MECÂNICA QUÂNTICA II

Exame de recurso, Quinta-feira, 12 de Julho de 2012

1. Um pacote de ondas que representa o movimento numa dimensão de uma partícula livre de massa m é, em unidades $\hbar = c = 1$, dado pela seguinte expressão:

$$\psi(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \varphi(k) \exp\left\{i\left(kx - \omega(k)t\right)\right\} \quad ,$$

onde $\omega(k) = \dfrac{k^2}{2m}$ e $\varphi(k) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2}$, com α e $\bar{k}$ constantes.

a Demonstre que $\psi(x,t)$ é solução da equação de Schrödinger.

b Demonstre que $-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2 + ikx = -\frac{1}{2}\alpha\left(k-\bar{k}-\frac{ix}{\alpha}\right)^2 + i\bar{k}x - \frac{x^2}{2\alpha}$.

c Demonstre que $\psi(x,t=0) = \left(\frac{1}{\pi\alpha}\right)^{1/4}\; e^{-\frac{1}{2}\frac{x^2}{\alpha}}\; e^{i\bar{k}x}$.
É de notar que

$$\int_{-\infty}^{\infty} dx\, e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}} \quad .$$

d Demonstre que

$$-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2 + i\left(kx - \frac{k^2}{2m}t\right) =$$

$$= -\frac{1}{2}\left(\alpha + \frac{it}{m}\right)\left(k-\bar{k}\right)^2 + i\left(k-\bar{k}\right)\left(x - \frac{\bar{k}}{m}t\right) + i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)$$

e demonstre ainda que, consequentemente,

$$\int_{-\infty}^{\infty} dk\; e^{-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2 + i\left(kx - \frac{k^2}{2m}t\right)} =$$

$$= e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} \int_{-\infty}^{\infty} dk\; e^{-\frac{1}{2}\left(\alpha + \frac{it}{m}\right)k^2}\; e^{ik\left(x - \frac{\bar{k}}{m}t\right)} \quad .$$

e Demonstre que $\psi(x,t) = \sqrt{\frac{1}{\beta}}\left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)}\; e^{-\frac{1}{2\beta}\left(x - \frac{\bar{k}}{m}t\right)^2}$, com $\beta = \alpha + \frac{it}{m}$.

f Demonstre que $\left|\psi(x,t)\right|^2 = \sqrt{\dfrac{\alpha m^2}{\pi\left(\alpha^2 m^2 + t^2\right)}}\; e^{-\frac{\alpha m^2}{\alpha^2 m^2 + t^2}\left(x - \frac{\bar{k}}{m}t\right)^2}$.

g Uma distribuição normal com média $\bar{x}$ e desvio padrão σ é definida por

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma^2}(x-\bar{x})^2} \quad .$$

Determine a média (valor expectável de k) e o desvio padrão (Δk) de $|\varphi(k)|^2$ e, ainda, a média (valor expectável de x) e o desvio padrão (Δx) de $|\psi(x,t)|^2$.

h Qual o significado do resultado do produto $\Delta k \Delta x$?

i Qual o significado de $x - \frac{\bar{k}}{m}t$?

j Qual o significado da expressão, obtida na alínea **g**, para o desvio padrão de $|\psi(x,t)|^2$?

2. Considere uma partícula que se desloca numa dimensão (de $x \downarrow -\infty$ para $x \uparrow +\infty$) e cuja função de onda é dada por (em unidades $\hbar = 1$):

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & \text{para} \quad x < 0 \\ Te^{ikx} & \text{para} \quad x > 0 \end{cases} \tag{1}$$

a Demonstre que a função de onda (1) é solução da seguinte equação de onda:

$$\left\{ -\frac{1}{2m}\frac{d^2}{dx^2} + \rho\,\delta(x) \right\} \psi(x) = E\psi(x) \quad ,$$

(ρ constante) com $k^2 = 2mE$.

b Mostre que para as derivadas da função de onda na posição $x = 0$ se tem a seguinte condição de fronteira:

$$-\frac{1}{2m}\left\{ \left(\frac{d\psi(x)}{dx}\right)_{x\downarrow 0} - \left(\frac{d\psi(x)}{dx}\right)_{x\uparrow 0} \right\} + \rho\psi(0) = 0 \quad ,$$

onde $\psi(0) = \psi(x \downarrow 0) = \psi(x \uparrow 0)$.

c Utilizando as condições de fronteira em $x = 0$, mostre que os coeficientes R e T da Equação (1) obedecem à relação $|R|^2 + |T|^2 = 1$.

d Mostre que para $\left|\frac{2E}{m\rho^2}\right| \ll 1$ se verifica $|T|^2 \approx \frac{2E}{m\rho^2}$.

e Um feixe de electrões choca contra uma barreira de potencial caracterizada por $\rho = 1.0 \times 10^{-3}$. Cada electrão tem uma massa igual a 0.5 MeV (em unidades $\hbar = c = 1$) e uma energia cinética $E = 1.0$ eV.
Determine a fracção dos electrões transmitidos pela barreira, isto é, $|T|^2$, e a fracção dos electrões reflectidos pela barreira, isto é, $|R|^2$.

3. Considere uma partícula de massa m cujo deslocamento unidimensional é descrito pelo seguinte Hamiltoniano:

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2 \quad . \tag{2}$$

a Mostre que, com a definição dos operadores (em unidades $\hbar = 1$)

$$a \;=\; \sqrt{\frac{m\omega}{2}}\left(x \;+\; i\frac{p}{m\omega}\right) \quad \text{e} \quad a^\dagger \;=\; \sqrt{\frac{m\omega}{2}}\left(x \;-\; i\frac{p}{m\omega}\right) \quad , \tag{3}$$

o Hamiltoniano H pode ser dado por

$$H = \omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

e determine os comutadores $\left[a \;,\; a^\dagger\right]$, $[H \;,\; a]$ e $\left[H \;,\; a^\dagger\right]$.

b Os estados próprios normalizados do Hamiltoniano H são designados por $|n\rangle$. O estado fundamental, $|0\rangle$, é dado por

$$\left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} \quad .$$

Determine (i) $a|0\rangle$, (ii) $|1\rangle = a^\dagger|0\rangle$ e (iii) $H|0\rangle$.

c Determine o valor próprio do estado $|1\rangle$ sob H e a normalização de $|1\rangle$.

d Demonstre, para os valores expectáveis de p, que

$$\langle 0\,|p|\,0\rangle = 0 \;\;,\;\; \langle 0\,|p|\,1\rangle = -i\sqrt{\frac{m\omega}{2}} \;\;,\;\; \langle 1\,|p|\,0\rangle = i\sqrt{\frac{m\omega}{2}} \;\; \text{e} \;\; \langle 1\,|p|\,1\rangle = 0 \quad .$$

e Considere um estado $\psi(0) = \sqrt{\frac{1}{2}}\,(\;|0\rangle + |1\rangle\;)$ cujo desenvolvimento temporal é dado por

$$\psi(t) = e^{-iHt}\psi(0) = \sqrt{\tfrac{1}{2}}\,e^{-iHt}\,(\;|0\rangle + |1\rangle\;) \quad .$$

Mostre que $\langle\psi(t)\,|p|\,\psi(t)\rangle = -\sqrt{\frac{m\omega}{2}}\,\sin(\omega t)$.

f Determine $\Delta x\Delta p$ para o estado fundamental, $|0\rangle$, de H, onde $\Delta x = \sqrt{\langle 0\,|x^2|\,0\rangle}$ e $\Delta p = \sqrt{\langle 0\,|p^2|\,0\rangle}$.

4. Considere no espaço de funções de três componentes a seguinte base de nove estados $|\ell = 1, m; s = 1, s_z = +1\rangle$, $|\ell = 1, m; s = 1, s_z = 0\rangle$ e $|\ell = 1, m; s = 1, s_z = -1\rangle$, respectivamente dados por

$$\begin{pmatrix} Y_m^{(1)}(\hat{r}) \\ \\ 0 \\ \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ \\ Y_m^{(1)}(\hat{r}) \\ \\ 0 \end{pmatrix} \quad \text{e} \quad \begin{pmatrix} 0 \\ \\ 0 \\ \\ Y_m^{(1)}(\hat{r}) \end{pmatrix} \quad , \quad \text{com} \;\; m = 0, \pm 1 \quad , \tag{4}$$

onde as funções $Y_m^{(1)}(\hat{r})$ representam funções harmónicas esféricas.
Os operadores L_x, L_y e L_z actuam no espaço das esféricas harmónicas da seguinte forma ($L_\pm = L_x \pm iL_y$) ($\hbar = 1$):

$$L_\pm Y_m^{(1)} = \sqrt{2 - m(m \pm 1)}\, Y_{m\pm 1}^{(1)} \quad \text{e} \quad L_z Y_m^{(1)} = m Y_m^{(1)} \quad .$$

Os operadores S_x, S_y e S_z são respectivamente dados pelas seguintes matrizes ($\hbar = 1$):

