MECÂNICA QUÂNTICA II

Exame Normal, Segunda-feira, 3 de Junho de 2013

1. Seja a dinâmica de um pacote de ondas numa dimensão e em unidades $\hbar = c = 1$ dada pela seguinte equação de Schrödinger

$$i\frac{\partial}{\partial t}\psi(x,t) \;=\; -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x,t) \quad .$$

No instante $t = 0$ a função de onda que representa uma partícula em repouso é dada por $\psi(x,0) = \mathcal{N}\, e^{-x^2}$.

a Determine a constante da normalização $\mathcal{N}$.

No instante genérico t a função de onda é dada por

$$\psi(x,t) = \mathcal{N} \exp\left\{\alpha(t) - \beta(t)x^2\right\} \quad .$$

b Determine $\alpha(t)$ e $\beta(t)$.

Nota que $\dfrac{df}{dt} = -cf^2$ para $f(t) = (1+ct)^{-1}$.

c Determine a probabilidade, $P(x,t)$, para encontrar a partícula na posição x no instante t.

Seja a incerteza, Δx, na posição da partícula dada por

$$P(x,t) = \frac{e^{-\frac{1}{2}\left(\frac{x}{\Delta x}\right)^2}}{\Delta x\sqrt{2\pi}} \quad .$$

d Determine Δx para a função de onda obtida na alínea (**b**).

2. Seja o cotangente do desvio de fase, $\delta_0(E)$, numa experiência de dispersão esférico simétrico em onda S e em unidades $\hbar = c = 1$ dado por

$$\mathrm{cotg}\left(\delta_0(E)\right) \;=\; -\mathrm{cotg}(ka) + \frac{2ka\,(E_0 - E)}{\lambda^2 \mu a \sin^2(ka)} \quad , \qquad (1)$$

onde E representa a energia cinética relativa entre o projéctil e o alvo, μ a massa reduzida do sistema projéctil-alvo, λ e a respectivamente a intensidade e a distância média da interacção entre o projéctil e o alvo e onde k é dado por $k^2 = 2\mu E$.

a Mostre que o limite $\lambda \to \infty$ representa o caso de dispersão em onda S ($\ell = 0$) contra uma esfera impenetrável de raio a.

A função de dispersão $S_0(E)$ e a amplitude de dispersão $T_0(E)$ são respectivamente dadas por

$$S_0(E) = e^{2i\delta_0(E)} \qquad \text{e} \quad S_0(E) = 1 + 2iT_0(E)$$

b Mostre que $T_0(E) = 1/\left(\mathrm{cotg}\left(\delta_0(E)\right) - i\right)$.

c Mostre que o caso $\mathrm{cotg}\left(\delta_0(E)\right) = i$, portanto $T_0(E) \to \infty$, resulta na seguinte equação

$$E = E_0 - \tfrac{1}{2}\lambda^2 \frac{\mu}{k}\left\{\sin(ka)\cos(ka) + i\sin^2(ka)\right\} \quad .$$

Para os casos de $\lambda^2 \ll 1$ podemos aproximar a solução desta equação, designada pelo pólo de T_0, por

$$E_{\text{pólo}} \approx E_0 - \tfrac{1}{2}\lambda^2 \frac{\mu}{k_0}\left\{\sin(k_0 a)\cos(k_0 a) + i\sin^2(k_0 a)\right\} \quad \text{onde} \quad k_0 = \sqrt{2\mu E_0} \quad .$$

A parte real, E_R, e a parte imaginária, $-\frac{1}{2}\Gamma_R$, do pólo de T_0 são respectivamente dadas por

$$E_R \approx E_0 - \tfrac{1}{2}\lambda^2 \frac{\mu}{k_0}\sin(k_0 a)\cos(k_0 a) \qquad \text{e} \qquad \Gamma_R \approx \lambda^2 \frac{\mu}{k_0}\sin^2(k_0 a) \quad .$$

d Substituindo apenas k por k_0 na equação (1), mostre que

$$\mathrm{cotg}\left(\delta_0(E)\right) \approx \frac{E_R - E}{\frac{1}{2}\Gamma_R}$$

e faça um esquema de $\left|T_0(E)\right|^2$, para o caso $E_0 = 0.8$ GeV, $\mu = 0.14$ GeV, $a = 2.5$ GeV^{-1} e $\lambda = 0.5$, no qual E_0, E_R e Γ_R são indicados.

