# MECÂNICA QUÂNTICA II

## Exame Normal, Terça-feira, 26 de Junho de 2012

**1.** Considere uma partícula de massa $m$ cujo deslocamento unidimensional é descrito pelo seguinte Hamiltoniano.

$$H = H_0 + U(x) \quad \text{com} \quad H_0 = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 \quad . \tag{1}$$

**a** Mostre que, com a definição dos operadores (em unidades $\hbar = 1$)

$$a \;=\; \sqrt{\frac{m\omega}{2}}\left(x \;+\; i\frac{p}{m\omega}\right) \quad \text{e} \quad a^\dagger \;=\; \sqrt{\frac{m\omega}{2}}\left(x \;-\; i\frac{p}{m\omega}\right) \quad , \tag{2}$$

o Hamiltoniano $H_0$ pode ser dado por

$$H_0 = \omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

e determine os comutadores $\left[a\ ,\ a^\dagger\right]$, $[H_0\ ,\ a]$ e $\left[H_0\ ,\ a^\dagger\right]$.

**b** Os estados próprios normalizados do Hamiltoniano $H_0$ são designados por $|n\rangle$. O estado fundamental, $|0\rangle$, é dado por

$$\left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} \quad .$$

Determine (i) $a|0\rangle$, (ii) $|1\rangle = a^\dagger|0\rangle$ e (iii) $H_0|0\rangle$.

**c** Determine o valor próprio do estado $|1\rangle$ sob $H_0$ e a normalização de $|1\rangle$.

**d** Determine o valor expectável $\langle 0\,|U(x)|\,1\rangle$ para $U(x) = gx$.

**e** Determine a correcção de primeira ordem em $g$ ao nível energético do estado fundamental de $H_0$ quando se aplica uma perturbação $U(x) = gx^2$.

**f** Compare o resultado da alínea (**e**) com o nível energético do estado fundamental de $H$, aplicando uma redefinição de $\omega$.

**2**. Considere uma partícula que se desloca numa dimensão (de $x \downarrow -\infty$ para $x \uparrow +\infty$) e cuja função de onda é dada por (em unidades $\hbar = 1$):

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & \text{para} \quad x < -L \\ Ae^{\kappa x} + Be^{-\kappa x} & \text{para} \quad -L \leq x \leq L \\ Te^{ikx} & \text{para} \quad x > L \end{cases} \tag{3}$$

**a** Demonstre que a função de onda (3) é solução da seguinte equação de onda:

$$\left\{-\frac{1}{2m}\frac{d^2}{dx^2} + U(x)\right\}\psi(x) = E\psi(x) \quad ,$$

com $k^2 = 2mE$, $\kappa^2 = 2m(V - E)$ e

$$U(x) = \begin{cases} 0 & \text{para} \quad x < -L \\ V > E > 0 & \text{para} \quad -L \leq x \leq L \\ 0 & \text{para} \quad x > L \end{cases} \quad .$$

**b** Utilizando as condições fronteiras em $x = -L$ e $x = L$, mostre que os coeficientes $R$ e $T$ da Equação (3) obedecem as seguintes relações:

$$T = \frac{2\sqrt{E(V-E)}\, e^{-2ikL}}{2\sqrt{E(V-E)}\, \cosh(2\kappa L) + i(V - 2E)\sinh(2\kappa L)}$$

e

$$R = \frac{-iVe^{-2ikL}\sinh(2\kappa L)}{2\sqrt{E(V-E)}\, \cosh(2\kappa L) + i(V - 2E)\sinh(2\kappa L)} \quad ,$$

onde $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ e $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$.

**c** Determine a expressão para $|R|^2 + |T|^2$ e mostre que para $2\kappa L \gg 1$ se verifica

$$|T|^2 \approx \frac{16E(V-E)e^{-4\kappa L}}{V^2} \quad .$$

**d** Uma feixe de electrões choca contra uma barreira potencial de $V = 2.0$ eV e com uma largura de $2L = 1.5$ nm. Cada electrão tem uma massa igual a 0.5 MeV (em unidades $\hbar = c = 1$) e uma energia cinética $E = 1.0$ eV.
Determine a fracção dos electrões transmitidos pela barreira, isto é $|T|^2$.
Notar que $1 = 2.0 \times 10^{-7}$ eVm em unidades $\hbar = c = 1$.

**3.** Considere um sistema de dois estados descrito pelo seguinte Hamiltoniano:

$$H = \begin{pmatrix} A & \frac{B}{r} \\ Br & A \end{pmatrix} \quad \text{com} \quad \begin{cases} A = M + \frac{1}{2}\left(\Delta M - \frac{1}{2} i \left(\gamma_1 + \gamma_2\right)\right) \\ B = -\frac{1}{2}\left(\Delta M + \frac{1}{2} i \left(\gamma_1 - \gamma_2\right)\right) \end{cases} \quad . \qquad (4)$$

O parâmetro complexo $r$ é dado pela expressão $r = 1 - 2\varepsilon$ onde $|\varepsilon| \ll 1$.
Notar que $H$ não é hermítico e portanto $H^\dagger \neq H$.

**a** Mostre que os dois estados próprios normalizados do Hamiltoniano (4) são dados por

$$\phi_\pm = \frac{1}{\sqrt{1 + |r|^2}} \begin{pmatrix} 1 \\ \pm r \end{pmatrix}$$

e determine os valores próprios $E_\pm$.

**b** Seja um estado abitrário $\psi(t)$ deste sistema no instant $t = 0$ dado pelo "*pacote de ondas*" ($c_\pm$ constantes)

$$\psi(0) = c_+ \phi_+ + c_- \phi_- \quad .$$

Mostre que o desenvolvimento temporal $\psi(t) = e^{-iHt}\psi(0)$ pode ser dado por

$$\psi(t) = \frac{1}{E_+ - E_-} \left\{ \left(E_+ - H\right) e^{-iE_- t} - \left(E_- - H\right) e^{-iE_+ t} \right\} \psi(0) \quad .$$

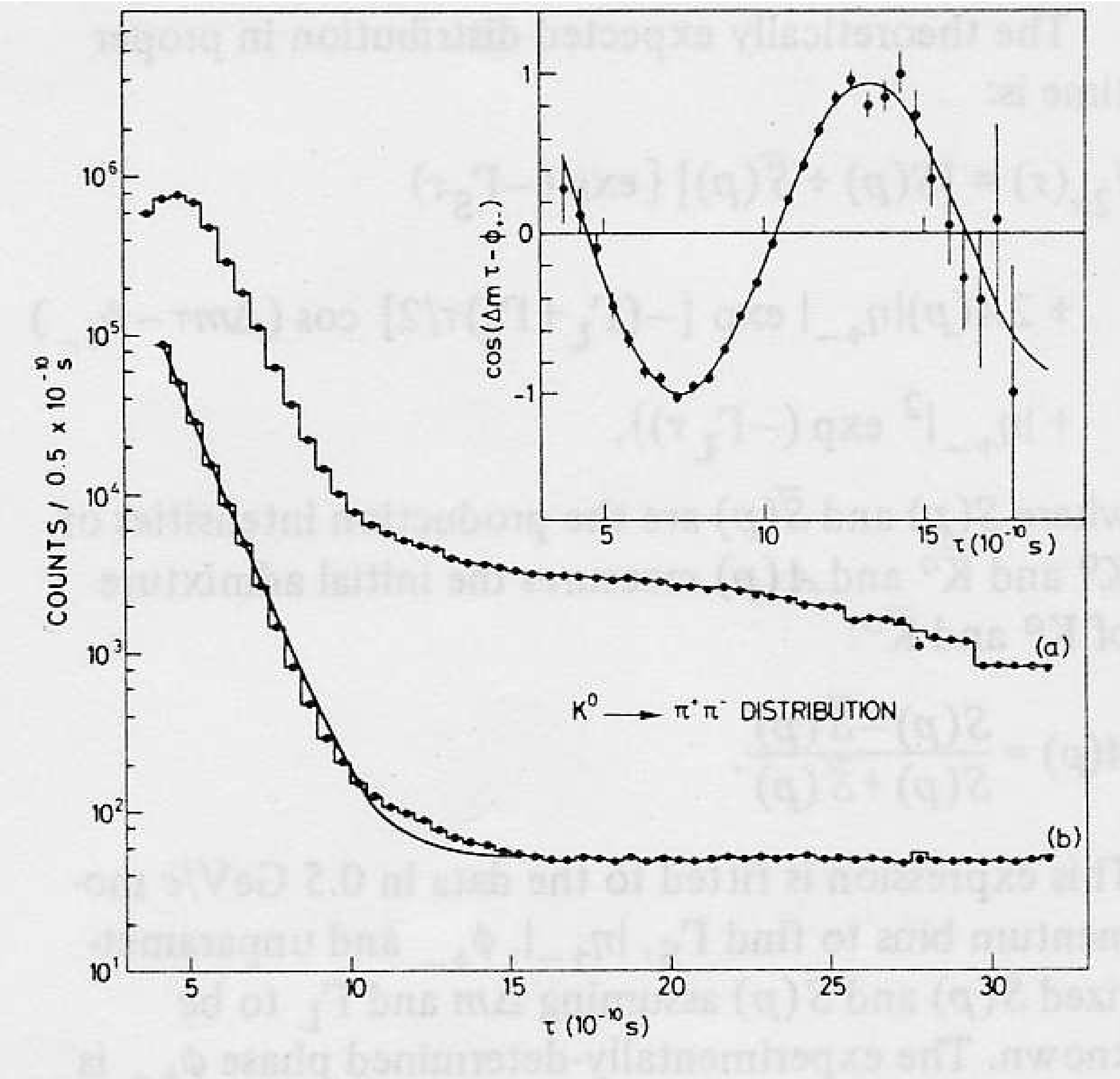
**c** Determine a amplitude da probabiblidade de transição, $\left\langle \downarrow \left| e^{-iHt} \right| \uparrow \right\rangle$, entre os estados

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{e} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e mostre que

$$\left| \left\langle \downarrow \left| e^{-iHt} \right| \uparrow \right\rangle \right|^2 = \tfrac{1}{4} |r|^2 \left\{ e^{-\gamma_1 t} + e^{-\gamma_2 t} - 2 e^{-\frac{1}{2}\left(\gamma_1 + \gamma_2\right) t} \cos(\Delta M t) \right\} \quad .$$

**d** A figura na página seguinte mostra as observações experimentais da desintegração do mesão Kaon neutro. O Kaon neutro é um estado misto do sistema de dois níveis que consiste dos estados próprios $K_S$ de vida curta e $K_L$ de vida longa. As massas do $K_S$ e $K_L$ são aproximadamente idênticas, nomeadamente $M_S \approx M_L \approx 500$ MeV. Os tempos de vida média de $K_S$ e $K_L$ são respectivamente $\gamma_1^{-1} = 0.90 \times 10^{-10}$ s e $\gamma_2^{-1} = 5.2 \times 10^{-8}$ s.
Determine, a partir das observações experimentais, a diferença de massa em eV entre os dois estados próprios, $\Delta M = M_L - M_S$.
Notar que $1 = 0.66 \times 10^{-15}$ eVs em unidades $\hbar = c = 1$.