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{e} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad .$$

a Determine a matriz que representa S^2.

b Utilizando $L_\pm = L_x \pm iL_y$ e $[L_+, L_-] = 2L_z$, demonstre que

$$L^2 = L_x^2 + L_y^2 + L_z^2 = \tfrac{1}{2}\left(L_+L_- + L_-L_+\right) + L_z^2 = L_-L_+ + L_z + L_z^2 \quad .$$

c Demonstre que $L_-L_+Y_m^{(1)} = \left(2 - m(m+1)\right) Y_m^{(1)} = \left(2 - L_z^2 - L_z\right) Y_m^{(1)}$.

d Determine os valores próprios de L^2, L_z, S^2 e S_z de cada um dos nove estados da base referida na Equação (4).

e Estes estados formam uma base ortonormal do espaço $\{s = 1\} \otimes \{\ell = 1\}$. Temos, por exemplo:

$$|s = 1, s_z = +1\rangle \otimes |\ell = 1, m = +1\rangle = \begin{pmatrix} Y_{+1}^{(1)} \\ \\ 0 \\ \\ 0 \end{pmatrix} \quad ,$$

ou simplesmente

$$|s_z\rangle \otimes |m = \ell_z\rangle = |+1\rangle \otimes |+1\rangle = \begin{pmatrix} Y_{+1}^{(1)} \\ \\ 0 \\ \\ 0 \end{pmatrix} \quad .$$

Determine relações semelhantes para os outros oito estados.

5. Uma outra base ortonormal do espaço $\{s=1\}\otimes\{\ell=1\}$, considerado no exercício 4, é dada por:

$$\begin{pmatrix} Y^{(1)}_{+1} \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} \sqrt{\frac{1}{2}}Y^{(1)}_{0} \\ \sqrt{\frac{1}{2}}Y^{(1)}_{+1} \\ 0 \end{pmatrix} , \begin{pmatrix} \sqrt{\frac{1}{6}}Y^{(1)}_{-1} \\ \sqrt{\frac{2}{3}}Y^{(1)}_{0} \\ \sqrt{\frac{1}{6}}Y^{(1)}_{+1} \end{pmatrix} , \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}}Y^{(1)}_{-1} \\ \sqrt{\frac{1}{2}}Y^{(1)}_{0} \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ Y^{(1)}_{-1} \end{pmatrix} ,$$

$$\begin{pmatrix} \sqrt{\frac{1}{2}}Y^{(1)}_{0} \\ -\sqrt{\frac{1}{2}}Y^{(1)}_{+1} \\ 0 \end{pmatrix} , \begin{pmatrix} \sqrt{\frac{1}{2}}Y^{(1)}_{-1} \\ 0 \\ -\sqrt{\frac{1}{2}}Y^{(1)}_{+1} \end{pmatrix} , \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}}Y^{(1)}_{-1} \\ -\sqrt{\frac{1}{2}}Y^{(1)}_{0} \end{pmatrix} \text{ e } \begin{pmatrix} \sqrt{\frac{1}{3}}Y^{(1)}_{-1} \\ -\sqrt{\frac{1}{3}}Y^{(1)}_{0} \\ \sqrt{\frac{1}{3}}Y^{(1)}_{+1} \end{pmatrix} .$$

Os operadores $\vec{J}=\vec{L}+\vec{S}$ representam neste espaço rotações das coordenadas $\hat{r}$.

a Utilizando os resultados das alíneas (4**b**) e (4**c**), demonstre que

$$J^2 = \begin{pmatrix} 4+2L_z & \sqrt{2}L_- & 0 \\ \sqrt{2}L_+ & 4 & \sqrt{2}L_- \\ 0 & \sqrt{2}L_+ & 4-2L_z \end{pmatrix} .$$

b Determine os valores próprios de J^2 e J_z de cada um dos nove estados da nova base.

c Verifica-se que podemos dividir o espaço $\{s=1\}\otimes\{\ell=1\}$ em três subespaços: $\{j=2\}$ de cinco dimensões, $\{j=1\}$ de três dimensões e $\{j=0\}$ de uma dimensão. Os estados destes últimos espaços são designados por $|j,j_z\rangle$. Temos, por exemplo:

$$|2,+2\rangle = \begin{pmatrix} Y^{(1)}_{+1} \\ 0 \\ 0 \end{pmatrix} = |s_z=+1\rangle \otimes |\ell_z=m=+1\rangle \ .$$

Determine relações semelhantes para os outros oito estados da nova base.

d Os coeficientes das relações mencionadas na alínea **c** chamam-se coeficientes de Clebsch-Gordan (CCG) e são designados por:

$$\begin{pmatrix} \ell & s & j \\ \ell_z & s_z & j_z \end{pmatrix} .$$

Da relação dada na alínea (**c**) determina-se por exemplo:

$$\begin{pmatrix} 1 & 1 & 2 \\ +1 & +1 & +2 \end{pmatrix} = 1 \ .$$

Determine os outros CCG das relações entre as bases dos subespaços $\{j = 2\}$, $\{j = 1\}$ e $\{j = 0\}$ e a base do espaço $\{s = 1\} \otimes \{\ell = 1\}$ do exercício 4.

Solutions

Exercício 1

a. The Schrödinger equation for a free particle is, in units $\hbar = c = 1$, given by

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} \quad .$$

Now,

$$-\frac{1}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \varphi(k)e^{i\,(kx-\omega(k)t)} =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \varphi(k)\left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2}\right)e^{i\,(kx-\omega(k)t)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \varphi(k)\frac{1}{2m}k^2 e^{i\,(kx-\omega(k)t)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \varphi(k)\omega(k)e^{i\,(kx-\omega(k)t)}$$

and

$$i\frac{\partial\psi(x,t)}{\partial t} = i\frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \varphi(k)e^{i\,(kx-\omega(k)t)} =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \varphi(k)i\frac{\partial}{\partial t}e^{i\,(kx-\omega(k)t)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ \varphi(k)\omega(k)e^{i\,(kx-\omega(k)t)}$$

$$= -\frac{1}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} \quad .$$

b.

$$-\frac{1}{2}\alpha\left(k-\bar{k}-\frac{ix}{\alpha}\right)^2 + i\bar{k}x - \frac{x^2}{2\alpha} =$$

$$= -\frac{1}{2}\alpha k^2 - \frac{1}{2}\alpha\bar{k}^2 + \frac{1}{2}\alpha\frac{x^2}{\alpha^2} + \alpha k\bar{k} + \alpha k\frac{ix}{\alpha} - \alpha\bar{k}\frac{ix}{\alpha} + i\bar{k}x - \frac{x^2}{2\alpha}$$

$$= -\frac{1}{2}\alpha k^2 - \frac{1}{2}\alpha\bar{k}^2 + \alpha k\bar{k} + \alpha k\frac{ix}{\alpha} = -\frac{1}{2}\alpha\left(k-\bar{k}\right)^2 + ikx \quad .$$

c.

$$\psi(x,t=0) = \int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\varphi(k)\,e^{ikx} = \int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2} e^{ikx}$$

$$= \int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}\left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha\left(k-\bar{k}-\frac{ix}{\alpha}\right)^2 + i\bar{k}x - \frac{x^2}{2\alpha}}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\bar{k}x}\, e^{-\frac{x^2}{2\alpha}}\int_{-\infty}^{\infty} dk\, e^{-\frac{1}{2}\alpha\left(k-\bar{k}-\frac{ix}{\alpha}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\bar{k}x} \, e^{-\frac{x^2}{2\alpha}} \int_{-\infty}^{\infty} dk \, e^{-\frac{1}{2}\alpha k^2}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\bar{k}x} \, e^{-\frac{x^2}{2\alpha}} \sqrt{\frac{2\pi}{\alpha}}$$

$$= \left(\frac{1}{\pi\alpha}\right)^{1/4} e^{-\frac{1}{2}\frac{x^2}{\alpha}} \, e^{i\bar{k}x} \quad .$$

d.

$$\left(k-\bar{k}\right)x - \left\{\frac{\bar{k}^2}{2m} + \frac{\bar{k}}{m}\left(k-\bar{k}\right) + \frac{1}{2m}\left(k-\bar{k}\right)^2\right\}t + \bar{k}x =$$

$$= kx - \bar{k}x - \frac{\bar{k}^2}{2m}t - \frac{k\bar{k}}{m}t + \frac{\bar{k}^2}{m}t - \frac{k^2}{2m}t - \frac{\bar{k}^2}{2m}t + \frac{k\bar{k}}{m}t + \bar{k}x = kx - \frac{k^2}{2m}t \quad .$$

Hence,

$$-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2 + i\left(kx - \frac{k^2}{2m}t\right) =$$

$$-\frac{1}{2}\left(\alpha + \frac{it}{m}\right)\left(k-\bar{k}\right)^2 + i\left(k-\bar{k}\right)\left(x - \frac{\bar{k}}{m}t\right) + i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right) \quad .$$

Furthermore, by changing the integration variable from k to $k-\bar{k}$, hence dk to dk, we obtain

$$\int_{-\infty}^{\infty} dk \, e^{-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2 + i\left(kx - \frac{k^2}{2m}t\right)} =$$

$$= \int_{-\infty}^{\infty} dk \, e^{-\frac{1}{2}\left(\alpha + \frac{it}{m}\right)\left(k-\bar{k}\right)^2 + i\left(k-\bar{k}\right)\left(x - \frac{\bar{k}}{m}t\right) + i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)}$$

$$= e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} \int_{-\infty}^{\infty} dk \, e^{-\frac{1}{2}\left(\alpha + \frac{it}{m}\right)\left(k-\bar{k}\right)^2 + i\left(k-\bar{k}\right)\left(x - \frac{\bar{k}}{m}t\right)}$$

$$= e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} \int_{-\infty}^{\infty} dk \, e^{-\frac{1}{2}\left(\alpha + \frac{it}{m}\right)k^2} \; e^{ik\left(x - \frac{\bar{k}}{m}t\right)} \quad .$$

e.