3. Considere a barreira de potencial numa dimensão da seguinte forma ($V_0 > 0$):

$$V(x) \; = \; \begin{cases} 0 & \text{para} \quad x < 0 \\ V_0 & \text{para} \quad 0 \leq x \leq a \\ 0 & \text{para} \quad x > a \end{cases}$$

Pretendemos estudar uma onda que vem de $x = -\infty$ e que se desloca no sentido positivo. A onda é suposta ser reflectida em parte (em $x = 0$) e na outra parte ser transmitida (em $x = a$) pela barreira de potencial. Esta situação é representada por uma função de onda para

$$E = \frac{\hbar^2 k_0^2}{2m} = V_0 + \frac{\hbar^2 k_1^2}{2m} > V_0 \quad ,$$

dada por (A e B constantes)

$$\psi(x) \; = \; \begin{cases} e^{ik_0 x} + r\, e^{-ik_0 x} & \text{para} \quad x < 0 \\ Ae^{ik_1 x} + Be^{-ik_1 x} & \text{para} \quad 0 \leq x \leq a \\ t\, e^{ik_0 x} & \text{para} \quad x > a \end{cases}$$

onde r representa o coeficiente de reflexão e t o coeficiente de transmissão.

a Mostre que $\psi(x)$ é solução da equação de Schrödinger e ainda que

$$r = \frac{(k_0^2 - k_1^2)\sin(k_1 a)}{(k_0^2 + k_1^2)\sin(k_1 a) + 2ik_0 k_1 \cos(k_1 a)} \quad ,$$

e

$$t = \frac{2k_0 k_1 \left\{\sin(k_0 a) + i\cos(k_0 a)\right\}}{(k_0^2 + k_1^2)\,\sin(k_1 a) + 2ik_0 k_1\,\cos(k_1 a)} \quad .$$

b A intensidade relativa T do sinal transmitido é igual ao quadrado do módulo da coeficiente de transmissão t. Mostre que

$$T(E > V_0 > 0) = \frac{1}{1 + \dfrac{V_0^2}{4E(E - V_0)}\sin^2(k_1 a)} \quad \text{e} \quad T + R = |t|^2 + |r|^2 = 1 \quad .$$

c O caso $0 < E < V_0$ obtém-se da alínea **a** através da substituição $k_1 \to i\kappa_1$. Mostre que

$$T(0 < E < V_0) = \frac{1}{1 + \dfrac{V_0^2}{4E(V_0 - E)}\sinh^2(\kappa_1 a)} \quad .$$

4. Considere a seguinte representação dos operadores J_x, J_y e J_z:

$$d\left(J_x\right)=\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad d\left(J_y\right)=\frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix} \quad \text{e} \quad d\left(J_z\right)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

a Determine os seguintes comutadores

$$\left[d\left(J_x\right)\ ,\ d\left(J_y\right)\right], \quad \left[d\left(J_y\right)\ ,\ d\left(J_z\right)\right] \quad \text{e} \quad \left[d\left(J_z\right)\ ,\ d\left(J_x\right)\right]$$

b Determine $J^2=d\left(J_x\right)^2+d\left(J_y\right)^2+d\left(J_z\right)^2$

c Determine os valores próprios e os estados próprios de J^2.

d Determine os valores próprios de $d\left(J_z\right)$ dos estados próprios da alínea (**c**) e tira uma conclusão do resultado.