De C. Geweniger *et al.*, Physics Letters 48B, 483 (1974).

**4.** Considere um sistema caracterizado pela seguinte base de dez estados.

$$\begin{pmatrix} Y_m^{(2)}(\hat{r}) \\ \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ \\ Y_m^{(2)}(\hat{r}) \end{pmatrix} \quad , \ \text{com} \ \ m = 0, \pm 1, \pm 2 \ \ . \tag{5}$$

Rotações no espaço de coordenadas $\hat{r}$ actuam nesta base da seguinte forma:

$$J_+ = L_+ \mathbf{1} + S_+ = \begin{pmatrix} L_+ & 0 \\ 0 & L_+ \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_+ & 1 \\ 0 & L_+ \end{pmatrix} \quad ,$$

$$J_- = L_- \mathbf{1} + S_- = \begin{pmatrix} L_- & 0 \\ 0 & L_- \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} L_- & 0 \\ 1 & L_- \end{pmatrix} \quad \text{e}$$

$$J_z = L_z \mathbf{1} + S_z = \begin{pmatrix} L_z & 0 \\ 0 & L_z \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} L_z + \frac{1}{2} & 0 \\ 0 & L_z - \frac{1}{2} \end{pmatrix}$$

com $(\hbar = 1)$

$$L_\pm Y_m^{(2)} = \sqrt{6 - m(m \pm 1)}\, Y_{m\pm1}^{(2)} \quad \text{e} \quad L_z Y_m^{(2)} = m Y_m^{(2)} \quad .$$

**a** Utilizando $S_\pm = S_x \pm i S_y$ e $[S_+, S_-] = 2S_z$, demonstre que

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \tfrac{1}{2}\left(S_+S_- + S_-S_+\right) + S_z^2 = S_-S_+ + S_z + S_z^2 \quad .$$

**b** Demonstre que $L_-L_+Y_m^{(2)} = (6 - m(m+1))\, Y_m^{(2)} = (6 - L_z^2 - L_z)\, Y_m^{(2)}$.

**c** Determine os valores próprios de $L^2$, $L_z$, $S^2$ e $S_z$ de cada um dos dez estados da base referida na Equação (5).

**d** Estes estados formam uma base ortonormal do espaço $\{s = \frac{1}{2}\} \otimes \{\ell = 2\}$. Temos por exemplo:

$$\left| s = \tfrac{1}{2}, s_z = +\tfrac{1}{2} \right\rangle \otimes |\ell = 2, \ell_z = +2\rangle = \begin{pmatrix} Y_{+2}^{(2)} \\ \\ 0 \end{pmatrix} \quad ,$$

ou simplesmente

$$\left| +\tfrac{1}{2} \right\rangle \otimes |+2\rangle = \begin{pmatrix} Y_{+2}^{(2)} \\ \\ 0 \end{pmatrix} \quad ,$$

Determine relações semelhantes para os outros nove estados.

**5**. Uma outra base ortonormal do espaço $\{s = \frac{1}{2}\} \otimes \{\ell = 2\}$, considerado no exercício 4, é dada por:

$$\begin{pmatrix} Y_{+2}^{(2)} \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{4}{5}}Y_{+1}^{(2)} \\ \sqrt{\frac{1}{5}}Y_{+2}^{(2)} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{3}{5}}Y_{0}^{(2)} \\ \sqrt{\frac{2}{5}}Y_{+1}^{(2)} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{2}{5}}Y_{-1}^{(2)} \\ \sqrt{\frac{3}{5}}Y_{0}^{(2)} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1}{5}}Y_{-2}^{(2)} \\ \sqrt{\frac{4}{5}}Y_{-1}^{(2)} \end{pmatrix}, \begin{pmatrix} 0 \\ Y_{-2}^{(2)} \end{pmatrix},$$

$$\begin{pmatrix} -\sqrt{\frac{1}{5}}Y_{+1}^{(2)} \\ \sqrt{\frac{4}{5}}Y_{+2}^{(2)} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{2}{5}}Y_{0}^{(2)} \\ \sqrt{\frac{3}{5}}Y_{+1}^{(2)} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{3}{5}}Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}Y_{0}^{(2)} \end{pmatrix} \text{ e } \begin{pmatrix} -\sqrt{\frac{4}{5}}Y_{-2}^{(2)} \\ \sqrt{\frac{1}{5}}Y_{-1}^{(2)} \end{pmatrix} .$$

**a** Utilizando os resultados das alíneas (4**a**) e (4**b**), demonstre que

$$J^2 \begin{pmatrix} aY_{m}^{(2)} \\ bY_{m+1}^{(2)} \end{pmatrix} = \begin{pmatrix} L_z + \frac{27}{4} & L_- \\ L_+ & -L_z + \frac{27}{4} \end{pmatrix} \begin{pmatrix} aY_{m}^{(2)} \\ bY_{m+1}^{(2)} \end{pmatrix} = \begin{pmatrix} \left(a\left(\frac{27}{4} + m\right) + b\sqrt{6 - m(m+1)}\right) Y_{m}^{(2)} \\ \left(a\sqrt{6 - m(m+1)} + b\left(\frac{23}{4} - m\right)\right) Y_{m+1}^{(2)} \end{pmatrix} .$$

**b** Determine os valores próprios de $J^2$ e $J_z$ de cada um dos dez estados da nova base.

**c** Verifica-se que podemos dividir o espaço $\{s = \frac{1}{2}\} \otimes \{\ell = 2\}$ em dois subespaços: $\left\{j = \frac{5}{2}\right\}$ de seis dimensões e $\left\{j = \frac{3}{2}\right\}$ de quatro dimensões. Os estados destes últimos espaços são designados por $|j, j_z\rangle$. Temos por exemplo:

$$\left|\tfrac{5}{2}, +\tfrac{5}{2}\right\rangle = \begin{pmatrix} Y_{+2}^{(2)} \\ 0 \end{pmatrix} = \left|+\tfrac{1}{2}\right\rangle \otimes |+2\rangle .$$

Determine relações semelhantes para os outros nove estados da nova base.

**d** Os coeficientes das relações mencionadas na alínea **c** chamam-se coeficientes de Clebsch-Gordan (CCG) e são designados por:

$$\begin{pmatrix} \ell & s & j \\ \ell_z & s_z & j_z \end{pmatrix} .$$

Da relação dada na alínea (**c**) determina-se por exemplo:

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ +2 & +\frac{1}{2} & +\frac{5}{2} \end{pmatrix} = 1 .$$

Determine os outros CCG das relações entre as bases dos subespaços $\left\{j = \frac{5}{2}\right\}$ e $\left\{j = \frac{3}{2}\right\}$ e a base do espaço $\{s = \frac{1}{2}\} \otimes \{\ell = 2\}$ do exercício 4.

# Solutions

## Exercício 1

**a.** We start with, remember $[x\ ,\ p] = i$,

$$a^\dagger a = \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)\sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right) =$$

$$= \frac{m\omega}{2}\left(x^2 + i\frac{1}{m\omega}\left[x\ ,\ p\right] + \frac{p^2}{m^2\omega^2}\right) = \frac{m\omega}{2}\left(x^2 - \frac{1}{m\omega} + \frac{p^2}{m^2\omega^2}\right)$$

$$= \frac{1}{\omega}\left(\frac{p^2}{2m} + \tfrac{1}{2}m\omega^2x^2\right) - \tfrac{1}{2} = \frac{H_0}{\omega} - \tfrac{1}{2} \quad\Longleftrightarrow\quad H_0 = \omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

and

$$aa^\dagger = \sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right)\sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right) =$$

$$= \frac{m\omega}{2}\left(x^2 - i\frac{1}{m\omega}\left[x\ ,\ p\right] + \frac{p^2}{m^2\omega^2}\right) = \frac{m\omega}{2}\left(x^2 + \frac{1}{m\omega} + \frac{p^2}{m^2\omega^2}\right)$$

$$= \frac{1}{\omega}\left(\frac{p^2}{2m} + \tfrac{1}{2}m\omega^2x^2\right) + \tfrac{1}{2} = \frac{H_0}{\omega} + \tfrac{1}{2} \quad\Longleftrightarrow\quad H_0 = \omega\left(aa^\dagger - \tfrac{1}{2}\right) \quad .$$

We find then

$$\left[a\ ,\ a^\dagger\right] = aa^\dagger - a^\dagger a = \frac{H_0}{\omega} + \tfrac{1}{2} - \left(\frac{H_0}{\omega} - \tfrac{1}{2}\right) = 1 \quad .$$

Also using the latter result, we obtain

$$\left[H_0\ ,\ a\right] = H_0 a - aH_0 = \omega\left(a^\dagger aa - aa^\dagger a\right) = \omega\left[a^\dagger\ ,\ a\right]a = -\omega a$$

and

$$\left[H_0\ ,\ a\right] = H_0 a^\dagger - a^\dagger H_0 = \omega\left(a^\dagger aa^\dagger - a^\dagger a^\dagger a\right) = \omega a^\dagger\left[a\ ,\ a^\dagger\right] = \omega a^\dagger \quad .$$

**b.** We determine

$$a|0\rangle = \sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right)\left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} =$$

$$= \sqrt{\frac{m\omega}{2}}\left[\frac{m\omega}{\pi}\right]^{1/4}\left(x + \frac{1}{m\omega}\frac{d}{dx}\right)e^{-\frac{1}{2}m\omega x^2}$$

$$= \sqrt{\frac{m\omega}{2}}\left[\frac{m\omega}{\pi}\right]^{1/4}\left(x + \frac{1}{m\omega}\left(-m\omega x\right)\right)e^{-\frac{1}{2}m\omega x^2} = 0$$

and

$$a^\dagger|0\rangle = \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)\left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} =$$

$$= \sqrt{\frac{m\omega}{2}} \left[\frac{m\omega}{\pi}\right]^{1/4} \left(x - \frac{1}{m\omega}\frac{d}{dx}\right) e^{-\frac{1}{2}m\omega x^2}$$

$$= \sqrt{\frac{m\omega}{2}} \left[\frac{m\omega}{\pi}\right]^{1/4} \left(x - \frac{1}{m\omega}\left(-m\omega x\right)\right) e^{-\frac{1}{2}m\omega x^2}$$

$$= 2x\sqrt{\frac{m\omega}{2}} \left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} \quad .$$

The eigenvalue of $|0\rangle$ for $H_0$ is obtained from

$$H_0|0\rangle = \omega\left(a^\dagger a + \tfrac{1}{2}\right)|0\rangle = \omega\left(a^\dagger a|0\rangle + \tfrac{1}{2}|0\rangle\right) = \omega\left(0 + \tfrac{1}{2}|0\rangle\right) = \tfrac{1}{2}\omega|0\rangle \quad .$$

Hence, the eigenvalue of $|0\rangle$ for $H_0$ equals $\frac{1}{2}\omega$.