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \, \varphi(k) \, e^{i\left(kx - \frac{k^2}{2m}t\right)} =$$

$$= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\alpha\left(k-\bar{k}\right)^2} \, e^{i\left(kx - \frac{k^2}{2m}t\right)}$$

$$= e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\left(\alpha + \frac{it}{m}\right)k^2 + ik\left(x - \frac{\bar{k}}{m}t\right)} \quad .$$

We define $\beta = \alpha + \frac{it}{m}$.

$$\psi(x,t) = e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{1}{2}\beta k^2 + ik\left(x - \frac{\bar{k}}{m}t\right)} =$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} \int_{-\infty}^{\infty} dk\, e^{-\frac{1}{2}\beta k^2} e^{ik\left(x - \frac{\bar{k}}{m}t\right)}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} \sqrt{\frac{2\pi}{\beta}}\, e^{-\frac{1}{2\beta}\left(x - \frac{\bar{k}}{m}t\right)^2}$$

$$= \sqrt{\frac{1}{\beta}} \left(\frac{\alpha}{\pi}\right)^{1/4} e^{i\left(\bar{k}x - \frac{\bar{k}^2}{2m}t\right)} e^{-\frac{1}{2\beta}\left(x - \frac{\bar{k}}{m}t\right)^2} \quad .$$

f.

$$|\psi(x,t)|^2 = \psi(x,t)\psi^*(x,t) = \sqrt{\frac{1}{\beta\beta^*}} \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\frac{1}{2}\frac{\left(x - \frac{\bar{k}}{m}t\right)^2}{\beta}} e^{-\frac{1}{2}\frac{\left(x - \frac{\bar{k}}{m}t\right)^2}{\beta^*}} =$$

$$= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\alpha}{\beta\beta^*}}\, e^{-\frac{\Re e(\beta)}{\beta\beta^*}\left(x - \frac{\bar{k}}{m}t\right)^2} = \sqrt{\frac{\alpha m^2}{\pi\left(\alpha^2 m^2 + t^2\right)}}\, e^{-\frac{\alpha m^2}{\alpha^2 m^2 + t^2}\left(x - \frac{\bar{k}}{m}t\right)^2} \quad .$$

g. $|\varphi(k)|^2 = \sqrt{\frac{\alpha}{\pi}}\ e^{-\alpha\left(k - \bar{k}\right)^2}$. Hence, the *mean* or *expectation value* of k equals $\bar{k}$, whereas the standard deviation for this distribution of k values is given by $\Delta k = \sqrt{\frac{1}{2\alpha}}$.

The expression for $|\psi(x,t)|^2$ has been obtained in line **f**. We read from that expression $\frac{\bar{k}}{m}t$ for the expectation value of x, whereas the standard deviation for this distribution of x values is given by $\Delta x = \sqrt{\frac{\alpha^2 m^2 + t^2}{2\alpha m^2}}$.

h. $\Delta k \Delta x = \sqrt{\frac{1}{2\alpha}}\sqrt{\frac{\alpha^2 m^2 + t^2}{2\alpha m^2}} = \frac{1}{2}\sqrt{1 + \left(\frac{t}{\alpha m}\right)^2}$.

For $t = 0$ we find $\Delta k \Delta x = \frac{1}{2}$, which is the minimum possible value for $\Delta k \Delta x$. However, when the time t increases $\Delta k \Delta x$ also increases. The reason is that the spreading (standard deviation), or the uncertainty in the particle's location, $\Delta x = \sqrt{\frac{\alpha^2 m^2 + t^2}{2\alpha m^2}}$, of the distribution $|\psi(x,t)|^2$ increases with time.

i. The particle has a velocity given by $v = \frac{\bar{k}}{m}$. Hence the peak value for its location, which is indicated by e^0, occurs for $x = vt = \frac{\bar{k}}{m}t$. The probability distribution is thus around $x - \frac{\bar{k}}{m}t$.

j. The spreading (standard deviation) of the probability distribution, or the uncertainty in the particle's location, is given by $\Delta x = \sqrt{\frac{\alpha^2 m^2 + t^2}{2\alpha m^2}}$. It increases with time. Hence, the particle's location becomes more uncertain with time.

Exercício 2

a.

$$\left\{-\frac{1}{2m}\frac{d^2}{dx^2}+\rho\,\delta(x)\right\}\psi(x)=\begin{cases}-\frac{1}{2m}\frac{d^2}{dx^2}\left(e^{ikx}+Re^{-ikx}\right) & \text{for}\quad x<0\\ -\frac{1}{2m}\frac{d^2}{dx^2}Te^{ikx} & \text{for}\quad x>0\end{cases}$$

$$=\begin{cases}\frac{k^2}{2m}\left(e^{ikx}+Re^{-ikx}\right) & \text{for}\quad x<0\\ \frac{k^2}{2m}Te^{ikx} & \text{for}\quad x>0\end{cases}$$

$$=\left\{\begin{array}{ll}E\left(e^{ikx}+Re^{-ikx}\right) & \text{for}\quad x<0\\ ETe^{ikx} & \text{for}\quad x>0\end{array}\right\}=E\psi(x)\quad .$$

b.

$$\int_{-\varepsilon}^{+\varepsilon}dx\ \left[\left\{-\frac{1}{2m}\frac{d^2}{dx^2}+\rho\,\delta(x)\right\}\psi(x)=E\psi(x)\right]$$

$$\Longleftrightarrow\quad \int_{-\varepsilon}^{+\varepsilon}dx\ \left\{-\frac{1}{2m}\frac{d^2\psi(x)}{dx^2}\right\}+\int_{-\varepsilon}^{+\varepsilon}dx\ \rho\,\delta(x)\psi(x)=\int_{-\varepsilon}^{+\varepsilon}dx\ E\psi(x)$$

$$\Longleftrightarrow\quad \left(-\frac{1}{2m}\frac{d\psi(x)}{dx}\right)_{x\,=\,+\varepsilon}-\left(-\frac{1}{2m}\frac{d\psi(x)}{dx}\right)_{x\,=\,-\varepsilon}+\rho\,\psi(x=0)=\int_{-\varepsilon}^{+\varepsilon}dx\ E\psi(x)\quad .$$

Taking, next, the limit $\varepsilon\downarrow 0$, we obtain:

$$-\frac{1}{2m}\left(\frac{d\psi(x)}{dx}\right)_{x\,\downarrow\,0}+\frac{1}{2m}\left(\frac{d\psi(x)}{dx}\right)_{x\,\uparrow\,0}+\rho\,\psi(0)=0\quad .$$

c.

$$\psi(x\uparrow 0)=1+R\quad\text{and}\quad\psi(x\downarrow 0)=T\quad .$$

Hence,

$$\psi(0)=\psi(x\uparrow 0)=\psi(x\downarrow 0)\quad\Longleftrightarrow\quad 1+R=T\quad .$$

$$\frac{1}{2m}\left(\frac{d\psi(x)}{dx}\right)_{x\,\uparrow\,0}=\frac{ik}{2m}(1-R)\quad\text{and}\quad -\frac{1}{2m}\left(\frac{d\psi(x)}{dx}\right)_{x\,\downarrow\,0}=-\frac{ik}{2m}T\quad .$$

Hence,

$$-\frac{1}{2m}\left(\frac{d\psi(x)}{dx}\right)_{x\,\downarrow\,0}+\frac{1}{2m}\left(\frac{d\psi(x)}{dx}\right)_{x\,\uparrow\,0}+\rho\,\psi(0)=0\quad\Longleftrightarrow\quad -\frac{ik}{2m}T+\frac{ik}{2m}(1-R)+\rho\,T=0$$

$$\Longleftrightarrow\quad -\frac{ik}{2m}T+\frac{ik}{2m}(2-T)+\rho\,T=0\quad\Longleftrightarrow\quad T=\frac{k}{k+im\rho}=\frac{\sqrt{2mE}}{\sqrt{2mE}+im\rho}\quad .$$

$$R = T - 1 = \frac{\sqrt{2mE}}{\sqrt{2mE} + im\rho} - 1 = \frac{-im\rho}{\sqrt{2mE} + im\rho} \quad .$$