Solutions

Exercício 1

a.

$$1 = \int_{-\infty}^{+\infty} dx \, |\psi(x,0)|^2 = |\mathcal{N}|^2 \int_{-\infty}^{+\infty} dx \, e^{-2x^2} = |\mathcal{N}|^2 \sqrt{\frac{\pi}{2}}$$

Consequently, $|\mathcal{N}| = \left[\frac{2}{\pi}\right]^{1/4}$.

b. From the Schrödinger equation we obtain

$$i\left\{\frac{d\alpha}{dt} - \frac{d\beta}{dt}x^2\right\}\mathcal{N}e^{\alpha - \beta x^2} = i\frac{\partial}{\partial t}\psi(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x,t) = -\frac{1}{2}\left\{-2\beta + 4\beta^2 x^2\right\}\mathcal{N}e^{\alpha - \beta x^2}$$

We end up with the following differential equations:

$$i\frac{d\alpha}{dt} = \beta \qquad \text{and} \qquad i\frac{d\beta}{dt} = 2\beta^2$$

We experiment for β with $\beta(t) = (1+ct)^{-1}$ to find

$$i\frac{d\beta}{dt} = -ic(1+ct)^{-2} = -ic\beta^2$$

Hence, with $c = 2i$ we obtain $i\frac{d\beta}{dt} = 2\beta^2$, whereas the expression $\beta(t) = (1+2it)^{-1}$ also satisfies the boundary condition $\beta(t=0) = 1$.
For $\alpha(t)$ we obtain then

$$i\frac{d\alpha}{dt} = (1+2it)^{-1}$$

which is solved by

$$\alpha(t) = -\tfrac{1}{2}\log(1+2it)$$

which, moreover, satisfies the boundary condition $\alpha(t=0) = 0$.
For the wave function we find thus

$$\psi(x,t) = \mathcal{N}e^{-\frac{1}{2}\log(1+2it) - (1+2it)^{-1}x^2} = \frac{\mathcal{N}e^{-(1+2it)^{-1}x^2}}{\sqrt{1+2it}}$$

c. For the probability, $P(x,t)$, to find the particle at position x at the instant t, we obtain

$$P(x,t) = |\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t) =$$

$$\frac{\mathcal{N}^* e^{-(1-2it)^{-1}x^2}}{\sqrt{1-2it}}\frac{\mathcal{N}e^{-(1+2it)^{-1}x^2}}{\sqrt{1+2it}} = \frac{|\mathcal{N}|^2 e^{-2\left(1+4t^2\right)^{-1}x^2}}{\sqrt{1+4t^2}} = \frac{\sqrt{2}\, e^{-2\left(1+4t^2\right)^{-1}x^2}}{\sqrt{\pi}\sqrt{1+4t^2}}$$

d. When we compare the expression of alinea (**c**) with the general expression given by

$$P(x,t) = \frac{e^{-\frac{1}{2}\left(\frac{x}{\Delta x}\right)^2}}{\Delta x\sqrt{2\pi}}$$

we obtain

$$\Delta x = \tfrac{1}{2}\sqrt{1+4t^2}$$

Note that Δx increases with time.

Exercício 2

a. Let us first recall that the radial part of the Schrödinger equation is given by ($\hbar = c = 1$)

$$\left\{-\frac{1}{2\mu}\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right] + V(r)\right\} u_\ell(r) = E u_\ell(r) = \frac{k^2}{2\mu} u_\ell(r)$$

For $\ell = 0$ this equation reduces to

$$\left\{-\frac{1}{2\mu}\frac{d^2}{dr^2} + V(r)\right\} u_0(r) = E u_0(r) = \frac{k^2}{2\mu} u_0(r)$$