**c.** The eigenvalue of $|1\rangle = a^\dagger|0\rangle$ for $H_0$ is obtained from

$$H_0|1\rangle = H_0 a^\dagger|0\rangle = \omega\left(a^\dagger a + \tfrac{1}{2}\right)a^\dagger|0\rangle = \omega\left(a^\dagger a a^\dagger|0\rangle + \tfrac{1}{2}a^\dagger|0\rangle\right) =$$

$$= \omega\left(a^\dagger\left(\left[a\ ,\ a^\dagger\right] + a^\dagger a\right)|0\rangle + \tfrac{1}{2}a^\dagger|0\rangle\right) = \omega\left(a^\dagger\left(1 + a^\dagger a\right)|0\rangle + \tfrac{1}{2}a^\dagger|0\rangle\right)$$

$$= \omega\left(a^\dagger\left(|0\rangle + 0\right) + \tfrac{1}{2}a^\dagger|0\rangle\right) = \tfrac{3}{2}a^\dagger\omega|0\rangle \quad .$$

Hence, the eigenvalue of $a^\dagger|0\rangle$ for $H_0$ equals $\frac{3}{2}\omega$.

The normalisation of $a^\dagger|0\rangle$ is determined by

$$\left|a^\dagger|0\rangle\right|^2 = \left\langle\left(a^\dagger|0\rangle\right)^\dagger\left|a^\dagger|0\rangle\right.\right\rangle =$$

$$= \left\langle 0\left|aa^\dagger\right|0\right\rangle = \left\langle 0\left|\frac{H_0}{\omega} + \tfrac{1}{2}\right|0\right\rangle = \left\langle 0\left|\tfrac{1}{2} + \tfrac{1}{2}\right|0\right\rangle = \langle 0\,|0\rangle = 1 \quad .$$

Hence, $a^\dagger|0\rangle$ is normalised.

**d.** We first determine

$$\frac{1}{\sqrt{2m\omega}}\left(a + a^\dagger\right) = \frac{1}{\sqrt{2m\omega}}\left(\sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right) + \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)\right) = \tfrac{1}{2}\left(2x\right) = x \quad .$$

Using this result, we find

$$\langle 0\,|U(x)|\,1\rangle = \langle 0\,|gx|\,1\rangle = g\langle 0\,|x|\,1\rangle = g\left\langle 0\left|\frac{1}{\sqrt{2m\omega}}\left(a + a^\dagger\right)\right|1\right\rangle =$$

$$= \frac{g}{\sqrt{2m\omega}}\left\langle 0\left|\left(a + a^\dagger\right)a^\dagger\right|0\right\rangle = \frac{g}{\sqrt{2m\omega}}\left(\left\langle 0\left|aa^\dagger\right|0\right\rangle + \left\langle 0\left|a^\dagger a^\dagger\right|0\right\rangle\right)$$

$$= \frac{g}{\sqrt{2m\omega}}\left(\left\langle 0\left|\frac{H_0}{\omega} + \tfrac{1}{2}\right|0\right\rangle + \left\langle (a|0\rangle)^\dagger\left|a^\dagger\right|0\right\rangle\right) = \frac{g}{\sqrt{2m\omega}}\left(\langle 0\,|0\rangle + 0\right) = \frac{g}{\sqrt{2m\omega}} \quad .$$

**e.** We first determine

$$x^2 = \frac{1}{2m\omega}\left(a + a^\dagger\right)^2 = \frac{1}{2m\omega}\left(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\right) \quad .$$

Using this result, we find

$$\begin{aligned} E_0^{(1)} &= \langle 0 \,| U(x) |\, 0\rangle = \langle 0 \,| gx^2 |\, 0\rangle = g\langle 0 \,| x^2 |\, 0\rangle = \\ &= g\left\langle 0 \left| \frac{1}{2m\omega}\left(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\right) \right| 0 \right\rangle \\ &= \frac{g}{2m\omega}\left(\langle 0 \,| aa |\, 0\rangle + \left\langle 0 \left| aa^\dagger + a^\dagger a \right| 0 \right\rangle + \left\langle 0 \left| a^\dagger a^\dagger \right| 0 \right\rangle\right) \\ &= \frac{g}{2m\omega}\left(0 + \left\langle 0 \left| 2\frac{H_0}{\omega} \right| 0 \right\rangle + 0\right) = \frac{g}{2m\omega} \quad . \end{aligned}$$

**f.** We may compare this result with the result of a redefinition of $\omega$ by $\omega'$, given by

$$\begin{aligned} H = H_0 + U(x) &= \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 + gx^2 = \\ &= \frac{p^2}{2m} + \tfrac{1}{2}m\left(\omega^2 + \frac{2g}{m}\right)x^2 = \frac{p^2}{2m} + \tfrac{1}{2}m\omega'^2 x^2 \quad . \end{aligned}$$

We find

$$\omega' = \sqrt{\omega^2 + \frac{2g}{m}} = \omega\sqrt{1 + \frac{2g}{m\omega^2}} \approx \omega\left(1 + \frac{g}{m\omega^2}\right) \quad .$$

Hence the ground-state energy of $H$ equals

$$\tfrac{1}{2}\omega' \approx \tfrac{1}{2}\omega + \frac{g}{2m\omega} = E_0^{(0)} + E_0^{(1)} \quad .$$

## Exercício 2

**a.**

$$\left\{-\frac{1}{2m}\frac{d^2}{dx^2}+U(x)\right\}\psi(x)=\begin{cases}-\frac{1}{2m}\frac{d^2}{dx^2}\left(e^{ikx}+Re^{-ikx}\right) & \text{for}\quad x<-L\\ \left\{-\frac{1}{2m}\frac{d^2}{dx^2}+V\right\}\left(Ae^{\kappa x}+Be^{-\kappa x}\right) & \text{for}\quad -L\le x\le L\\ -\frac{1}{2m}\frac{d^2}{dx^2}Te^{ikx} & \text{for}\quad x>L\end{cases}$$

$$=\begin{cases}\frac{k^2}{2m}\left(e^{ikx}+Re^{-ikx}\right) & \text{for}\quad x<-L\\ \left\{-\frac{\kappa^2}{2m}+V\right\}\left(Ae^{\kappa x}+Be^{-\kappa x}\right) & \text{for}\quad -L\le x\le L\\ \frac{k^2}{2m}Te^{ikx} & \text{for}\quad x>L\end{cases}$$

$$=\left.\begin{cases}E\left(e^{ikx}+Re^{-ikx}\right) & \text{for}\quad x<-L\\ E\left(Ae^{\kappa x}+Be^{-\kappa x}\right) & \text{for}\quad -L\le x\le L\\ ETe^{ikx} & \text{for}\quad x>L\end{cases}\right\}=E\psi(x)$$

**b**. The boundary conditions at $x=-L$ are given by

$$\psi(x\uparrow -L)=\psi(x\downarrow -L)\qquad\Leftrightarrow\qquad e^{-ikL}+Re^{ikL}=Ae^{-\kappa L}+Be^{\kappa L}$$

and

$$\frac{d\psi}{dx}(x\uparrow -L)=\frac{d\psi}{dx}(x\downarrow -L)\qquad\Leftrightarrow\qquad ik\left(e^{-ikL}-Re^{ikL}\right)=\kappa\left(Ae^{-\kappa L}-Be^{\kappa L}\right)\quad .$$

We rewrite the boundary conditions

$$e^{-ikL}+Re^{ikL}\;=\;Ae^{-\kappa L}+Be^{\kappa L}$$

and

$$e^{-ikL}-Re^{ikL}\;=\;\frac{-i\kappa}{k}\left(Ae^{-\kappa L}-Be^{\kappa L}\right)\quad .$$

The sum gives

$$2e^{-ikL}\;=\;A\left(1-\frac{i\kappa}{k}\right)e^{-\kappa L}+B\left(1+\frac{i\kappa}{k}\right)e^{\kappa L}\quad .$$

The difference gives

$$2Re^{ikL}\;=\;A\left(1+\frac{i\kappa}{k}\right)e^{-\kappa L}+B\left(1-\frac{i\kappa}{k}\right)e^{\kappa L}\quad .$$

Hence,

$$\begin{pmatrix}1\\ R\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}\left(1-\frac{i\kappa}{k}\right)e^{-\kappa L+ikL} & \left(1+\frac{i\kappa}{k}\right)e^{\kappa L+ikL}\\ \left(1+\frac{i\kappa}{k}\right)e^{-\kappa L-ikL} & \left(1-\frac{i\kappa}{k}\right)e^{\kappa L-ikL}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}$$

The boundary conditions at $x = +L$ are given by

$$\psi(x \uparrow L) = \psi(x \downarrow L) \qquad \Leftrightarrow \qquad Ae^{\kappa L} + Be^{-\kappa L} = Te^{ikL}$$

$$\frac{d\psi}{dx}(x \uparrow L) = \frac{d\psi}{dx}(x \downarrow L) \qquad \Leftrightarrow \qquad \kappa\left(Ae^{\kappa L} - Be^{-\kappa L}\right) = ikTe^{ikL}$$

Hence,

$$\begin{pmatrix} A \\ B \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} T\left(1 + \frac{ik}{\kappa}\right)e^{-\kappa L + ikL} \\ T\left(1 - \frac{ik}{\kappa}\right)e^{\kappa L + ikL} \end{pmatrix}$$