For the transmission probability $|T|^2$ we obtain

$$|T|^2 = TT^* = \frac{k}{k + im\rho}\,\frac{k}{k - im\rho} = \frac{k^2}{k^2 + m^2\rho^2} = \frac{2mE}{2mE + m^2\rho^2} = \frac{E}{E + \frac{1}{2}m\rho^2}$$

whereas, for the reflection probabiblity $|R|^2$ we find

$$|R|^2 = RR^* = \frac{-im\rho}{k + im\rho}\,\frac{im\rho}{k - im\rho} = \frac{m^2\rho^2}{k^2 + m^2\rho^2} = \frac{m^2\rho^2}{2mE + m^2\rho^2} = \frac{\frac{1}{2}m\rho^2}{E + \frac{1}{2}m\rho^2} \quad .$$

Consequently,

$$|R|^2 + |T|^2 = \frac{\frac{1}{2}m\rho^2}{E + \frac{1}{2}m\rho^2} + \frac{E}{E + \frac{1}{2}m\rho^2} = \frac{\frac{1}{2}m\rho^2 + E}{E + \frac{1}{2}m\rho^2} = 1 \quad .$$

d. For $\left|\frac{2E}{m\rho^2}\right| \ll 1$ one has

$$|T|^2 = \frac{E}{E + \frac{1}{2}m\rho^2} = \frac{\frac{2E}{m\rho^2}}{1 + \frac{2E}{m\rho^2}} \approx \frac{2E}{m\rho^2} \quad .$$

and

$$|R|^2 = \frac{\frac{1}{2}m\rho^2}{E + \frac{1}{2}m\rho^2} = \frac{1}{1 + \frac{2E}{m\rho^2}} \approx 1 - \frac{2E}{m\rho^2} \quad .$$

e. We determine

$$\frac{2E}{m\rho^2} = \frac{2 \times (1.0 \text{ eV})}{(0.5 \times 10^6 \text{ eV}) \times (1.0 \times 10^{-3})^2} = 4 \quad .$$

This result does not satisfy $\left|\frac{2E}{m\rho^2}\right| \ll 1$. Hence, we have to use the full expression for $|T|^2$:

$$|T|^2 = \frac{\frac{2E}{m\rho^2}}{1 + \frac{2E}{m\rho^2}} = \frac{4}{1 + 4} = 0.8 \quad .$$

Furthermore,

$$|R|^2 = 1 - |T|^2 = 1 - 0.8 = 0.2 \quad .$$

80% of the beam is transmitted, 20% of the beam is reflected.

Exercício 3

a. We start with, remember $[x\ ,\ p] = i$,

$$a^\dagger a = \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)\sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right) =$$

$$= \frac{m\omega}{2}\left(x^2 + i\frac{1}{m\omega}[x\ ,\ p] + \frac{p^2}{m^2\omega^2}\right) = \frac{m\omega}{2}\left(x^2 - \frac{1}{m\omega} + \frac{p^2}{m^2\omega^2}\right)$$

$$= \frac{1}{\omega}\left(\frac{p^2}{2m} + \tfrac{1}{2}m\omega^2x^2\right) - \tfrac{1}{2} = \frac{H}{\omega} - \tfrac{1}{2} \quad \Longleftrightarrow \quad H = \omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

and

$$aa^\dagger = \sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right)\sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right) =$$

$$= \frac{m\omega}{2}\left(x^2 - i\frac{1}{m\omega}[x\ ,\ p] + \frac{p^2}{m^2\omega^2}\right) = \frac{m\omega}{2}\left(x^2 + \frac{1}{m\omega} + \frac{p^2}{m^2\omega^2}\right)$$

$$= \frac{1}{\omega}\left(\frac{p^2}{2m} + \tfrac{1}{2}m\omega^2x^2\right) + \tfrac{1}{2} = \frac{H}{\omega} + \tfrac{1}{2} \quad \Longleftrightarrow \quad H = \omega\left(aa^\dagger - \tfrac{1}{2}\right) \quad .$$

We find then

$$\left[a\ ,\ a^\dagger\right] = aa^\dagger - a^\dagger a = \frac{H}{\omega} + \tfrac{1}{2} - \left(\frac{H}{\omega} - \tfrac{1}{2}\right) = 1 \quad .$$

Also using the latter result, we obtain

$$[H\ ,\ a] = Ha - aH = \omega\left(a^\dagger aa - aa^\dagger a\right) = \omega\left[a^\dagger\ ,\ a\right]a = -\omega a$$

and

$$[H\ ,\ a] = Ha^\dagger - a^\dagger H = \omega\left(a^\dagger aa^\dagger - a^\dagger a^\dagger a\right) = \omega a^\dagger\left[a\ ,\ a^\dagger\right] = \omega a^\dagger \quad .$$

b. We determine

$$a|0\rangle = \sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right)\left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} =$$

$$= \sqrt{\frac{m\omega}{2}}\left[\frac{m\omega}{\pi}\right]^{1/4}\left(x + \frac{1}{m\omega}\frac{d}{dx}\right)e^{-\frac{1}{2}m\omega x^2}$$

$$= \sqrt{\frac{m\omega}{2}}\left[\frac{m\omega}{\pi}\right]^{1/4}\left(x + \frac{1}{m\omega}(-m\omega x)\right)e^{-\frac{1}{2}m\omega x^2} = 0$$

and

$$a^\dagger|0\rangle = \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)\left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} =$$

$$= \sqrt{\frac{m\omega}{2}} \left[\frac{m\omega}{\pi}\right]^{1/4} \left(x - \frac{1}{m\omega}\frac{d}{dx}\right) e^{-\frac{1}{2}m\omega x^2}$$

$$= \sqrt{\frac{m\omega}{2}} \left[\frac{m\omega}{\pi}\right]^{1/4} \left(x - \frac{1}{m\omega}\left(-m\omega x\right)\right) e^{-\frac{1}{2}m\omega x^2}$$

$$= 2x\sqrt{\frac{m\omega}{2}} \left[\frac{m\omega}{\pi}\right]^{1/4} e^{-\frac{1}{2}m\omega x^2} \quad .$$

The eigenvalue of $|0\rangle$ for H is obtained from

$$H|0\rangle = \omega\left(a^\dagger a + \tfrac{1}{2}\right)|0\rangle = \omega\left(a^\dagger a|0\rangle + \tfrac{1}{2}|0\rangle\right) = \omega\left(0 + \tfrac{1}{2}|0\rangle\right) = \tfrac{1}{2}\omega|0\rangle \quad .$$

Hence, the eigenvalue of $|0\rangle$ for H equals $\frac{1}{2}\omega$.

c. The eigenvalue of $|1\rangle = a^\dagger|0\rangle$ for H is obtained from

$$H|1\rangle = Ha^\dagger|0\rangle = \omega\left(a^\dagger a + \tfrac{1}{2}\right)a^\dagger|0\rangle = \omega\left(a^\dagger a a^\dagger|0\rangle + \tfrac{1}{2}a^\dagger|0\rangle\right) =$$

$$= \omega\left(a^\dagger\left(\left[a\ ,\ a^\dagger\right] + a^\dagger a\right)|0\rangle + \tfrac{1}{2}a^\dagger|0\rangle\right) = \omega\left(a^\dagger\left(1 + a^\dagger a\right)|0\rangle + \tfrac{1}{2}a^\dagger|0\rangle\right)$$

$$= \omega\left(a^\dagger\left(|0\rangle + 0\right) + \tfrac{1}{2}a^\dagger|0\rangle\right) = \tfrac{3}{2}a^\dagger\omega|0\rangle \quad .$$

Hence, the eigenvalue of $a^\dagger|0\rangle$ for H equals $\frac{3}{2}\omega$.