The projectile cannot reach distances smaller than the radius a of the sphere. Consequently, one has the condition $u_0(r) = 0$ for $r \leq a$ and thus $u_0(a) = 0$. For $r > a$ the potential $V(r)$ vanishes. Hence, for $r > a$ the Schrödinger equation reduces to

$$-\frac{1}{2\mu}\frac{d^2 u_0(r)}{dr^2} = E u_0(r) = \frac{k^2}{2\mu} u_0(r)$$

which is solved by

$$u_0(r) = \sin(kr) \qquad \text{or} \qquad u_0(r) = \cos(kr)$$

Consequently, a general solution for the wave function is given by (A and B constants)

$$\begin{aligned} u_0(r) &= A\sin(kr) + B\cos(kr) = \sqrt{A^2+B^2}\left\{\frac{A}{\sqrt{A^2+B^2}}\sin(kr) + \frac{B}{\sqrt{A^2+B^2}}\cos(kr)\right\} \\ &= \sqrt{A^2+B^2}\sin\left(kr + \delta_0(E)\right) \end{aligned}$$

with $\cos\left(\delta_0(E)\right) = A/\sqrt{A^2+B^2}$ and $\sin\left(\delta_0(E)\right) = B/\sqrt{A^2+B^2}$.
From the boundary condition $u_0(a) = 0$ we obtain

$$\begin{aligned} 0 = u_0(a) &= \sqrt{A^2+B^2}\sin\left(ka + \delta_0(E)\right) = \\ &= \sqrt{A^2+B^2}\left\{\cos\left(\delta_0(E)\right)\sin(ka) + \sin\left(\delta_0(E)\right)\cos(ka)\right\} \end{aligned}$$

Hence,

$$\text{cotg}\left(\delta_0(E)\right) = -\text{cotg}(ka)$$

which is the limit of equation (1) for $\lambda \to \infty$.

b.

$$\begin{aligned} T_0(E) &= \frac{S_0(E) - 1}{2i} = \frac{e^{2i\delta_0(E)} - 1}{2i} = \frac{e^{i\delta_0(E)} - e^{-i\delta_0(E)}}{2ie^{-i\delta_0(E)}} = \\ &= \frac{\sin\left(\delta_0(E)\right)}{e^{-i\delta_0(E)}} = \frac{\sin\left(\delta_0(E)\right)}{\cos\left(\delta_0(E)\right) - i\sin\left(\delta_0(E)\right)} = \frac{1}{\text{cotg}\left(\delta_0(E)\right) - i} \end{aligned}$$

c.

$$i = \text{cotg}\left(\delta_0(E)\right) = -\text{cotg}(ka) + \frac{2ka\,(E_0 - E)}{\lambda^2 \mu a \sin^2(ka)}$$

$$\Longleftrightarrow \quad i\lambda^2 \mu a \sin^2(ka) + \lambda^2 \mu a \sin(ka)\cos(ka) = 2ka\,(E_0 - E)$$

$$\Longleftrightarrow \quad E_0 - E = \lambda^2 \frac{\mu a}{2ka} \sin(ka)\cos(ka) + i\lambda^2 \frac{\mu a}{2ka} \sin^2(ka)$$

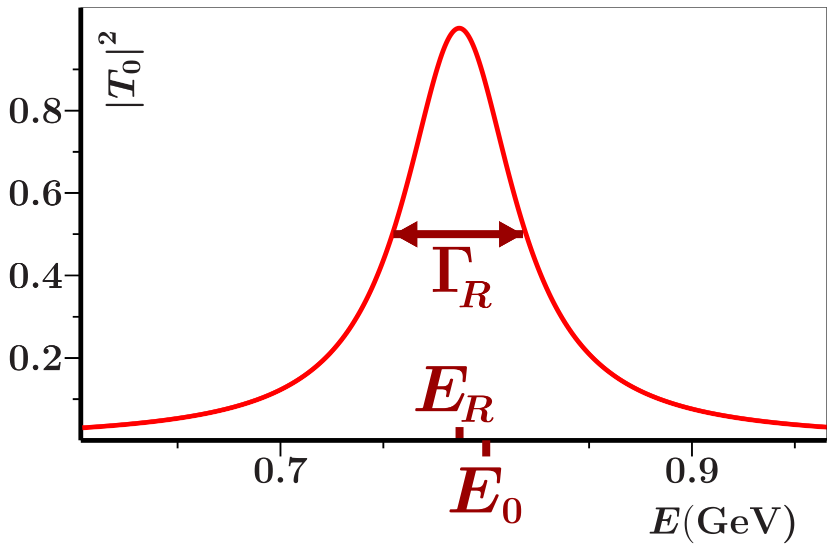
$$\Longleftrightarrow \quad E = E_0 - \tfrac{1}{2}\lambda^2 \frac{\mu}{k} \left\{\sin(ka)\cos(ka) + i\sin^2(ka)\right\}$$

d.