We thus obtain

$$\begin{pmatrix} 1 \\ R \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} \left(1 - \frac{i\kappa}{k}\right)e^{-\kappa L + ikL} & \left(1 + \frac{i\kappa}{k}\right)e^{\kappa L + ikL} \\ \left(1 + \frac{i\kappa}{k}\right)e^{-\kappa L - ikL} & \left(1 - \frac{i\kappa}{k}\right)e^{\kappa L - ikL} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} =$$

$$= \tfrac{1}{4}\begin{pmatrix} \left(1 - \frac{i\kappa}{k}\right)e^{-\kappa L + ikL} & \left(1 + \frac{i\kappa}{k}\right)e^{\kappa L + ikL} \\ \left(1 + \frac{i\kappa}{k}\right)e^{-\kappa L - ikL} & \left(1 - \frac{i\kappa}{k}\right)e^{\kappa L - ikL} \end{pmatrix}\begin{pmatrix} T\left(1 + \frac{ik}{\kappa}\right)e^{-\kappa L + ikL} \\ T\left(1 - \frac{ik}{\kappa}\right)e^{\kappa L + ikL} \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} T\left\{\left(1 - \frac{i\kappa}{k}\right)\left(1 + \frac{ik}{\kappa}\right)e^{-2\kappa L} + \left(1 + \frac{i\kappa}{k}\right)\left(1 - \frac{ik}{\kappa}\right)e^{\kappa L}\right\}e^{2ikL} \\ T\left\{\left(1 + \frac{i\kappa}{k}\right)\left(1 + \frac{ik}{\kappa}\right)e^{-2\kappa L} + \left(1 - \frac{i\kappa}{k}\right)\left(1 - \frac{ik}{\kappa}\right)e^{\kappa L}\right\} \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} T\left\{\left(2 - \frac{i\kappa}{k} + \frac{ik}{\kappa}\right)e^{-2\kappa L} + \left(2 + \frac{i\kappa}{k} - \frac{ik}{\kappa}\right)e^{\kappa L}\right\}e^{2ikL} \\ T\left\{\left(\frac{i\kappa}{k} + \frac{ik}{\kappa}\right)e^{-2\kappa L} - \left(\frac{i\kappa}{k} + \frac{ik}{\kappa}\right)e^{\kappa L}\right\} \end{pmatrix}$$

$$= \begin{pmatrix} T\left\{\tfrac{1}{2}\left(e^{\kappa L} + e^{-2\kappa L}\right) + \tfrac{1}{2}i\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right)\tfrac{1}{2}\left(e^{\kappa L} - e^{-2\kappa L}\right)\right\}e^{2ikL} \\ T\left\{-\tfrac{1}{2}i\left(\frac{\kappa}{k} + \frac{k}{\kappa}\right)\tfrac{1}{2}\left(e^{\kappa L} - e^{-2\kappa L}\right)\right\} \end{pmatrix}$$

$$= \begin{pmatrix} T\left\{\cosh(2\kappa L) + \tfrac{1}{2}i\frac{\kappa^2 - k^2}{\kappa k}\sinh(2\kappa L)\right\}e^{2ikL} \\ -\tfrac{1}{2}iT\frac{\kappa^2 + k^2}{\kappa k}\sinh(2\kappa L) \end{pmatrix}$$

$$= \begin{pmatrix} T\left\{\cosh(2\kappa L) + \tfrac{1}{2}i\frac{V - 2E}{\sqrt{E(V-E)}}\sinh(2\kappa L)\right\}e^{2ikL} \\ -\tfrac{1}{2}iT\frac{V}{\sqrt{E(V-E)}}\sinh(2\kappa L) \end{pmatrix}$$

Hence,

$$T = \frac{e^{-2ikL}}{\cosh(2\kappa L) + \frac{1}{2} i \frac{V-2E}{\sqrt{E(V-E)}} \sinh(2\kappa L)} =$$

$$= \frac{2\sqrt{E(V-E)}\, e^{-2ikL}}{2\sqrt{E(V-E)}\, \cosh(2\kappa L) + i(V-2E)\sinh(2\kappa L)}$$

and

$$R = -\tfrac{1}{2} i T \frac{V}{\sqrt{E(V-E)}} \sinh(2\kappa L) =$$

$$= \frac{-iVe^{-2ikL}\sinh(2\kappa L)}{2\sqrt{E(V-E)}\, \cosh(2\kappa L) + i(V-2E)\sinh(2\kappa L)}$$

**c.** Using the relation

$$\cosh^2(x) - \sinh^2(x) = \tfrac{1}{4}\left(e^{x} + e^{-x}\right)^2 - \tfrac{1}{4}\left(e^{x} - e^{-x}\right)^2 = 1 \quad ,$$

we determine $|R|^2$ and $|T|^2$.

$$|R|^2 = \left| \frac{-iVe^{-2ikL}\sinh(2\kappa L)}{2\sqrt{E(V-E)}\cosh(2\kappa L) + i(V-2E)\sinh(2\kappa L)} \right|^2 =$$

$$= \frac{V^2\sinh^2(2\kappa L)}{\left(2\sqrt{E(V-E)}\cosh(2\kappa L)\right)^2 + ((V-2E)\sinh(2\kappa L))^2}$$

$$= \frac{V^2\sinh^2(2\kappa L)}{4E(V-E)\left(\cosh^2(2\kappa L) - \sinh^2(2\kappa L)\right) + V^2\sinh^2(2\kappa L)}$$

$$= \frac{V^2\sinh^2(2\kappa L)}{4E(V-E) + V^2\sinh^2(2\kappa L)}$$

and

$$|T|^2 = \left| \frac{2\sqrt{E(V-E)}e^{-2ikL}}{2\sqrt{E(V-E)}\cosh(2\kappa L) + i(V-2E)\sinh(2\kappa L)} \right|^2 = \frac{4E(V-E)}{4E(V-E) + V^2\sinh^2(2\kappa L)} \quad .$$

We readily find

$$|R|^2 + |T|^2 = 1 \quad .$$

Furthermore,

$$\sinh(x) = \tfrac{1}{2}\left(e^{x} - e^{-x}\right) \xrightarrow{x \gg 1} \tfrac{1}{2}e^{x} \quad .$$

Hence, for $2\kappa L \gg 1$ we obtain

$$|T|^2 = \frac{4E(V-E)}{4E(V-E) + V^2 \sinh^2(2\kappa L)} \approx \frac{4E(V-E)}{4E(V-E) + V^2 \frac{1}{4} e^{4\kappa L}} =$$

$$= \frac{16E(V-E)e^{-4\kappa L}}{16E(V-E)e^{-4\kappa L} + V^2} \approx \frac{16E(V-E)e^{-4\kappa L}}{V^2} \quad .$$

**d.** We first determine $\kappa$ for this case.

$$\kappa = \sqrt{2m(V-E)} =$$

$$= \sqrt{2 \times (0.5 \times 10^6 \text{ eV}) \times ((2.0 \text{ eV}) - (1.0 \text{ eV}))} = 1.0 \times 10^3 \text{ eV} \quad .$$

Then we determine $2\kappa L$.

$$2\kappa L = \kappa \times (2L) = \left(1.0 \times 10^3 \text{ eV}\right) \times \left(1.5 \times 10^{-9} \text{ m}\right) =$$

$$= 1.5 \times 10^{-6} \text{ eVm} = \frac{1.5 \times 10^{-6} \text{ eVm}}{2.0 \times 10^{-7} \text{ eVm}} = 7.5 \gg 1 \quad .$$

We are thus allowed to use the approximations for $2\kappa L \gg 1$ in order to determine $|T|^2$.

$$|T|^2 \approx \frac{16E(V-E)e^{-4\kappa L}}{V^2} =$$

$$= \frac{16\,(1.0 \text{ eV})\,((2.0 \text{ eV}) - (1.0 \text{ eV}))\, e^{-2 \times 7.5}}{(2.0 \text{ eV})^2} = 4.0 \times e^{-15} = 1.33 \times 10^{-6} \quad .$$

See also http://en.wikipedia.org/wiki/File:EffetTunnel.gif at WikipediA.

## Exercício 3

**a.**

$$H\begin{pmatrix}1\\ \pm r\end{pmatrix}=\begin{pmatrix}A & \frac{B}{r}\\ Br & A\end{pmatrix}\begin{pmatrix}1\\ \pm r\end{pmatrix}=\begin{pmatrix}A\pm B\\ Br\pm Ar\end{pmatrix}=(A\pm B)\begin{pmatrix}1\\ \pm r\end{pmatrix}\quad .$$

Hence, the eigenvalues are given by $E_\pm = A \pm B$. For the normalization we determine

$$\phi_\pm^\dagger\phi_\pm=\frac{1}{1+|r|^2}\begin{pmatrix}1 & \pm r^*\end{pmatrix}\begin{pmatrix}1\\ \pm r\end{pmatrix}=\frac{1}{1+|r|^2}(1+r^*r)=1\quad .$$

**b.**

$$\frac{1}{E_+-E_-}\left\{(E_+-H)\,e^{-iE_-t}-(E_--H)\,e^{-iE_+t}\right\}\psi(0)=$$

$$=\frac{1}{E_+-E_-}\left\{(E_+-H)\,e^{-iE_-t}-(E_--H)\,e^{-iE_+t}\right\}(c_+\phi_++c_-\phi_-)$$

$$=\frac{c_+}{E_+-E_-}\left\{(E_+-E_+)\,e^{-iE_-t}-(E_--E_+)\,e^{-iE_+t}\right\}\phi_++$$

$$+\frac{c_-}{E_+-E_-}\left\{(E_+-E_-)\,e^{-iE_-t}-(E_--E_-)\,e^{-iE_+t}\right\}\phi_-$$

$$=c_+e^{-iE_+t}\phi_++c_-e^{-iE_-t}\phi_-=c_+e^{-iHt}\phi_++c_-e^{-iHt}\phi_-$$

$$=e^{-iHt}(c_+\phi_++c_-\phi_-)=e^{-iHt}\psi(0)\quad .$$

**c.** First, we use

$$\langle\downarrow|H|\uparrow\rangle=\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}A & \frac{B}{r}\\ Br & A\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}A\\ Br\end{pmatrix}=Br\quad .$$

Next, we use

$$\left\langle\downarrow\left|e^{-iHt}\right|\uparrow\right\rangle=$$

$$=\frac{1}{E_+-E_-}\left\{(\langle\downarrow|E_+|\uparrow\rangle-\langle\downarrow|H|\uparrow\rangle)\,e^{-iE_-t}-(\langle\downarrow|E_-|\uparrow\rangle-\langle\downarrow|H|\uparrow\rangle)\,e^{-iE_+t}\right\}$$

$$=\frac{1}{E_+-E_-}\left\{(E_+\langle\downarrow|\uparrow\rangle-Br)\,e^{-iE_-t}-(E_-\langle\downarrow|\uparrow\rangle-Br)\,e^{-iE_+t}\right\}$$

$$=\frac{1}{E_+-E_-}\left\{(0-Br)\,e^{-iE_-t}-(0-Br)\,e^{-iE_+t}\right\}$$

$$=\frac{Br}{E_+-E_-}\left\{-e^{-iE_-t}+e^{-iE_+t}\right\}=\frac{Br}{2B}\left\{-e^{-iE_-t}+e^{-iE_+t}\right\}$$

$$=\frac{r}{2}\left\{-e^{-iE_-t}+e^{-iE_+t}\right\}\quad .$$

Furthermore

$$E_+ = A + B = M - \frac{1}{2} i\gamma_1 \quad \text{and} \quad E_- = A - B = M + \Delta M - \frac{1}{2} i\gamma_2 \quad .$$