The normalisation of $a^\dagger|0\rangle$ is determined by

$$\left|a^\dagger|0\rangle\right|^2 = \left\langle\left(a^\dagger|0\rangle\right)^\dagger\left|a^\dagger|0\rangle\right.\right\rangle =$$

$$= \left\langle 0\left|aa^\dagger\right|0\right\rangle = \left\langle 0\left|\frac{H}{\omega} + \tfrac{1}{2}\right|0\right\rangle = \left\langle 0\left|\tfrac{1}{2} + \tfrac{1}{2}\right|0\right\rangle = \langle 0\,|0\rangle = 1 \quad .$$

Hence, $a^\dagger|0\rangle$ is normalised.

d. We first determine

$$-i\sqrt{\frac{m\omega}{2}}\left(a - a^\dagger\right) = -i\sqrt{\frac{m\omega}{2}}\left(\sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right) - \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)\right) =$$

$$= -i\frac{m\omega}{2}\left(\left(x + i\frac{p}{m\omega}\right) - \left(x - i\frac{p}{m\omega}\right)\right) = -i\frac{m\omega}{2}\left(2i\frac{p}{m\omega}\right) = p \quad .$$

Using this result, we find
(remember $H|0\rangle = \frac{1}{2}\omega|0\rangle$ and $\langle 0|H = \left(H^\dagger|0\rangle\right)^\dagger = \left(H|0\rangle\right)^\dagger = \left(\frac{1}{2}\omega|0\rangle\right)^\dagger = \frac{1}{2}\omega\langle 0|$):

$$\langle 0\,|p|\,0\rangle = \left\langle 0\left|-i\sqrt{\tfrac{m\omega}{2}}\left(a - a^\dagger\right)\right|0\right\rangle = -i\sqrt{\frac{m\omega}{2}}\left\langle 0\left|\left(a - a^\dagger\right)\right|0\right\rangle =$$

$$= -i\sqrt{\frac{m\omega}{2}}\left(\langle 0|\left\{a|0\rangle\right\} - \left\langle\left(a|0\rangle\right)^\dagger|0\right\rangle\right) = -i\sqrt{\frac{m\omega}{2}}\left(0 - 0\right) = 0 \quad ,$$

$$\langle 0\,|p|\,1\rangle = \left\langle 0\left|-i\sqrt{\tfrac{m\omega}{2}}\left(a-a^\dagger\right)\right|1\right\rangle = -i\sqrt{\frac{m\omega}{2}}\left\langle 0\left|\left(a-a^\dagger\right)\right|1\right\rangle =$$

$$= -i\sqrt{\frac{m\omega}{2}}\left\langle 0\left|\left(a-a^\dagger\right)a^\dagger\right|0\right\rangle = -i\sqrt{\frac{m\omega}{2}}\left(\left\langle 0\left|aa^\dagger\right|0\right\rangle - \left\langle 0\left|a^\dagger a^\dagger\right|0\right\rangle\right)$$

$$= -i\sqrt{\frac{m\omega}{2}}\left(\left\langle 0\left|\frac{H}{\omega}+\tfrac{1}{2}\right|0\right\rangle - \left\langle (a|0\rangle)^\dagger\left|a^\dagger\right|0\right\rangle\right) = -i\sqrt{\frac{m\omega}{2}}\left(\langle 0\,|0\rangle + 0\right) = -i\sqrt{\frac{m\omega}{2}} \quad ,$$

$$\langle 1\,|p|\,0\rangle = \left\langle 1\left|-i\sqrt{\tfrac{m\omega}{2}}\left(a-a^\dagger\right)\right|0\right\rangle = -i\sqrt{\frac{m\omega}{2}}\left\langle 1\left|\left(a-a^\dagger\right)\right|0\right\rangle =$$

$$= -i\sqrt{\frac{m\omega}{2}}\left\langle 0\left|a\left(a-a^\dagger\right)\right|0\right\rangle = -i\sqrt{\frac{m\omega}{2}}\left(\langle 0\,|aa|\,0\rangle - \left\langle 0\left|aa^\dagger\right|0\right\rangle\right)$$

$$= -i\sqrt{\frac{m\omega}{2}}\left(0 - \left\langle 0\left|\frac{H}{\omega}+\tfrac{1}{2}\right|0\right\rangle\right) = -i\sqrt{\frac{m\omega}{2}}\left(0 - \langle 0\,|0\rangle\right) = +i\sqrt{\frac{m\omega}{2}} = \langle 0\,|p|\,1\rangle^*$$

and

$$\langle 1\,|p|\,1\rangle = \left\langle 1\left|-i\sqrt{\tfrac{m\omega}{2}}\left(a-a^\dagger\right)\right|1\right\rangle = -i\sqrt{\frac{m\omega}{2}}\left\langle 1\left|\left(a-a^\dagger\right)\right|1\right\rangle =$$

$$= -i\sqrt{\frac{m\omega}{2}}\left\langle 0\left|a\left(a-a^\dagger\right)a^\dagger\right|0\right\rangle = -i\sqrt{\frac{m\omega}{2}}\left(\left\langle 0\left|aaa^\dagger\right|0\right\rangle - \left\langle 0\left|aa^\dagger a^\dagger\right|0\right\rangle\right)$$

$$= -i\sqrt{\frac{m\omega}{2}}\left(\left\langle 0\left|a\left(\frac{H}{\omega}+\tfrac{1}{2}\right)\right|0\right\rangle - \left\langle 0\left|\left(\frac{H}{\omega}+\tfrac{1}{2}\right)a^\dagger\right|0\right\rangle\right)$$

$$= -i\sqrt{\frac{m\omega}{2}}\left(\langle 0\,|a|\,0\rangle - \left\langle 0\left|a^\dagger\right|0\right\rangle\right) = -i\sqrt{\frac{m\omega}{2}}\left(0-0\right) = 0 \quad .$$

e. For the time development of the state we have

$$\psi(t) = \sqrt{\tfrac{1}{2}}\,e^{-iHt}\left(|0\rangle + |1\rangle\right) = \sqrt{\tfrac{1}{2}}\left(e^{-i\frac{1}{2}\omega t}|0\rangle + e^{-i\frac{3}{2}\omega t}|1\rangle\right) \quad .$$

Hence, the time development of the expation value for p reads

$$\langle \psi(t)\,|p|\,\psi(t)\rangle =$$

$$= \sqrt{\tfrac{1}{2}}\left(e^{i\frac{1}{2}\omega t}\langle 0| + e^{i\frac{3}{2}\omega t}\langle 1|\right)\ p\ \sqrt{\tfrac{1}{2}}\left(e^{-i\frac{1}{2}\omega t}|0\rangle + e^{-i\frac{3}{2}\omega t}|1\rangle\right)$$

$$= \tfrac{1}{2}\left\{\langle 0\,|p|\,0\rangle + e^{-i\omega t}\langle 0\,|p|\,1\rangle + e^{+i\omega t}\langle 1\,|p|\,0\rangle + \langle 1\,|p|\,1\rangle\right\}$$

$$= \tfrac{1}{2}\left\{0 + e^{-i\omega t}\left(-i\sqrt{\frac{m\omega}{2}}\right) + e^{+i\omega t}\left(i\sqrt{\frac{m\omega}{2}}\right) + 0\right\}$$

$$= -\sqrt{\frac{m\omega}{2}}\frac{1}{2i}\left(e^{+i\omega t} - e^{-i\omega t}\right) = -\sqrt{\frac{m\omega}{2}}\ \sin(\omega t) \quad .$$

f. We first determine

$$\frac{1}{\sqrt{2m\omega}}\left(a+a^\dagger\right) = \frac{1}{\sqrt{2m\omega}}\left(\sqrt{\frac{m\omega}{2}}\left(x + i\frac{p}{m\omega}\right) + \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)\right) = \tfrac{1}{2}\left(2x\right) = x \quad ,$$

$$x^2 = \frac{1}{2m\omega}\left(a + a^\dagger\right)^2 = \frac{1}{2m\omega}\left(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\right)$$

and (see paragraph **d**)

$$p^2 = -\frac{m\omega}{2}\left(a - a^\dagger\right)^2 = -\frac{m\omega}{2}\left(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger\right) \quad .$$

Using those results, we find (remember $H|0\rangle = \frac{1}{2}\omega|0\rangle$)

$$\begin{aligned}
&\langle 0\,|x^2|\,0\rangle = \\
&\quad = \left\langle 0\left|\frac{1}{2m\omega}\left(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\right)\right|0\right\rangle \\
&\quad = \frac{1}{2m\omega}\left(\langle 0\,|aa|\,0\rangle + \left\langle 0\left|aa^\dagger + a^\dagger a\right|0\right\rangle + \left\langle 0\left|a^\dagger a^\dagger\right|0\right\rangle\right) \\
&\quad = \frac{1}{2m\omega}\left(0 + \left\langle 0\left|2\frac{H}{\omega}\right|0\right\rangle + 0\right) = \frac{1}{2m\omega} \quad .
\end{aligned}$$

and

$$\begin{aligned}
&\langle 0\,|p^2|\,0\rangle = \\
&\quad = \left\langle 0\left|-\frac{m\omega}{2}\left(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger\right)\right|0\right\rangle \\
&\quad = -\frac{m\omega}{2}\left(\langle 0\,|aa|\,0\rangle - \left\langle 0\left|aa^\dagger + a^\dagger a\right|0\right\rangle + \left\langle 0\left|a^\dagger a^\dagger\right|0\right\rangle\right) \\
&\quad = -\frac{m\omega}{2}\left(0 - \left\langle 0\left|2\frac{H}{\omega}\right|0\right\rangle + 0\right) = \frac{m\omega}{2} \quad .
\end{aligned}$$

Hence,

$$\Delta x \Delta p = \sqrt{\langle 0\,|x^2|\,0\rangle}\,\sqrt{\langle 0\,|p^2|\,0\rangle} = \tfrac{1}{2} \quad .$$