$$\text{cotg}\left(\delta_0(E)\right) = -\text{cotg}(k_0 a) + \frac{2k_0 a\,(E_0 - E)}{\lambda^2 \mu a \sin^2(k_0 a)} =$$

$$= \frac{-\lambda^2 \mu a \cos(k_0 a)\sin(k_0 a) + 2k_0 a\,(E_0 - E)}{\lambda^2 \mu a \sin^2(k_0 a)}$$

$$= \frac{E_0 - \tfrac{1}{2}\lambda^2 \frac{\mu}{k_0}\cos(k_0 a)\sin(k_0 a) - E}{\tfrac{1}{2}\lambda^2 \frac{\mu}{k_0}\sin^2(k_0 a)} = \frac{E_R - E}{\tfrac{1}{2}\Gamma_R} \quad .$$



$$k_0 = \sqrt{2\mu E_0} = 0.473 \ \text{GeV}, \quad k_0 a = 1.183, \quad \sin(k_0 a) = 0.926, \quad \cos(k_0 a) = 0.378,$$

$$E_R = E_0 - \tfrac{1}{2}\lambda^2 \frac{\mu}{k_0}\cos(k_0 a)\sin(k_0 a) = 0.787 \ \text{GeV}$$

$$\Gamma_R = \lambda^2 \frac{\mu}{k_0}\sin^2(k_0 a) = 0.063 \ \text{GeV}$$

Exercício 3

a. The Schrödinger equation in one-dimension reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left\{\begin{array}{cl} 0 & (x<0) \\ V_0 & (0\leq x\leq a) \\ 0 & (x>a) \end{array}\right\}\psi = E\psi$$

which can be casted in the form

$$\hbar^2\frac{d^2\psi}{dx^2} = \left\{\begin{array}{cl} -2mE & (x<0) \\ -2m(E-V_0) & (0\leq x\leq a) \\ -2mE & (x>a) \end{array}\right\}\psi$$

Solutions for $d^2\psi/dx^2 = -2mE\psi$ are given by $\psi = \exp(\pm ik_0x)$ with $\hbar^2k_0^2 = 2mE$, whereas solutions for $d^2\psi/dx^2 = -2m(E-V_0)\psi$ are given by $\psi = \exp(\pm ik_1x)$ with $\hbar^2k_1^2 = 2m(E-V_0)$. Linear combinations are also solutions of the Schrödinger equation.
From the boundary conditions at $x = 0$ we find

$$\begin{aligned} 1+r = \psi(x\uparrow 0) &= \psi(x\downarrow 0) = A+B \\ ik_0(1-r) = \left[\frac{d\psi}{dx}\right]_{x\uparrow 0} &= \left[\frac{d\psi}{dx}\right]_{x\downarrow 0} = ik_1(A-B) \end{aligned}$$

whereas, from the boundary conditions at $x = a$, we find

$$\begin{aligned} Ae^{ik_1a}+Be^{-ik_1a} = \psi(x\uparrow a) &= \psi(x\downarrow a) = te^{ik_0a} \\ ik_1\left(Ae^{ik_1a}-Be^{-ik_1a}\right) = \left[\frac{d\psi}{dx}\right]_{x\uparrow a} &= \left[\frac{d\psi}{dx}\right]_{x\downarrow a} = ik_0te^{ik_0a} \end{aligned}$$