Hence the transition amplitude equals

$$\begin{aligned}
&\left\langle \downarrow \left| e^{-iHt} \right| \uparrow \right\rangle = \\
&= \frac{r}{2} \left\{ -e^{-i\left( M + \Delta M - \frac{1}{2} i\gamma_2 \right) t} + e^{-i\left( M - \frac{1}{2} i\gamma_1 \right) t} \right\} \\
&= \frac{r}{2} \left\{ -e^{-i(M + \Delta M)t - \frac{1}{2}\gamma_2 t} + e^{-iMt - \frac{1}{2}\gamma_1 t} \right\} \\
&= \frac{r}{2} e^{-iMt} \left\{ -e^{-i\Delta M t - \frac{1}{2}\gamma_2 t} + e^{-\frac{1}{2}\gamma_1 t} \right\} \quad .
\end{aligned}$$

The decay rate is given by

$$\begin{aligned}
&\left| \left\langle \downarrow \left| e^{-iHt} \right| \uparrow \right\rangle \right|^2 = \\
&= \tfrac{1}{4} |r|^2 \left\{ -e^{-i\Delta M t - \frac{1}{2}\gamma_2 t} + e^{-\frac{1}{2}\gamma_1 t} \right\} \left\{ -e^{i\Delta M t - \frac{1}{2}\gamma_2 t} + e^{-\frac{1}{2}\gamma_1 t} \right\} \\
&= \tfrac{1}{4} |r|^2 \left\{ e^{-\gamma_1 t} + e^{-\gamma_2 t} - e^{-i\Delta M t - \frac{1}{2}(\gamma_1 + \gamma_2) t} - e^{i\Delta M t - \frac{1}{2}(\gamma_1 + \gamma_2) t} \right\} \\
&= \tfrac{1}{4} |r|^2 \left\{ e^{-\gamma_1 t} + e^{-\gamma_2 t} - e^{-\frac{1}{2}(\gamma_1 + \gamma_2) t} \left( e^{i\Delta M t} + e^{-i\Delta M t} \right) \right\} \\
&= \tfrac{1}{4} |r|^2 \left\{ e^{-\gamma_1 t} + e^{-\gamma_2 t} - 2e^{-\frac{1}{2}(\gamma_1 + \gamma_2) t} \cos(\Delta M t) \right\} \quad .
\end{aligned}$$

**d.** From the inset of the figure we read

$$\Delta M T = 2\pi \quad \text{for} \quad T = 12.0 \times 10^{-10} \text{ s} \quad .$$

Hence,

$$\begin{aligned}
&\Delta M = 2\pi / (12.0 \times 10^{-10} \text{ s}) = 0.52 \times 10^{10} \text{ s}^{-1} = \\
&\quad = 0.52 \times 10^{10} \text{ s}^{-1} \times 0.66 \times 10^{-15} \text{ eVs} = 3.5 \times 10^{-6} \text{ eV} \quad .
\end{aligned}$$

## Exercício 4

**a.** Using $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_+ - S_-)$, we obtain

$$\begin{aligned}
S^2 &= S_x^2 + S_y^2 + S_z^2 \\
&= \left\{\frac{1}{2}(S_+ + S_-)\right\}^2 + \left\{\frac{1}{2i}(S_+ - S_-)\right\}^2 + S_z^2 \\
&= \left\{\frac{1}{4}\left(S_+^2 + S_+S_- + S_-S_+ + S_-^2\right)\right\} + \left\{-\frac{1}{4}\left(S_+^2 - S_+S_- - S_-S_+ + S_-^2\right)\right\} + S_z^2 \\
&= \frac{1}{2}(S_+S_- + S_-S_+) + S_z^2 \\
&= \frac{1}{2}(S_+S_- - S_-S_+ + 2S_-S_+) + S_z^2 \\
&= \frac{1}{2}(2S_z + 2S_-S_+) + S_z^2 = S_z + S_-S_+ + S_z^2 \quad .
\end{aligned}$$

Note that, since $L_\pm = L_x \pm iL_y$ and $[L_+, L_-] = 2L_z$ and, moreover, $J_\pm = J_x \pm iJ_y$ and $[J_+, J_-] = 2J_z$, one also has

$$L^2 = L_x^2 + L_y^2 + L_z^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2 = L_-L_+ + L_z + L_z^2$$

and

$$J^2 = J_x^2 + J_y^2 + J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 = J_-J_+ + J_z + J_z^2 \quad .$$

We will use those expressions in the following.

**b.**

$$\begin{aligned}
L_-L_+Y_m^{(2)} &= L_-\left(L_+Y_m^{(2)}\right) = L_-\left(\sqrt{6 - m(m+1)}\, Y_{m+1}^{(2)}\right) \\
&= \sqrt{6 - m(m+1)}\, L_-Y_{m+1}^{(2)} \\
&= \sqrt{6 - m(m+1)}\, \sqrt{6 - (m+1)(m+1-1)}\, Y_{m+1-1}^{(2)} \\
&= (6 - m(m+1))\; Y_m^{(2)} = \left(6 - m^2 - m\right)\; Y_m^{(2)} = \left(6 - L_z^2 - L_z\right) Y_m^{(2)} \quad .
\end{aligned}$$

As a consequence, one has

$$\begin{aligned}
L^2Y_m^{(2)} &= \left(L_-L_+ + L_z + L_z^2\right) Y_m^{(2)} \\
&= \left(6 - L_z^2 - L_z + L_z + L_z^2\right) Y_m^{(2)} = 6Y_m^{(2)} \quad .
\end{aligned}$$

**c.** At the $2 \times 2$ basis $L^2 = L^2\mathbf{1}$. Hence,

$$L^2\begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} L^2 & 0 \\ 0 & L^2 \end{pmatrix}\begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} L^2Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} 6Y_m^{(2)} \\ 0 \end{pmatrix} = 6\begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix}$$

and

$$L^2 \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} L^2 & 0 \\ 0 & L^2 \end{pmatrix} \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ L^2 Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 6Y_m^{(2)} \end{pmatrix} = 6 \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} \quad .$$

Similar for $L_z$:

$$L_z \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} L_z & 0 \\ 0 & L_z \end{pmatrix} \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} L_z Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} mY_m^{(2)} \\ 0 \end{pmatrix} = m \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix}$$

and

$$L_z \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} L_z & 0 \\ 0 & L_z \end{pmatrix} \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ L_z Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ mY_m^{(2)} \end{pmatrix} = m \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} \quad .$$

For $S^2$ one has

$$\begin{aligned} S^2 &= S_- S_+ + S_z + S_z^2 \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} + \frac{1}{4} & 0 \\ 0 & 1 - \frac{1}{2} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} = \frac{3}{4}\mathbf{1} \quad . \end{aligned}$$

Hence,

$$S^2 \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} = \frac{3}{4} \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} \quad \text{and} \quad S^2 \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} = \frac{3}{4} \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} \quad .$$

Furthermore, for $S_z$ we have

$$S_z \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} Y_m^{(2)} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} Y_m^{(2)} \\ 0 \end{pmatrix}$$

and

$$S_z \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} Y_m^{(2)} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ Y_m^{(2)} \end{pmatrix} \quad .$$

**d.** Using the results of (**c**), with $L^2 = \ell(\ell+1) = 6$, hence $\ell = 2$ and $S^2 = s(s+1) = \frac{3}{4}$, hence $s = \frac{1}{2}$, we obtain

| basis | $\ell_z$ | $s_z$ | state |
|---|---|---|---|
| $\begin{pmatrix} Y^{(2)}_{+2} \\ 0 \end{pmatrix}$ | $+2$ | $+\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=+\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=+2\rangle \quad = \left\|+\frac{1}{2}\right\rangle \otimes \|+2\rangle$ |
| $\begin{pmatrix} Y^{(2)}_{+1} \\ 0 \end{pmatrix}$ | $+1$ | $+\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=+\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=+1\rangle \quad = \left\|+\frac{1}{2}\right\rangle \otimes \|+1\rangle$ |
| $\begin{pmatrix} Y^{(2)}_{0} \\ 0 \end{pmatrix}$ | $0$ | $+\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=+\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=0\rangle \quad = \left\|+\frac{1}{2}\right\rangle \otimes \|0\rangle$ |
| $\begin{pmatrix} Y^{(2)}_{-1} \\ 0 \end{pmatrix}$ | $-1$ | $+\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=+\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=-1\rangle \quad = \left\|+\frac{1}{2}\right\rangle \otimes \|-1\rangle$ |
| $\begin{pmatrix} Y^{(2)}_{-2} \\ 0 \end{pmatrix}$ | $-2$ | $+\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=+\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=-2\rangle \quad = \left\|+\frac{1}{2}\right\rangle \otimes \|-2\rangle$ |
| $\begin{pmatrix} 0 \\ Y^{(2)}_{+2} \end{pmatrix}$ | $+2$ | $-\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=-\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=+2\rangle \quad = \left\|-\frac{1}{2}\right\rangle \otimes \|+2\rangle$ |
| $\begin{pmatrix} 0 \\ Y^{(2)}_{+1} \end{pmatrix}$ | $+1$ | $-\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=-\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=+1\rangle \quad = \left\|-\frac{1}{2}\right\rangle \otimes \|+1\rangle$ |
| $\begin{pmatrix} 0 \\ Y^{(2)}_{0} \end{pmatrix}$ | $0$ | $-\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=-\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=0\rangle \quad = \left\|-\frac{1}{2}\right\rangle \otimes \|0\rangle$ |
| $\begin{pmatrix} 0 \\ Y^{(2)}_{-1} \end{pmatrix}$ | $-1$ | $-\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=-\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=-1\rangle \quad = \left\|-\frac{1}{2}\right\rangle \otimes \|-1\rangle$ |
| $\begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix}$ | $-2$ | $-\frac{1}{2}$ | $\left\|s=\frac{1}{2}, s_z=-\frac{1}{2}\right\rangle \otimes \|\ell=2, \ell_z=-2\rangle \quad = \left\|-\frac{1}{2}\right\rangle \otimes \|-2\rangle$ |