Exercício 4

a.

$$S_x^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad , \quad S_y^2 = -\frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

and

$$S_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

Hence,

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\mathbf{1} = s(s+1)\mathbf{1} \quad \text{with} \quad s = 1 \quad .$$

b. Using $L_x = \frac{1}{2}(L_+ + L_-)$ and $L_y = \frac{1}{2i}(L_+ - L_-)$, we obtain

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 = \left\{\tfrac{1}{2}(L_+ + L_-)\right\}^2 + \left\{\tfrac{1}{2i}(L_+ - L_-)\right\}^2 + L_z^2 \\ &= \left\{\tfrac{1}{4}\left(L_+^2 + L_+L_- + L_-L_+ + L_-^2\right)\right\} + \left\{-\tfrac{1}{4}\left(L_+^2 - L_+L_- - L_-L_+ + L_-^2\right)\right\} + L_z^2 \\ &= \tfrac{1}{2}(L_+L_- + L_-L_+) + L_z^2 = \tfrac{1}{2}(L_+L_- - L_-L_+ + 2L_-L_+) + L_z^2 \\ &= \tfrac{1}{2}(2L_z + 2L_-L_+) + L_z^2 = L_z + L_-L_+ + L_z^2 \quad . \end{aligned}$$

c.

$$\begin{aligned} L_-L_+Y_m^{(1)} &= L_-\left(L_+Y_m^{(1)}\right) = L_-\left(\sqrt{2 - m(m+1)}\; Y_{m+1}^{(1)}\right) \\ &= \sqrt{2 - m(m+1)}\; L_-Y_{m+1}^{(1)} \\ &= \sqrt{2 - m(m+1)}\; \sqrt{2 - (m+1)(m+1-1)}\; Y_{m+1-1}^{(1)} \\ &= (2 - m(m+1))\; Y_m^{(1)} = \left(2 - m^2 - m\right)\; Y_m^{(1)} = \left(2 - L_z^2 - L_z\right) Y_m^{(1)} \quad . \end{aligned}$$

As a consequence, one has

$$\begin{aligned} L^2Y_m^{(1)} &= \left(L_-L_+ + L_z + L_z^2\right) Y_m^{(1)} \\ &= \left(2 - L_z^2 - L_z + L_z + L_z^2\right) Y_m^{(1)} = 2Y_m^{(1)} = \ell(\ell+1)Y_m^{(1)} \quad \text{with} \quad \ell = 1 \quad . \end{aligned}$$

c. At the 3×3 basis $L^2 = L^2\mathbf{1}$. Hence,

$$L^2\mathbf{1}\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L^2 Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = 2\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix}$$

and similar for the other six states.
Furthermore for $L_z = L_z\mathbf{1}$:

$$L_z\mathbf{1}\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_z Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} mY_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = m\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix}$$

and similar for the other six states.
For $S^2 = 2\mathbf{1}$ one has

$$S^2\mathbf{1}\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = 2\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix}$$

and similar for the other six states.
Furthermore, for S_z we have

$$S_z\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} = +1\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix} \quad ,$$

$$S_z\begin{pmatrix} 0 \\ Y_m^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ Y_m^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 0 \\ Y_m^{(1)} \\ 0 \end{pmatrix}$$

and

$$S_z\begin{pmatrix} 0 \\ 0 \\ Y_m^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ Y_m^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -Y_m^{(1)} \end{pmatrix} = -1\begin{pmatrix} 0 \\ 0 \\ Y_m^{(1)} \end{pmatrix} \quad .$$

d. Using the results of (**c**), with $L^2 = \ell(\ell+1) = 2$, hence $\ell = 1$ and $S^2 = s(s+1) = 2$, hence $s = 1$, we obtain

basis	s_z	state	
$\begin{pmatrix} Y_m^{(1)} \\ 0 \\ 0 \end{pmatrix}$	$+1$	$m=+1$	$\lvert s=1, s_z=+1\rangle \otimes \lvert \ell=1, m=+1\rangle = \lvert +1\rangle \otimes \lvert +1\rangle$
		$m=0$	$\lvert s=1, s_z=+1\rangle \otimes \lvert \ell=1, m=0\rangle = \lvert +1\rangle \otimes \lvert 0\rangle$
		$m=-1$	$\lvert s=1, s_z=+1\rangle \otimes \lvert \ell=1, m=-1\rangle = \lvert +1\rangle \otimes \lvert -1\rangle$
$\begin{pmatrix} 0 \\ Y_m^{(1)} \\ 0 \end{pmatrix}$	0	$m=+1$	$\lvert s=1, s_z=0\rangle \otimes \lvert \ell=1, m=+1\rangle = \lvert 0\rangle \otimes \lvert +1\rangle$
		$m=0$	$\lvert s=1, s_z=0\rangle \otimes \lvert \ell=1, m=0\rangle = \lvert 0\rangle \otimes \lvert 0\rangle$
		$m=-1$	$\lvert s=1, s_z=0\rangle \otimes \lvert \ell=1, m=-1\rangle = \lvert 0\rangle \otimes \lvert -1\rangle$
$\begin{pmatrix} 0 \\ 0 \\ Y_m^{(1)} \end{pmatrix}$	-1	$m=+1$	$\lvert s=1, s_z=-1\rangle \otimes \lvert \ell=1, m=+1\rangle = \lvert -1\rangle \otimes \lvert +1\rangle$
		$m=0$	$\lvert s=1, s_z=-1\rangle \otimes \lvert \ell=1, m=0\rangle = \lvert -1\rangle \otimes \lvert 0\rangle$
		$m=-1$	$\lvert s=1, s_z=-1\rangle \otimes \lvert \ell=1, m=-1\rangle = \lvert -1\rangle \otimes \lvert -1\rangle$

Exercício 5

a. The operators $J_\pm$ and J_z are represented by

$$J_+ = L_+\mathbf{1} + S_+ = L_+\mathbf{1} + S_x + iS_y = \begin{pmatrix} L_+ & \sqrt{2} & 0 \\ 0 & L_+ & \sqrt{2} \\ 0 & 0 & L_+ \end{pmatrix} ,$$

$$J_- = L_-\mathbf{1} + S_- = L_-\mathbf{1} + S_x - iS_y = \begin{pmatrix} L_- & 0 & 0 \\ \sqrt{2} & L_- & 0 \\ 0 & \sqrt{2} & L_- \end{pmatrix}$$

and

$$J_z = L_z\mathbf{1} + S_z = \begin{pmatrix} L_z+1 & 0 & 0 \\ 0 & L_z & 0 \\ 0 & 0 & L_z-1 \end{pmatrix} .$$

Using the results of exercises (4**a** e **b**) one has for J^2 the following result:

$J^2 = J_-J_+ + J_z + J_z^2 =$

$$= \begin{pmatrix} L_- & 0 & 0 \\ \sqrt{2} & L_- & 0 \\ 0 & \sqrt{2} & L_- \end{pmatrix} \begin{pmatrix} L_+ & \sqrt{2} & 0 \\ 0 & L_+ & \sqrt{2} \\ 0 & 0 & L_+ \end{pmatrix} + \begin{pmatrix} L_z+1 & 0 & 0 \\ 0 & L_z & 0 \\ 0 & 0 & L_z-1 \end{pmatrix} + \begin{pmatrix} L_z+1 & 0 & 0 \\ 0 & L_z & 0 \\ 0 & 0 & L_z-1 \end{pmatrix}^2$$

$$= \begin{pmatrix} L_-L_+ & \sqrt{2}L_- & 0 \\ \sqrt{2}L_+ & 2+L_-L_+ & \sqrt{2}L_- \\ 0 & \sqrt{2}L_+ & 2+L_-L_+ \end{pmatrix} +$$

$$+ \begin{pmatrix} L_z+1 & 0 & 0 \\ 0 & L_z & 0 \\ 0 & 0 & L_z-1 \end{pmatrix} + \begin{pmatrix} L_z^2+2L_z+1 & 0 & 0 \\ 0 & L_z^2 & 0 \\ 0 & 0 & L_z^2-2L_z+1 \end{pmatrix}$$

$$= \begin{pmatrix} L_-L_+ + L_z^2 + 3L_z + 2 & \sqrt{2}L_- & 0 \\ \sqrt{2}L_+ & L_-L_+ + L_z^2 + L_z + 2 & \sqrt{2}L_- \\ 0 & \sqrt{2}L_+ & L_-L_+ + L_z^2 - L_z + 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+2L_z & \sqrt{2}L_- & 0 \\ \sqrt{2}L_+ & 4 & \sqrt{2}L_- \\ 0 & \sqrt{2}L_+ & 4-2L_z \end{pmatrix} .$$