From those equations we deduce

$$\begin{aligned} 2A &= (1+r)+\frac{k_0}{k_1}(1-r) \\ 2B &= (1+r)-\frac{k_0}{k_1}(1-r) \\ 2Ae^{ik_1a} &= \left(1+\frac{k_0}{k_1}\right)te^{ik_0a} \\ 2Be^{-ik_1a} &= \left(1-\frac{k_0}{k_1}\right)te^{ik_0a} \end{aligned}$$

Hence

$$\begin{aligned} (1+r)+\frac{k_0}{k_1}(1-r) &= \left(1+\frac{k_0}{k_1}\right)te^{i(k_0-k_1)a} \\ (1+r)-\frac{k_0}{k_1}(1-r) &= \left(1-\frac{k_0}{k_1}\right)te^{i(k_0+k_1)a} \end{aligned}$$

from the latter relations we obtain

$$\frac{(k_1+k_0)+(k_1-k_0)\,r}{(k_1-k_0)+(k_1+k_0)\,r}=\frac{(1+r)+\dfrac{k_0}{k_1}(1-r)}{(1+r)-\dfrac{k_0}{k_1}(1-r)} \;=\; \frac{\left(1+\dfrac{k_0}{k_1}\right)e^{-ik_1a}}{\left(1-\dfrac{k_0}{k_1}\right)e^{ik_1a}}=\frac{(k_1+k_0)\,e^{-ik_1a}}{(k_1-k_0)\,e^{ik_1a}}$$

which is equivalent to

$$\left(k_1^2-k_0^2\right)\left(e^{ik_1a}-e^{-ik_1a}\right) \;=\; \left\{(k_1+k_0)^2\,e^{-ik_1a}-(k_1-k_0)^2\,e^{ik_1a}\right\}r$$

Now,

$$\begin{aligned}
&(k_1+k_0)^2\,e^{-ik_1a}-(k_1-k_0)^2\,e^{ik_1a}=\\
&\quad=\left(k_1^2+k_0^2\right)\left(e^{-ik_1a}-e^{ik_1a}\right)+2k_1k_0\left(e^{-ik_1a}+e^{ik_1a}\right)\\
&\quad=-\left(k_1^2+k_0^2\right)2i\sin(k_1a)+4k_1k_0\cos(k_1a)
\end{aligned}$$

Hence,

$$r \;=\; \frac{(k_1^2-k_0^2)\,2i\sin(k_1a)}{-\,(k_1^2+k_0^2)\,2i\sin(k_1a)+4k_1k_0\cos(k_1a)}$$

or

$$r \;=\; \frac{(k_0^2-k_1^2)\sin(k_1a)}{(k_1^2+k_0^2)\sin(k_1a)+2ik_1k_0\cos(k_1a)}$$

Furthermore, for t, we have

$$\begin{aligned}
&(1+r)+\frac{k_0}{k_1}(1-r)=\left(1+\frac{k_0}{k_1}\right)te^{i(k_0-k_1)a}\\
\Longleftrightarrow\quad &(k_1+k_0)+(k_1-k_0)\,r=(k_1+k_0)\,te^{i(k_0-k_1)a}\\
\Longleftrightarrow\quad &t=\left(1+\frac{k_1-k_0}{k_1+k_0}r\right)e^{-i(k_0-k_1)a}
\end{aligned}$$

Hence, by substitution of r, we find

$$t=\left(1+\frac{-\,(k_1-k_0)^2\sin(k_1a)}{(k_1^2+k_0^2)\sin(k_1a)+2ik_1k_0\cos(k_1a)}\right)e^{-i(k_0-k_1)a}=$$

$$=\left(\frac{(k_1^2+k_0^2)\sin(k_1a)+2ik_1k_0\cos(k_1a)-(k_1-k_0)^2\sin(k_1a)}{(k_1^2+k_0^2)\sin(k_1a)+2ik_1k_0\cos(k_1a)}\right)e^{-i(k_0-k_1)a}$$