## Exercício 5

**a.** Using the results of exercises (4**a** e **b**) one has for $J^2$ the following resul.

$$
\begin{aligned}
J^2 &= J_-J_+ + J_z + J_z^2 \\
&= \begin{pmatrix} L_- & 0 \\ 1 & L_- \end{pmatrix}\begin{pmatrix} L_+ & 1 \\ 0 & L_+ \end{pmatrix} + \begin{pmatrix} L_z + \frac{1}{2} & 0 \\ 0 & L_z - \frac{1}{2} \end{pmatrix} + \begin{pmatrix} L_z + \frac{1}{2} & 0 \\ 0 & L_z - \frac{1}{2} \end{pmatrix}^2 \\
&= \begin{pmatrix} L_-L_+ & L_- \\ L_+ & 1 + L_-L_+ \end{pmatrix} + \begin{pmatrix} L_z + \frac{1}{2} & 0 \\ 0 & L_z - \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \left(L_z + \frac{1}{2}\right)^2 & 0 \\ 0 & \left(L_z - \frac{1}{2}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{4} + L_-L_+ + 2L_z + L_z^2 & L_- \\ L_+ & \frac{3}{4} + L_-L_+ + L_z^2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{4} + L^2 + L_z & L_- \\ L_+ & \frac{3}{4} + L^2 - L_z \end{pmatrix} = \begin{pmatrix} \frac{3}{4} + 6 + L_z & L_- \\ L_+ & \frac{3}{4} + 6 - L_z \end{pmatrix} \\
&= \begin{pmatrix} \frac{27}{4} + L_z & L_- \\ L_+ & \frac{27}{4} - L_z \end{pmatrix} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
J^2 \begin{pmatrix} aY_m^{(2)} \\ bY_{m+1}^{(2)} \end{pmatrix} &= \begin{pmatrix} \frac{27}{4} + L_z & L_- \\ L_+ & \frac{27}{4} - L_z \end{pmatrix}\begin{pmatrix} aY_m^{(2)} \\ bY_{m+1}^{(2)} \end{pmatrix} = \\
&= \begin{pmatrix} a\left(\frac{27}{4} + L_z\right)Y_m^{(2)} + bL_-Y_{m+1}^{(2)} \\ aL_+Y_m^{(2)} + b\left(\frac{27}{4} - L_z\right)Y_{m+1}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} a\left(\frac{27}{4} + m\right)Y_m^{(2)} + b\sqrt{6 - (m+1)(m+1-1)}\,Y_{m+1-1}^{(2)} \\ a\sqrt{6 - m(m+1)}\,Y_{m+1}^{(2)} + b\left(\frac{27}{4} - (m+1)\right)Y_{m+1}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} \left\{a\left(\frac{27}{4} + m\right) + b\sqrt{6 - m(m+1)}\right\}\, Y_m^{(2)} \\ \left\{a\sqrt{6 - m(m+1)} + b\left(\frac{23}{4} - m\right)\right\}\, Y_{m+1}^{(2)} \end{pmatrix} .
\end{aligned}
$$

**b.**

1. $a = 1$, $b = 0$, $m = +2$ for $\begin{pmatrix} Y_{+2}^{(2)} \\ 0 \end{pmatrix}$, hence

$$a\left(\frac{27}{4} + m\right) + b\sqrt{6 - m(m+1)} = 1 \times \left(\frac{27}{4} + 2\right) + 0 = \frac{35}{4}$$

and

$$a\sqrt{6 - m(m+1)} + b\left(\frac{23}{4} - m\right) = 1 \times \sqrt{6-6} + 0 = 0 \ .$$

Consequently

$$J^2\begin{pmatrix} Y^{(2)}_{+2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{35}{4}Y^{(2)}_{+2} \\ 0 \end{pmatrix} = \frac{5}{2}\left(\frac{5}{2}+1\right)\begin{pmatrix} Y^{(2)}_{+2} \\ 0 \end{pmatrix} \quad .$$

Hence $j = \frac{5}{2}$.
Furthermore

$$J_z\begin{pmatrix} Y^{(2)}_{+2} \\ 0 \end{pmatrix} = \begin{pmatrix} L_z+\frac{1}{2} & 0 \\ 0 & L_z-\frac{1}{2} \end{pmatrix}\begin{pmatrix} Y^{(2)}_{+2} \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \left(L_z+\frac{1}{2}\right)Y^{(2)}_{+2} \\ 0 \end{pmatrix} = \begin{pmatrix} \left(2+\frac{1}{2}\right)Y^{(2)}_{+2} \\ 0 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} Y^{(2)}_{+2} \\ 0 \end{pmatrix} \quad .$$

Hence, $j_z = +\frac{5}{2}$.

2. $a = \sqrt{\frac{4}{5}}$, $b = \sqrt{\frac{1}{5}}$, $m = +1$ for $\begin{pmatrix} \sqrt{\frac{4}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\,Y^{(2)}_{+2} \end{pmatrix}$, hence

$$a\left(\frac{27}{4}+m\right)+b\sqrt{6-m(m+1)} = \sqrt{\frac{4}{5}}\left(\frac{27}{4}+2\right)+\sqrt{\frac{1}{5}}\sqrt{6-2} = \frac{35}{4}\sqrt{\frac{4}{5}}$$

and

$$a\sqrt{6-m(m+1)}+b\left(\frac{23}{4}-m\right) = \sqrt{\frac{4}{5}}\sqrt{6-2}+\sqrt{\frac{1}{5}}\left(\frac{23}{4}-1\right) = \frac{35}{4}\sqrt{\frac{1}{5}} \quad .$$

Consequently

$$J^2\begin{pmatrix} \sqrt{\frac{4}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\,Y^{(2)}_{+2} \end{pmatrix} = \begin{pmatrix} \frac{35}{4}\sqrt{\frac{4}{5}}\,Y^{(2)}_{+1} \\ \frac{35}{4}\sqrt{\frac{1}{5}}\,Y^{(2)}_{+2} \end{pmatrix} = \frac{5}{2}\left(\frac{5}{2}+1\right)\begin{pmatrix} \sqrt{\frac{4}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\,Y^{(2)}_{+2} \end{pmatrix} \quad .$$

Hence $j = \frac{5}{2}$.
Furthermore

$$J_z\begin{pmatrix} \sqrt{\frac{4}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\,Y^{(2)}_{+2} \end{pmatrix} = \begin{pmatrix} L_z+\frac{1}{2} & 0 \\ 0 & L_z-\frac{1}{2} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{4}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\,Y^{(2)}_{+2} \end{pmatrix} =$$

$$= \begin{pmatrix} \sqrt{\frac{4}{5}}\left(L_z+\frac{1}{2}\right)Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\left(L_z-\frac{1}{2}\right)Y^{(2)}_{+2} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{4}{5}}\left(1+\frac{1}{2}\right)Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\left(2-\frac{1}{2}\right)Y^{(2)}_{+2} \end{pmatrix} = \frac{3}{2}\begin{pmatrix} \sqrt{\frac{4}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\,Y^{(2)}_{+2} \end{pmatrix} \quad .$$

Hence, $j_z = +\frac{3}{2}$.

3. $a = \sqrt{\frac{3}{5}}$, $b = \sqrt{\frac{2}{5}}$, $m = 0$ for $\begin{pmatrix} \sqrt{\frac{3}{5}}\,Y^{(2)}_{0} \\ \sqrt{\frac{2}{5}}\,Y^{(2)}_{+1} \end{pmatrix}$, hence

$$a\left(\frac{27}{4}+m\right)+b\sqrt{6-m(m+1)} = \sqrt{\frac{3}{5}}\left(\frac{27}{4}+0\right)+\sqrt{\frac{2}{5}}\sqrt{6-0} = \frac{35}{4}\sqrt{\frac{3}{5}}$$

and

$$a\sqrt{6-m(m+1)}+b\left(\tfrac{23}{4}-m\right)=\sqrt{\tfrac{3}{5}}\sqrt{6-0}+\sqrt{\tfrac{2}{5}}\left(\tfrac{23}{4}-0\right)=\tfrac{35}{4}\sqrt{\tfrac{2}{5}}\ .$$

Consequently

$$J^2\begin{pmatrix}\sqrt{\frac{3}{5}}\,Y_0^{(2)}\\ \sqrt{\frac{2}{5}}\,Y_{+1}^{(2)}\end{pmatrix}=\begin{pmatrix}\frac{35}{4}\sqrt{\frac{3}{5}}\,Y_0^{(2)}\\ \frac{35}{4}\sqrt{\frac{2}{5}}\,Y_{+1}^{(2)}\end{pmatrix}=\tfrac{5}{2}\left(\tfrac{5}{2}+1\right)\begin{pmatrix}\sqrt{\frac{3}{5}}\,Y_0^{(2)}\\ \sqrt{\frac{2}{5}}\,Y_{+1}^{(2)}\end{pmatrix}\ .$$

Hence $j=\frac{5}{2}$.
Furthermore

$$J_z\begin{pmatrix}\sqrt{\frac{3}{5}}\,Y_0^{(2)}\\ \sqrt{\frac{2}{5}}\,Y_{+1}^{(2)}\end{pmatrix}=\begin{pmatrix}L_z+\frac{1}{2} & 0\\ 0 & L_z-\frac{1}{2}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{3}{5}}\,Y_0^{(2)}\\ \sqrt{\frac{2}{5}}\,Y_{+1}^{(2)}\end{pmatrix}=$$

$$=\begin{pmatrix}\sqrt{\frac{3}{5}}\left(L_z+\frac{1}{2}\right)Y_0^{(2)}\\ \sqrt{\frac{2}{5}}\left(L_z-\frac{1}{2}\right)Y_{+1}^{(2)}\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{3}{5}}\left(0+\frac{1}{2}\right)Y_0^{(2)}\\ \sqrt{\frac{2}{5}}\left(1-\frac{1}{2}\right)Y_{+1}^{(2)}\end{pmatrix}=\tfrac{1}{2}\begin{pmatrix}\sqrt{\frac{3}{5}}\,Y_0^{(2)}\\ \sqrt{\frac{2}{5}}\,Y_{+1}^{(2)}\end{pmatrix}\ .$$

Hence, $j_z=+\frac{1}{2}$.

4. $a=\sqrt{\frac{2}{5}}$, $b=\sqrt{\frac{3}{5}}$, $m=-1$ for $\begin{pmatrix}\sqrt{\frac{2}{5}}\,Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\,Y_0^{(2)}\end{pmatrix}$, hence

$$a\left(\tfrac{27}{4}+m\right)+b\sqrt{6-m(m+1)}=\sqrt{\tfrac{2}{5}}\left(\tfrac{27}{4}-1\right)+\sqrt{\tfrac{3}{5}}\sqrt{6-0}=\tfrac{35}{4}\sqrt{\tfrac{2}{5}}$$

and

$$a\sqrt{6-m(m+1)}+b\left(\tfrac{23}{4}-m\right)=\sqrt{\tfrac{2}{5}}\sqrt{6-0}+\sqrt{\tfrac{3}{5}}\left(\tfrac{23}{4}+1\right)=\tfrac{35}{4}\sqrt{\tfrac{3}{5}}\ .$$

Consequently

$$J^2\begin{pmatrix}\sqrt{\frac{2}{5}}\,Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\,Y_0^{(2)}\end{pmatrix}=\begin{pmatrix}\frac{35}{4}\sqrt{\frac{2}{5}}\,Y_{-1}^{(2)}\\ \frac{35}{4}\sqrt{\frac{3}{5}}\,Y_0^{(2)}\end{pmatrix}=\tfrac{5}{2}\left(\tfrac{5}{2}+1\right)\begin{pmatrix}\sqrt{\frac{2}{5}}\,Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\,Y_0^{(2)}\end{pmatrix}\ .$$

Hence $j=\frac{5}{2}$.
Furthermore

$$J_z\begin{pmatrix}\sqrt{\frac{2}{5}}\,Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\,Y_0^{(2)}\end{pmatrix}=\begin{pmatrix}L_z+\frac{1}{2} & 0\\ 0 & L_z-\frac{1}{2}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2}{5}}\,Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\,Y_0^{(2)}\end{pmatrix}=$$

$$=\begin{pmatrix}\sqrt{\frac{2}{5}}\left(L_z+\frac{1}{2}\right)Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\left(L_z-\frac{1}{2}\right)Y_0^{(2)}\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{2}{5}}\left(-1+\frac{1}{2}\right)Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\left(0-\frac{1}{2}\right)Y_0^{(2)}\end{pmatrix}=-\tfrac{1}{2}\begin{pmatrix}\sqrt{\frac{2}{5}}\,Y_{-1}^{(2)}\\ \sqrt{\frac{3}{5}}\,Y_0^{(2)}\end{pmatrix}\ .$$

Hence, $j_z=-\frac{1}{2}$.