We must remember:

$$L_+Y_{+1}^{(1)} = 0 \quad , \quad L_+Y_0^{(1)} = \sqrt{2}Y_{+1}^{(1)} \quad \text{and} \quad L_+Y_{-1}^{(1)} = \sqrt{2}Y_0^{(1)} \quad ,$$

and

$$L_-Y_{+1}^{(1)} = \sqrt{2}Y_0^{(1)} \quad , \quad L_-Y_0^{(1)} = \sqrt{2}Y_{-1}^{(1)} \quad \text{and} \quad L_-Y_{-1}^{(1)} = 0 \quad .$$

We study next each of the nine states of the new basis.

$$J^2 \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4+2L_z & \sqrt{2}L_- & 0 \\ \sqrt{2}L_+ & 4 & \sqrt{2}L_- \\ 0 & \sqrt{2}L_+ & 4-2L_z \end{pmatrix} \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (4+2L_z)Y_{+1}^{(1)} \\ \sqrt{2}L_+Y_{+1}^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 6Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} .$$

Hence,

$$j(j+1) \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = J^2 \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = 6 \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad j = 2 \; .$$

Furthermore,

$$J_z \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_z+1 & 0 & 0 \\ 0 & L_z & 0 \\ 0 & 0 & L_z-1 \end{pmatrix} \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (L_z+1)Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} .$$

Hence,

$$j_z \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = J_z \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad j_z = +2 \; .$$

Conclusion:

$$|j=2, j_z=+2\rangle = \begin{pmatrix} Y_{+1}^{(1)} \\ 0 \\ 0 \end{pmatrix} = |s_z=+1\rangle \otimes |m=+1\rangle \; .$$

Next, we study two states at once:

$$J^2 \begin{pmatrix} aY_0^{(1)} \\ bY_{+1}^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} a(4+2L_z)Y_0^{(1)} + b\sqrt{2}L_-Y_{+1}^{(1)} \\ a\sqrt{2}L_+Y_0^{(1)} + 4bY_{+1}^{(1)} \\ b\sqrt{2}L_+Y_{+1}^{(1)} \end{pmatrix} = \begin{pmatrix} (4a+2b)Y_0^{(1)} \\ (2a+4b)Y_{+1}^{(1)} \\ 0 \end{pmatrix}$$

and

$$J_z \begin{pmatrix} aY_0^{(1)} \\ bY_{+1}^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} a(L_z+1)Y_0^{(1)} \\ bL_zY_{+1}^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} aY_0^{(1)} \\ bY_{+1}^{(1)} \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad j_z = +1 \; .$$

1. For $b = a = \sqrt{\frac{1}{2}}$ we find

$$J^2 \begin{pmatrix} aY_0^{(1)} \\ aY_{+1}^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 6aY_0^{(1)} \\ 6aY_{+1}^{(1)} \\ 0 \end{pmatrix} = 6 \begin{pmatrix} aY_0^{(1)} \\ aY_{+1}^{(1)} \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad j = 2 \ .$$

Conclusion:

$$|j = 2, j_z = +1\rangle = \begin{pmatrix} \sqrt{\frac{1}{2}}Y_0^{(1)} \\ \sqrt{\frac{1}{2}}Y_{+1}^{(1)} \\ 0 \end{pmatrix} = \sqrt{\tfrac{1}{2}}|s_z = +1\rangle \otimes |m = 0\rangle + \sqrt{\tfrac{1}{2}}|s_z = 0\rangle \otimes |m = +1\rangle \ .$$

2. For $b = -a = -\sqrt{\frac{1}{2}}$ we find

$$J^2 \begin{pmatrix} aY_0^{(1)} \\ -aY_{+1}^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 2aY_0^{(1)} \\ -2aY_{+1}^{(1)} \\ 0 \end{pmatrix} = 2 \begin{pmatrix} aY_0^{(1)} \\ -aY_{+1}^{(1)} \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad j = 1 \ .$$

Conclusion:

$$|j = 1, j_z = +1\rangle = \begin{pmatrix} \sqrt{\frac{1}{2}}Y_0^{(1)} \\ -\sqrt{\frac{1}{2}}Y_{+1}^{(1)} \\ 0 \end{pmatrix} = \sqrt{\tfrac{1}{2}}|s_z = +1\rangle \otimes |m = 0\rangle - \sqrt{\tfrac{1}{2}}|s_z = 0\rangle \otimes |m = +1\rangle \ .$$

Next, we study three states at once:

$$J^2 \begin{pmatrix} aY_{-1}^{(1)} \\ bY_0^{(1)} \\ cY_{+1}^{(1)} \end{pmatrix} = \begin{pmatrix} a(4 + 2L_z)Y_{-1}^{(1)} + b\sqrt{2}L_-Y_0^{(1)} \\ a\sqrt{2}L_+Y_{-1}^{(1)} + 4bY_0^{(1)} + c\sqrt{2}L_-Y_{+1}^{(1)} \\ b\sqrt{2}L_+Y_0^{(1)} + c(4 - 2L_z)Y_{+1}^{(1)} \end{pmatrix} = \begin{pmatrix} (2a + 2b)Y_{-1}^{(1)} \\ (2a + 4b + 2c)Y_0^{(1)} \\ (2b + 2c)Y_{+1}^{(1)} \end{pmatrix}$$

and

$$J_z \begin{pmatrix} aY_{-1}^{(1)} \\ bY_0^{(1)} \\ cY_{+1}^{(1)} \end{pmatrix} = \begin{pmatrix} a(L_z + 1)Y_{-1}^{(1)} \\ bL_zY_0^{(1)} \\ a(L_z - 1)Y_{+1}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad j_z = 0 \ .$$

1. For $c = a = \sqrt{\frac{1}{6}}$ and $b = 2a = \sqrt{\frac{2}{3}}$ we find

$$J^2 \begin{pmatrix} aY_{-1}^{(1)} \\ 2aY_0^{(1)} \\ aY_{+1}^{(1)} \end{pmatrix} = \begin{pmatrix} 6aY_{-1}^{(1)} \\ 12aY_0^{(1)} \\ 6aY_{+1}^{(1)} \end{pmatrix} = 6 \begin{pmatrix} aY_{-1}^{(1)} \\ 2aY_0^{(1)} \\ aY_{+1}^{(1)} \end{pmatrix} \quad \Longleftrightarrow \quad j = 2 \ .$$

Conclusion:

$$|j=2, j_z=0\rangle = \begin{pmatrix} \sqrt{\frac{1}{6}}Y^{(1)}_{-1} \\ \\ \sqrt{\frac{2}{3}}Y^{(1)}_{0} \\ \\ \sqrt{\frac{1}{6}}Y^{(1)}_{+1} \end{pmatrix} =$$

$$= \sqrt{\frac{1}{6}}|s_z=+1\rangle \otimes |m=-1\rangle + \sqrt{\frac{2}{3}}|s_z=0\rangle \otimes |m=0\rangle + \sqrt{\frac{1}{6}}|s_z=-1\rangle \otimes |m=+1\rangle \ .$$

2. For $c=-a=-\sqrt{\frac{1}{2}}$ and $b=0$ we find

$$J^2 \begin{pmatrix} aY^{(1)}_{-1} \\ \\ 0 \\ \\ -aY^{(1)}_{+1} \end{pmatrix} = \begin{pmatrix} 2aY^{(1)}_{-1} \\ \\ 0 \\ \\ -2aY^{(1)}_{+1} \end{pmatrix} = 2 \begin{pmatrix} aY^{(1)}_{-1} \\ \\ 0 \\ \\ -aY^{(1)}_{+1} \end{pmatrix} \quad \Longleftrightarrow \quad j=1\ .$$

Conclusion:

$$|j=1, j_z=0\rangle = \begin{pmatrix} \sqrt{\frac{1}{2}}Y^{(1)}_{-1} \\ \\ 0 \\ \\ -\sqrt{\frac{1}{2}}Y^{(1)}_{+1} \end{pmatrix} =$$

$$= \sqrt{\frac{1}{2}}|s_z=+1\rangle \otimes |m=-1\rangle + 0|s_z=0\rangle \otimes |m=0\rangle - \sqrt{\frac{1}{2}}|s_z=-1\rangle \otimes |m=+1\rangle \ .$$

3. For $c=a=\sqrt{\frac{1}{3}}$ and $b=-a=-\sqrt{\frac{1}{3}}$ we find

$$J^2 \begin{pmatrix} aY^{(1)}_{-1} \\ \\ -aY^{(1)}_{0} \\ \\ aY^{(1)}_{+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \\ \\ 0 \end{pmatrix} = 0 \begin{pmatrix} aY^{(1)}_{-1} \\ \\ -aY^{(1)}_{0} \\ \\ aY^{(1)}_{+1} \end{pmatrix} \quad \Longleftrightarrow \quad j=0\ .$$

