(\vec{x}+\Delta\vec{x}, t+\Delta t) = v_i(\vec{x}, t) + \left(\left. \frac{\partial v_i}{\partial x_j} \right|_{(\vec{x},\, t)} \right) \Delta x_j + \left(\left. \frac{\partial v_i}{\partial t} \right|_{(\vec{x},\, t)} \right) \Delta t + \cdots.$$

The accelleration is then given by:

$$a_i(\vec{x}, t) = \lim_{\Delta t \to 0} \frac{v_i(\vec{x}+\Delta\vec{x}, t+\Delta t) - v_i(\vec{x}, t)}{\Delta t},$$

which, by means of the above Taylor expansion, yields:

$$a_i(\vec{x}, t) = \left(\left. \frac{\partial v_i}{\partial x_j} \right|_{(\vec{x},\, t)} \right) \frac{dx_j}{dt} + \left(\left. \frac{\partial v_i}{\partial t} \right|_{(\vec{x},\, t)} \right). \tag{6.17}$$

This expression is just equal to the total derivative with respect to time t of the velocity vector field. It can be rewritten, using the usual definition of velocity, *i.e.* $v = dx/dt$, according to:

$$a_i(\vec{x}, t) = \left(\left. \frac{\partial v_i}{\partial x_j} \right|_{(\vec{x},\, t)} \right) v_j(\vec{x}, t) + \left(\left. \frac{\partial v_i}{\partial t} \right|_{(\vec{x},\, t)} \right). \tag{6.18}$$

In the following we will relax our notation and no longer refer to the coordinates $(\vec{x}, t)$, which from now on are understood implicitly. Inserting the above formula (6.18) for the acceleration in expression (6.16), we obtain:

$$\Delta M \left\{ v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial t} \right\} = \Delta V \left\{ \frac{\partial t_{ij}}{\partial x_j} + \rho f_i^{(b)} \right\}. \tag{6.19}$$

Now, the relation between ΔM, ρ and ΔV, given by: $\Delta M = \rho \Delta V$, leads finally to the expression:

$$\rho \left\{ v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial t} \right\} = \frac{\partial t_{ij}}{\partial x_j} + \rho f_i^{(b)}. \tag{6.20}$$

A non-viscous (frictionless) or *perfect* fluid does not support shear forces. So, the stress tensor becomes diagonal. Moreover, using the property of isotropy for a fluid, we are led to the same form for the stress tensor as in the case of a fluid at rest, *i.e.*:

$$t_{ij} = -p\delta_{ij}. \tag{6.21}$$

Inserting this relation in equation (6.20), one finds:

$$\rho\left\{v_j\frac{\partial v_i}{\partial x_j}+\frac{\partial v_i}{\partial t}\right\}=-\frac{\partial p}{\partial x_i}+\rho f_i^{(b)}. \tag{6.22}$$

These equations can be further modified, but that is left as an excercise to the reader.

Problem 29:
Proof the following identity:

$$v_j\frac{\partial v_i}{\partial x_j}=\frac{1}{2}\frac{\partial}{\partial x_i}\{\mid\vec{v}\mid^2\}-[\vec{v}\times\mathrm{curl}(\vec{v})]_i. \tag{6.23}$$

Using the above identity, one finds for (6.22) the usual expression for *Euler's equation of motion for a fluid*:

$$\rho\{\frac{\partial\vec{v}}{\partial t}-\vec{v}\times\mathrm{curl}(\vec{v})+\frac{1}{2}\mathrm{grad}(\mid\vec{v}\mid^2)\}+\mathrm{grad}(p)=\rho\vec{f}^{(b)}. \tag{6.24}$$

This is a nonlinear equation in the velocity of the fluid and hence in general difficult to solve. But in some special situations one might obtain solutions as we will discuss below.

6.5 Bernoulli's equation.

When one puts the following conditions on the form of the fluid flow, then one obtains from Euler's equation the equation of Bernoulli, which has been discussed in section (1.7):

1. The flow is irrotational.

The vorticity $\vec{\omega}$ of a fluid flow is defined by:

$$\vec{\omega}=\mathrm{curl}(\vec{v})=\nabla\times\vec{v}. \tag{6.25}$$

In order to interpret this quantity, let us return to a fluid flow in two dimensions. For a flow in the (x,y)-plane the vorticity is in the z-direction, according to:

$$\vec{\omega}=\omega\hat{z}\quad\text{, with }\ \omega=\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}. \tag{6.26}$$

Problem 30:

(i) Show that for the velocity vector fields which follow from the stream-functions (1.7), (1.8) and (1.9), the vorticity vanishes.

(ii) Show that in the case (1.10) the vorticity is given by:

$$\omega(x, y) = -2\pi\delta(x)\delta(y),$$

where $\delta(x)$ and $\delta(y)$ represent the Dirac delta function.

The latter result can be obtained by integration of $\text{curl}\{\vec{v}(x, y)\}$ over the surface of a circle with arbitrary radius. This integral can, by means of Stoke's theorem, be converted into the integral of the velocity vector field over the circumference of the circle, *i.e.*

$$\int_{\text{surface}} dxdy\hat{z}\cdot(\nabla\times\vec{v}) = \int_{\text{circumference}} d\vec{s}\cdot\vec{v} = -2\pi.$$

The rest of the proof is then simple.

The conclusion of the second part (ii) of the above problem is, that the vorticity ω does not vanish at the position of a vortex. If we are careful enough to stay away from vortices, then the remaining part of the fluid flow may still be considered to have vanishing vorticity.

One defines, in general, for a fluid to be considered free of vortices, or irrotational, when:

$$\nabla\times\vec{v} = \vec{\omega} = 0. \tag{6.27}$$

For an irrotational flow one has moreover that the velocity may be written as the gradient of the *velocity potential* ϕ, *i.e.*:

$$\vec{v} = \text{grad}(\phi) = \nabla(\phi). \tag{6.28}$$

Problem 31:

Show that the definition (6.28) is a consequence of the identity (6.27).

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2. The body force is conservative.

When the body force is conservative, then we know from Newtonian mechanics that a potential field $U^{(b)}(\vec{x})$ can be defined, such that:

$$\vec{f}^{(b)} = -\text{grad}(U^{(b)}) = -\nabla U^{(b)}. \tag{6.29}$$

3. The fluid is incompressible.

In the case of incompressible fluids one has moreover that the density is constant and independent of the pressure.

For an incompressible fluid, free of vortices and subject to a conservative body force, we find for Euler's equation of motion (6.24), using the formulas (6.27), (6.28) and (6.29), the following form:

$$\rho \text{grad}\{\frac{\partial \phi}{\partial t} + \frac{1}{2} \mid \vec{v} \mid^2 + \frac{p}{\rho}\} = -\rho \text{grad}(U^{(b)}),$$

or equivalently:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \mid \vec{v} \mid^2 + \frac{p}{\rho} + U^{(b)} = \text{constant}. \tag{6.30}$$

4. The flow is steady.

In case that the velocity vector field is independent of time (*steady flow*), one has moreover that the velocity potential ϕ is constant in time, which reduces the above formula to the Bernoulli equation (compare formula 1.40):

$$\frac{1}{2}\rho \mid \vec{v} \mid^2 + p + \rho U^{(b)} = \text{constant}. \tag{6.31}$$

This leaves us with the final conclusion that in chapter 1 we studied fluid flows which satisfy all the above conditions, *i.e.* a perfect fluid, free of vortices, incompressible and steady, in the case that moreover the body forces can be ignored.