5. $a=\sqrt{\frac{1}{5}}$, $b=\sqrt{\frac{4}{5}}$, $m=-2$ for $\begin{pmatrix} \sqrt{\frac{1}{5}}\,Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{-1} \end{pmatrix}$, hence

$$a\left(\tfrac{27}{4}+m\right)+b\sqrt{6-m(m+1)}=\sqrt{\tfrac{1}{5}}\left(\tfrac{27}{4}-2\right)+\sqrt{\tfrac{4}{5}}\sqrt{6-2}=\tfrac{35}{4}\sqrt{\tfrac{1}{5}}$$

and

$$a\sqrt{6-m(m+1)}+b\left(\tfrac{23}{4}-m\right)=\sqrt{\tfrac{1}{5}}\sqrt{6-2}+\sqrt{\tfrac{4}{5}}\left(\tfrac{23}{4}+2\right)=\tfrac{35}{4}\sqrt{\tfrac{4}{5}}\ .$$

Consequently

$$J^2\begin{pmatrix} \sqrt{\frac{1}{5}}\,Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{-1} \end{pmatrix}=\begin{pmatrix} \frac{35}{4}\sqrt{\frac{1}{5}}\,Y^{(2)}_{-2} \\ \frac{35}{4}\sqrt{\frac{4}{5}}\,Y^{(2)}_{-1} \end{pmatrix}=\tfrac{5}{2}\left(\tfrac{5}{2}+1\right)\begin{pmatrix} \sqrt{\frac{1}{5}}\,Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{-1} \end{pmatrix}\ .$$

Hence $j=\frac{5}{2}$.
Furthermore

$$J_z\begin{pmatrix} \sqrt{\frac{1}{5}}\,Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{-1} \end{pmatrix}=\begin{pmatrix} L_z+\frac{1}{2} & 0 \\ 0 & L_z-\frac{1}{2} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{1}{5}}\,Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{-1} \end{pmatrix}=$$

$$=\begin{pmatrix} \sqrt{\frac{1}{5}}\left(L_z+\frac{1}{2}\right)Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\left(L_z-\frac{1}{2}\right)Y^{(2)}_{-1} \end{pmatrix}=\begin{pmatrix} \sqrt{\frac{1}{5}}\left(-2+\frac{1}{2}\right)Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\left(-1-\frac{1}{2}\right)Y^{(2)}_{-1} \end{pmatrix}=-\tfrac{3}{2}\begin{pmatrix} \sqrt{\frac{1}{5}}\,Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{-1} \end{pmatrix}\ .$$

Hence, $j_z=-\frac{3}{2}$.

6. $a=0$, $b=1$, $m=-3$ for $\begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix}$, hence

$$a\left(\tfrac{27}{4}+m\right)+b\sqrt{6-m(m+1)}=0+1\times\sqrt{6-6}=0$$

and

$$a\sqrt{6-m(m+1)}+b\left(\tfrac{23}{4}-m\right)=0+1\times\left(\tfrac{23}{4}+3\right)=\tfrac{35}{4}\ .$$

Consequently

$$J^2\begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix}=\begin{pmatrix} 0 \\ \frac{35}{4}\,Y^{(2)}_{-2} \end{pmatrix}=\tfrac{5}{2}\left(\tfrac{5}{2}+1\right)\begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix}\ .$$

Hence $j=\frac{5}{2}$.
Furthermore

$$J_z\begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix}=\begin{pmatrix} L_z+\frac{1}{2} & 0 \\ 0 & L_z-\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix}=$$

$$=\begin{pmatrix} 0 \\ \left(L_z-\frac{1}{2}\right)Y^{(2)}_{-2} \end{pmatrix}=\begin{pmatrix} 0 \\ \left(-2-\frac{1}{2}\right)Y^{(2)}_{-2} \end{pmatrix}=-\tfrac{5}{2}\begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix}\ .$$

Hence, $j_z=-\frac{5}{2}$.

7. $a=-\sqrt{\frac{1}{5}}$, $b=\sqrt{\frac{4}{5}}$, $m=+1$ for $\begin{pmatrix} -\sqrt{\frac{1}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{+2} \end{pmatrix}$, hence

$$a\left(\frac{27}{4}+m\right)+b\sqrt{6-m(m+1)}=-\sqrt{\frac{1}{5}}\left(\frac{27}{4}+2\right)+\sqrt{\frac{4}{5}}\sqrt{6-2}=-\frac{15}{4}\sqrt{\frac{1}{5}}$$

and

$$a\sqrt{6-m(m+1)}+b\left(\frac{23}{4}-m\right)=-\sqrt{\frac{1}{5}}\sqrt{6-2}+\sqrt{\frac{4}{5}}\left(\frac{23}{4}-1\right)=\frac{15}{4}\sqrt{\frac{4}{5}}\ \ .$$

Consequently

$$J^2\begin{pmatrix} -\sqrt{\frac{1}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{+2} \end{pmatrix}=\begin{pmatrix} -\frac{15}{4}\sqrt{\frac{1}{5}}\,Y^{(2)}_{+1} \\ \frac{15}{4}\sqrt{\frac{4}{5}}\,Y^{(2)}_{+2} \end{pmatrix}=\frac{3}{2}\left(\frac{3}{2}+1\right)\begin{pmatrix} -\sqrt{\frac{1}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{+2} \end{pmatrix}\ \ .$$

Hence $j=\frac{3}{2}$.
Furthermore

$$J_z\begin{pmatrix} -\sqrt{\frac{1}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{+2} \end{pmatrix}=\begin{pmatrix} L_z+\frac{1}{2} & 0 \\ 0 & L_z-\frac{1}{2} \end{pmatrix}\begin{pmatrix} -\sqrt{\frac{1}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{+2} \end{pmatrix}=$$

$$=\begin{pmatrix} -\sqrt{\frac{1}{5}}\left(L_z+\frac{1}{2}\right)Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\left(L_z-\frac{1}{2}\right)Y^{(2)}_{+2} \end{pmatrix}=\begin{pmatrix} -\sqrt{\frac{1}{5}}\left(1+\frac{1}{2}\right)Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\left(2-\frac{1}{2}\right)Y^{(2)}_{+2} \end{pmatrix}=\frac{3}{2}\begin{pmatrix} -\sqrt{\frac{1}{5}}\,Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\,Y^{(2)}_{+2} \end{pmatrix}\ \ .$$

Hence, $j_z=+\frac{3}{2}$.

8. $a=-\sqrt{\frac{2}{5}}$, $b=\sqrt{\frac{3}{5}}$, $m=0$ for $\begin{pmatrix} -\sqrt{\frac{2}{5}}\,Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\,Y^{(2)}_{+1} \end{pmatrix}$, hence

$$a\left(\frac{27}{4}+m\right)+b\sqrt{6-m(m+1)}=-\sqrt{\frac{2}{5}}\left(\frac{27}{4}+0\right)+\sqrt{\frac{3}{5}}\sqrt{6-0}=\frac{15}{4}\sqrt{\frac{2}{5}}$$

and

$$a\sqrt{6-m(m+1)}+b\left(\frac{23}{4}-m\right)=-\sqrt{\frac{2}{5}}\sqrt{6-0}+\sqrt{\frac{3}{5}}\left(\frac{23}{4}-0\right)=\frac{15}{4}\sqrt{\frac{2}{5}}\ \ .$$

Consequently

$$J^2\begin{pmatrix} -\sqrt{\frac{2}{5}}\,Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\,Y^{(2)}_{+1} \end{pmatrix}=\begin{pmatrix} -\frac{15}{4}\sqrt{\frac{2}{5}}\,Y^{(2)}_{0} \\ \frac{15}{4}\sqrt{\frac{3}{5}}\,Y^{(2)}_{+1} \end{pmatrix}=\frac{3}{2}\left(\frac{3}{2}+1\right)\begin{pmatrix} -\sqrt{\frac{2}{5}}\,Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\,Y^{(2)}_{+1} \end{pmatrix}\ \ .$$

Hence $j=\frac{3}{2}$.
Furthermore

$$J_z\begin{pmatrix} -\sqrt{\frac{2}{5}}\,Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\,Y^{(2)}_{+1} \end{pmatrix}=\begin{pmatrix} L_z+\frac{1}{2} & 0 \\ 0 & L_z-\frac{1}{2} \end{pmatrix}\begin{pmatrix} -\sqrt{\frac{2}{5}}\,Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\,Y^{(2)}_{+1} \end{pmatrix}=$$

$$=\begin{pmatrix} -\sqrt{\frac{2}{5}}\left(L_z+\frac{1}{2}\right)Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\left(L_z-\frac{1}{2}\right)Y^{(2)}_{+1} \end{pmatrix}=\begin{pmatrix} -\sqrt{\frac{2}{5}}\left(0+\frac{1}{2}\right)Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\left(1-\frac{1}{2}\right)Y^{(2)}_{+1} \end{pmatrix}=\frac{1}{2}\begin{pmatrix} -\sqrt{\frac{2}{5}}\,Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\,Y^{(2)}_{+1} \end{pmatrix}\ \ .$$

Hence, $j_z = +\frac{1}{2}$.