Conclusion:

$$|j=0, j_z=0\rangle = \begin{pmatrix} \sqrt{\frac{1}{3}}Y^{(1)}_{-1} \\ \\ -\sqrt{\frac{1}{3}}Y^{(1)}_{0} \\ \\ \sqrt{\frac{1}{3}}Y^{(1)}_{+1} \end{pmatrix} =$$

$$= \sqrt{\frac{1}{3}}|s_z=+1\rangle \otimes |m=-1\rangle - \sqrt{\frac{1}{3}}|s_z=0\rangle \otimes |m=0\rangle + \sqrt{\frac{1}{3}}|s_z=-1\rangle \otimes |m=+1\rangle \ .$$

Next, we study two states at once:

$$J^2 \begin{pmatrix} 0 \\ bY_{-1}^{(1)} \\ cY_0^{(1)} \end{pmatrix} = \begin{pmatrix} b\sqrt{2}L_- Y_{-1}^{(1)} \\ 4bY_{-1}^{(1)} + c\sqrt{2}L_- Y_0^{(1)} \\ b\sqrt{2}L_+ Y_{-1}^{(1)} + c(4-2L_z)Y_0^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ (4b+2c)Y_{-1}^{(1)} \\ (2b+4c)Y_0^{(1)} \end{pmatrix}$$

and

$$J_z \begin{pmatrix} 0 \\ bY_{-1}^{(1)} \\ cY_0^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ bL_z Y_{-1}^{(1)} \\ c(L_z-1)Y_0^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ -bY_{-1}^{(1)} \\ -cY_0^{(1)} \end{pmatrix} = - \begin{pmatrix} 0 \\ bY_{-1}^{(1)} \\ cY_0^{(1)} \end{pmatrix} \iff j_z = -1 \ .$$

1. For $c = b = \sqrt{\frac{1}{2}}$ we find

$$J^2 \begin{pmatrix} 0 \\ bY_{-1}^{(1)} \\ bY_0^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 6bY_{-1}^{(1)} \\ 6bY_0^{(1)} \end{pmatrix} = 6 \begin{pmatrix} 0 \\ bY_{-1}^{(1)} \\ bY_0^{(1)} \end{pmatrix} \iff j = 2 \ .$$

   Conclusion:

$$|j=2, j_z=-1\rangle = \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}}Y_{-1}^{(1)} \\ \sqrt{\frac{1}{2}}Y_0^{(1)} \end{pmatrix} = \sqrt{\frac{1}{2}}|s_z=0\rangle \otimes |m=-1\rangle + \sqrt{\frac{1}{2}}|s_z=-1\rangle \otimes |m=0\rangle \ .$$

2. For $c = -b = -\sqrt{\frac{1}{2}}$ we find

$$J^2 \begin{pmatrix} 0 \\ bY_{-1}^{(1)} \\ -bY_0^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 2bY_{-1}^{(1)} \\ -2bY_0^{(1)} \end{pmatrix} = 2 \begin{pmatrix} 0 \\ bY_{-1}^{(1)} \\ -bY_0^{(1)} \end{pmatrix} \iff j = 1 \ .$$

   Conclusion:

$$|j=1, j_z=-1\rangle = \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}}Y_{-1}^{(1)} \\ -\sqrt{\frac{1}{2}}Y_0^{(1)} \end{pmatrix} = \sqrt{\frac{1}{2}}|s_z=0\rangle \otimes |m=-1\rangle - \sqrt{\frac{1}{2}}|s_z=-1\rangle \otimes |m=0\rangle \ .$$

Finally, we are left with one more state.

$$J^2 \begin{pmatrix} 0 \\ 0 \\ Y_{-1}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}L_- Y_{-1}^{(1)} \\ (4-2L_z)Y_{-1}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6Y_{-1}^{(1)} \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 0 \\ Y_{-1}^{(1)} \end{pmatrix} \iff j = 2 \ .$$

Furthermore,

$$J_z \begin{pmatrix} 0 \\ 0 \\ Y^{(1)}_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (L_z - 1) Y^{(1)}_{-1} \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 0 \\ Y^{(1)}_{-1} \end{pmatrix} \quad \Longleftrightarrow \quad j_z = -2 \ .$$

Conclusion:

$$|j = 2, j_z = -2\rangle = \begin{pmatrix} 0 \\ 0 \\ Y^{(1)}_{-1} \end{pmatrix} = |s_z = -1\rangle \otimes |m = -1\rangle \ .$$

c. We found ($|j, j_z\rangle$ and $|s_z\rangle \otimes |\ell_z\rangle$):

1. Five states with $j = 2$:

$$\begin{aligned} |2, +2\rangle &= |+1\rangle \otimes |+1\rangle \\ |2, +1\rangle &= \sqrt{\tfrac{1}{2}} |+1\rangle \otimes |0\rangle + \sqrt{\tfrac{1}{2}} |0\rangle \otimes |+1\rangle \\ |2, 0\rangle &= \sqrt{\tfrac{1}{6}} |+1\rangle \otimes |-1\rangle + \sqrt{\tfrac{2}{3}} |0\rangle \otimes |0\rangle + \sqrt{\tfrac{1}{6}} |-1\rangle \otimes |+1\rangle \\ |2, -1\rangle &= \sqrt{\tfrac{1}{2}} |0\rangle \otimes |-1\rangle + \sqrt{\tfrac{1}{2}} |-1\rangle \otimes |0\rangle \\ |2, -2\rangle &= |-1\rangle \otimes |-1\rangle \end{aligned}$$

2. Three states with $j = 1$:

$$\begin{aligned} |1, +1\rangle &= \sqrt{\tfrac{1}{2}} |+1\rangle \otimes |0\rangle - \sqrt{\tfrac{1}{2}} |0\rangle \otimes |+1\rangle \\ |1, 0\rangle &= \sqrt{\tfrac{1}{2}} |+1\rangle \otimes |-1\rangle + 0 |0\rangle \otimes |0\rangle - \sqrt{\tfrac{1}{2}} |-1\rangle \otimes |+1\rangle \\ |1, -1\rangle &= \sqrt{\tfrac{1}{2}} |0\rangle \otimes |-1\rangle - \sqrt{\tfrac{1}{2}} |-1\rangle \otimes |0\rangle \end{aligned}$$

3. One state with $j = 0$:

$$|0, 0\rangle = \sqrt{\tfrac{1}{3}} |+1\rangle \otimes |-1\rangle - \sqrt{\tfrac{1}{3}} |0\rangle \otimes |0\rangle + \sqrt{\tfrac{1}{3}} |-1\rangle \otimes |+1\rangle \ .$$

d. From the solutions of (**c**) we read the Clebsch-Gordan coefficients:

$$\begin{pmatrix} \ell & s & j \\ \ell_z & s_z & j_z \end{pmatrix} .$$

	$s_z = +1$	$s_z = 0$	$s_z = -1$
$j = 2$, $j_z = +2$	$\begin{pmatrix} 1 & 1 & 2 \\ +1 & +1 & +2 \end{pmatrix} = 1$		
$j_z = +1$	$\begin{pmatrix} 1 & 1 & 2 \\ +1 & 0 & +1 \end{pmatrix} = \sqrt{\frac{1}{2}}$	$\begin{pmatrix} 1 & 1 & 2 \\ 0 & +1 & +1 \end{pmatrix} = \sqrt{\frac{1}{2}}$	
$j_z = 0$	$\begin{pmatrix} 1 & 1 & 2 \\ +1 & -1 & 0 \end{pmatrix} = \sqrt{\frac{1}{6}}$	$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{2}{3}}$	$\begin{pmatrix} 1 & 1 & 2 \\ -1 & +1 & 0 \end{pmatrix} = \sqrt{\frac{1}{6}}$
$j_z = -1$		$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \end{pmatrix} = \sqrt{\frac{1}{2}}$	$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \sqrt{\frac{1}{2}}$
$j_z = -2$			$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix} = 1$
$j = 1$, $j_z = +1$	$\begin{pmatrix} 1 & 1 & 1 \\ +1 & 0 & +1 \end{pmatrix} = \sqrt{\frac{1}{2}}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & +1 & +1 \end{pmatrix} = -\sqrt{\frac{1}{2}}$	
$j_z = 0$	$\begin{pmatrix} 1 & 1 & 1 \\ +1 & -1 & 0 \end{pmatrix} = \sqrt{\frac{1}{6}}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$	$\begin{pmatrix} 1 & 1 & 1 \\ -1 & +1 & 0 \end{pmatrix} = -\sqrt{\frac{1}{6}}$
$j_z = -1$		$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} = \sqrt{\frac{1}{2}}$	$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} = -\sqrt{\frac{1}{2}}$
$j = 0$, $j_z = 0$	$\begin{pmatrix} 1 & 1 & 0 \\ +1 & -1 & 0 \end{pmatrix} = \sqrt{\frac{1}{3}}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\sqrt{\frac{1}{3}}$	$\begin{pmatrix} 1 & 1 & 0 \\ -1 & +1 & 0 \end{pmatrix} = \sqrt{\frac{1}{3}}$

See also
http://en.wikipedia.org/wiki/Table_of_Clebsch-Gordan_coefficients#j1.3D1.2C_j2.3D1