9. $a = -\sqrt{\frac{3}{5}}$, $b = \sqrt{\frac{2}{5}}$, $m = -1$ for $\begin{pmatrix} -\sqrt{\frac{3}{5}}\, Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\, Y_0^{(2)} \end{pmatrix}$, hence

$$a\left(\tfrac{27}{4} + m\right) + b\sqrt{6 - m(m+1)} = -\sqrt{\tfrac{3}{5}}\left(\tfrac{27}{4} - 1\right) + \sqrt{\tfrac{2}{5}}\sqrt{6-0} = -\tfrac{15}{4}\sqrt{\tfrac{3}{5}}$$

and

$$a\sqrt{6 - m(m+1)} + b\left(\tfrac{23}{4} - m\right) = -\sqrt{\tfrac{3}{5}}\sqrt{6-0} + \sqrt{\tfrac{2}{5}}\left(\tfrac{23}{4} + 1\right) = \tfrac{15}{4}\sqrt{\tfrac{2}{5}} \ .$$

Consequently

$$J^2 \begin{pmatrix} -\sqrt{\frac{3}{5}}\, Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\, Y_0^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{15}{4}\sqrt{\frac{3}{5}}\, Y_{-1}^{(2)} \\ \frac{15}{4}\sqrt{\frac{2}{5}}\, Y_0^{(2)} \end{pmatrix} = \tfrac{3}{2}\left(\tfrac{3}{2}+1\right) \begin{pmatrix} -\sqrt{\frac{3}{5}}\, Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\, Y_0^{(2)} \end{pmatrix} \ .$$

Hence $j = \frac{3}{2}$.
Furthermore

$$J_z \begin{pmatrix} -\sqrt{\frac{3}{5}}\, Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\, Y_0^{(2)} \end{pmatrix} = \begin{pmatrix} L_z + \frac{1}{2} & 0 \\ 0 & L_z - \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{3}{5}}\, Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\, Y_0^{(2)} \end{pmatrix} =$$

$$= \begin{pmatrix} -\sqrt{\frac{3}{5}}\left(L_z + \frac{1}{2}\right) Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\left(L_z - \frac{1}{2}\right) Y_0^{(2)} \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{3}{5}}\left(-1 + \frac{1}{2}\right) Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\left(0 - \frac{1}{2}\right) Y_0^{(2)} \end{pmatrix} = -\tfrac{1}{2} \begin{pmatrix} -\sqrt{\frac{3}{5}}\, Y_{-1}^{(2)} \\ \sqrt{\frac{2}{5}}\, Y_0^{(2)} \end{pmatrix} \ .$$

Hence, $j_z = -\frac{1}{2}$.

10. $a = -\sqrt{\frac{4}{5}}$, $b = \sqrt{\frac{1}{5}}$, $m = -2$ for $\begin{pmatrix} -\sqrt{\frac{4}{5}}\, Y_{-2}^{(2)} \\ \sqrt{\frac{1}{5}}\, Y_{-1}^{(2)} \end{pmatrix}$, hence

$$a\left(\tfrac{27}{4} + m\right) + b\sqrt{6 - m(m+1)} = -\sqrt{\tfrac{4}{5}}\left(\tfrac{27}{4} - 2\right) + \sqrt{\tfrac{1}{5}}\sqrt{6-2} = -\tfrac{15}{4}\sqrt{\tfrac{4}{5}}$$

and

$$a\sqrt{6 - m(m+1)} + b\left(\tfrac{23}{4} - m\right) = -\sqrt{\tfrac{4}{5}}\sqrt{6-2} + \sqrt{\tfrac{1}{5}}\left(\tfrac{23}{4} + 2\right) = \tfrac{15}{4}\sqrt{\tfrac{1}{5}} \ .$$

Consequently

$$J^2 \begin{pmatrix} -\sqrt{\frac{4}{5}}\, Y_{-2}^{(2)} \\ \sqrt{\frac{1}{5}}\, Y_{-1}^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{15}{4}\sqrt{\frac{4}{5}}\, Y_{-2}^{(2)} \\ \frac{15}{4}\sqrt{\frac{1}{5}}\, Y_{-1}^{(2)} \end{pmatrix} = \tfrac{3}{2}\left(\tfrac{3}{2}+1\right) \begin{pmatrix} -\sqrt{\frac{4}{5}}\, Y_{-2}^{(2)} \\ \sqrt{\frac{1}{5}}\, Y_{-1}^{(2)} \end{pmatrix} \ .$$

Hence $j = \frac{3}{2}$.
Furthermore

$$J_z \begin{pmatrix} -\sqrt{\frac{4}{5}}\, Y_{-2}^{(2)} \\ \sqrt{\frac{1}{5}}\, Y_{-1}^{(2)} \end{pmatrix} = \begin{pmatrix} L_z + \frac{1}{2} & 0 \\ 0 & L_z - \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{4}{5}}\, Y_{-2}^{(2)} \\ \sqrt{\frac{1}{5}}\, Y_{-1}^{(2)} \end{pmatrix} =$$

$$= \begin{pmatrix} -\sqrt{\frac{4}{5}}\left(L_z + \frac{1}{2}\right) Y^{(2)}_{-2} \\ \sqrt{\frac{1}{5}}\left(L_z - \frac{1}{2}\right) Y^{(2)}_{-1} \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{4}{5}}\left(-2 + \frac{1}{2}\right) Y^{(2)}_{-2} \\ \sqrt{\frac{1}{5}}\left(-1 - \frac{1}{2}\right) Y^{(2)}_{-1} \end{pmatrix} = -\frac{3}{2} \begin{pmatrix} -\sqrt{\frac{4}{5}}\, Y^{(2)}_{-2} \\ \sqrt{\frac{1}{5}}\, Y^{(2)}_{-1} \end{pmatrix} \quad .$$

Hence, $j_z = -\frac{3}{2}$.

**c.** We found ($|j, j_z\rangle$ and $|s_z\rangle \otimes |\ell_z\rangle$)

$$\left|\tfrac{5}{2}, +\tfrac{5}{2}\right\rangle = \begin{pmatrix} Y^{(2)}_{+2} \\ 0 \end{pmatrix} = \left|+\tfrac{1}{2}\right\rangle \otimes |+2\rangle \ ,$$

$$\left|\tfrac{5}{2}, +\tfrac{3}{2}\right\rangle = \begin{pmatrix} \sqrt{\frac{4}{5}}\, Y^{(2)}_{+1} \\ \sqrt{\frac{1}{5}}\, Y^{(2)}_{+2} \end{pmatrix} = \sqrt{\tfrac{4}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |+1\rangle + \sqrt{\tfrac{1}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |+2\rangle \ ,$$

$$\left|\tfrac{5}{2}, +\tfrac{1}{2}\right\rangle = \begin{pmatrix} \sqrt{\frac{3}{5}}\, Y^{(2)}_{0} \\ \sqrt{\frac{2}{5}}\, Y^{(2)}_{+1} \end{pmatrix} = \sqrt{\tfrac{3}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |0\rangle + \sqrt{\tfrac{2}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |+1\rangle \ ,$$

$$\left|\tfrac{5}{2}, -\tfrac{1}{2}\right\rangle = \begin{pmatrix} \sqrt{\frac{2}{5}}\, Y^{(2)}_{-1} \\ \sqrt{\frac{3}{5}}\, Y^{(2)}_{0} \end{pmatrix} = \sqrt{\tfrac{2}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |-1\rangle + \sqrt{\tfrac{3}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |0\rangle \ ,$$

$$\left|\tfrac{5}{2}, -\tfrac{3}{2}\right\rangle = \begin{pmatrix} \sqrt{\frac{1}{5}}\, Y^{(2)}_{-2} \\ \sqrt{\frac{4}{5}}\, Y^{(2)}_{-1} \end{pmatrix} = \sqrt{\tfrac{1}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |-2\rangle + \sqrt{\tfrac{4}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |-1\rangle \ ,$$

$$\left|\tfrac{5}{2}, -\tfrac{5}{2}\right\rangle = \begin{pmatrix} 0 \\ Y^{(2)}_{-2} \end{pmatrix} = \left|-\tfrac{1}{2}\right\rangle \otimes |-2\rangle \ ,$$

$$\left|\tfrac{3}{2}, +\tfrac{3}{2}\right\rangle = \begin{pmatrix} -\sqrt{\frac{1}{5}}\, Y^{(2)}_{+1} \\ \sqrt{\frac{4}{5}}\, Y^{(2)}_{+2} \end{pmatrix} = -\sqrt{\tfrac{1}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |+1\rangle + \sqrt{\tfrac{4}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |+2\rangle \ ,$$

$$\left|\tfrac{3}{2}, +\tfrac{1}{2}\right\rangle = \begin{pmatrix} -\sqrt{\frac{2}{5}}\, Y^{(2)}_{0} \\ \sqrt{\frac{3}{5}}\, Y^{(2)}_{+1} \end{pmatrix} = -\sqrt{\tfrac{2}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |0\rangle + \sqrt{\tfrac{3}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |+1\rangle \ ,$$

$$\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \begin{pmatrix} -\sqrt{\frac{3}{5}}\, Y^{(2)}_{-1} \\ \sqrt{\frac{2}{5}}\, Y^{(2)}_{0} \end{pmatrix} = -\sqrt{\tfrac{3}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |-1\rangle + \sqrt{\tfrac{2}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |0\rangle \ ,$$

$$\left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = \begin{pmatrix} -\sqrt{\frac{4}{5}}\, Y^{(2)}_{-2} \\ \sqrt{\frac{1}{5}}\, Y^{(2)}_{-1} \end{pmatrix} = -\sqrt{\tfrac{4}{5}}\left|+\tfrac{1}{2}\right\rangle \otimes |-2\rangle + \sqrt{\tfrac{1}{5}}\left|-\tfrac{1}{2}\right\rangle \otimes |-1\rangle \ .$$

**d.** From the solutions of (**c**) we read the Clebsch-Gordan coefficients:

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ +2 & +\frac{1}{2} & +\frac{5}{2} \end{pmatrix} = 1$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ +1 & +\frac{1}{2} & +\frac{3}{2} \end{pmatrix} = \sqrt{\tfrac{4}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ +2 & -\frac{1}{2} & +\frac{3}{2} \end{pmatrix} = \sqrt{\tfrac{1}{5}}$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ 0 & +\frac{1}{2} & +\frac{1}{2} \end{pmatrix} = \sqrt{\tfrac{3}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ +1 & -\frac{1}{2} & +\frac{1}{2} \end{pmatrix} = \sqrt{\tfrac{2}{5}}$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ -1 & +\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \sqrt{\tfrac{2}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \sqrt{\tfrac{3}{5}}$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ -2 & +\frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \sqrt{\tfrac{1}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \sqrt{\tfrac{4}{5}}$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{5}{2} \\ -2 & -\frac{1}{2} & -\frac{5}{2} \end{pmatrix} = 1$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ +1 & +\frac{1}{2} & +\frac{3}{2} \end{pmatrix} = -\sqrt{\tfrac{1}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ +2 & -\frac{1}{2} & +\frac{3}{2} \end{pmatrix} = \sqrt{\tfrac{4}{5}}$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ 0 & +\frac{1}{2} & +\frac{1}{2} \end{pmatrix} = -\sqrt{\tfrac{2}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ +1 & -\frac{1}{2} & +\frac{1}{2} \end{pmatrix} = \sqrt{\tfrac{3}{5}}$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ -1 & +\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\sqrt{\tfrac{3}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \sqrt{\tfrac{2}{5}}$$

$$\begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ -2 & +\frac{1}{2} & -\frac{3}{2} \end{pmatrix} = -\sqrt{\tfrac{4}{5}} \qquad \begin{pmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ -1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} = \sqrt{\tfrac{1}{5}}$$

See also
http://en.wikipedia.org/wiki/Table_of_Clebsch-Gordan_coefficients#j1.3D2.2C_j2.3D1.2F2