$$= \left(\frac{2ik_1k_0\cos(k_1a) + 2k_1k_0\sin(k_1a)}{(k_1^2+k_0^2)\sin(k_1a) + 2ik_1k_0\cos(k_1a)} \right) e^{-i(k_0-k_1)a}$$

$$= \left(\frac{2ik_1k_0 e^{-ik_1a}}{(k_1^2+k_0^2)\sin(k_1a) + 2ik_1k_0\cos(k_1a)} \right) e^{-i(k_0-k_1)a}$$

$$= \frac{2ik_1k_0 e^{-ik_0a}}{(k_1^2+k_0^2)\sin(k_1a) + 2ik_1k_0\cos(k_1a)} = \frac{2k_1k_0\left(i\cos(k_0a)+\sin(k_0a)\right)}{(k_1^2+k_0^2)\sin(k_1a) + 2ik_1k_0\cos(k_1a)}$$

b. The intensity T of the transmitted signal is obtained from the transmission amplitude by

$$T = |t|^2 = \frac{4k_0^2k_1^2}{\left(k_0^2+k_1^2\right)^2\,\sin^2\left(k_1a\right) + 4k_0^2k_1^2\cos^2\left(k_1a\right)}$$

$$= \frac{k_0^2k_1^2}{k_0^2k_1^2\cos^2\left(k_1a\right) + \frac{1}{4}\left(k_0^2+k_1^2\right)^2\,\sin^2\left(k_1a\right)}$$

$$= \frac{k_0^2k_1^2}{k_0^2k_1^2 + \frac{1}{4}\left(k_0^2-k_1^2\right)^2\,\sin^2\left(k_1a\right)} = \frac{1}{1+\frac{\left(k_0^2-k_1^2\right)^2}{4k_0^2k_1^2}\,\sin^2\left(k_1a\right)}$$

$$= \frac{1}{1+\frac{V_0^2}{4E(E-V_0)}\,\sin^2\left(k_1a\right)} \quad .$$

This formula works also for the other cases as we will see below.

But, first let us determine the intensity of the reflected signal $R = |r|^2$.

We obtained before

$$r = \frac{\left(k_0^2-k_1^2\right)\sin\left(k_1a\right)}{\left(k_0^2+k_1^2\right)\sin\left(k_1a\right) + 2ik_0k_1\cos\left(k_1a\right)} \quad .$$

This can also be written in the form

$$r = \frac{-\frac{i}{2}\left(k_0^2-k_1^2\right)\sin\left(k_1a\right)}{k_0k_1\,\cos\left(k_1a\right) - \frac{i}{2}\left(k_0^2+k_1^2\right)\,\sin\left(k_1a\right)} \quad .$$

Consequently R takes the form

$$R = |r|^2 = \frac{\frac{1}{4}\left(k_0^2-k_1^2\right)^2\sin^2\left(k_1a\right)}{k_0^2k_1^2\cos^2\left(k_1a\right) + \frac{1}{4}\left(k_0^2+k_1^2\right)^2\,\sin^2\left(k_1a\right)} \quad .$$

If we add this expression to

$$T = \frac{k_0^2 k_1^2}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)} \quad ,$$

then we obtain

$$R+T = \frac{k_0^2 k_1^2 + \frac{1}{4}\left(k_0^2 - k_1^2\right)^2 \sin^2(k_1 a)}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)} = \frac{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)}{k_0^2 k_1^2 \cos^2(k_1 a) + \frac{1}{4}\left(k_0^2 + k_1^2\right)^2 \sin^2(k_1 a)} = 1 \quad .$$

The sum of the intensities of the reflected and the transmitted signals, $R + T$, equals the initial intensity, which is the coefficient, equal to 1, of the incoming wave for $x < 0$.

d. For $0 < E < V_0$, we just perform the substitution $k_1 \to i\kappa_1$ and also observe

$$\sin(ix) = \frac{1}{2i}\left(e^{-x} - e^{x}\right) = i \sinh(x) \quad .$$

We obtain

$$T = \frac{1}{1 - \frac{V_0^2}{4E(E - V_0)} \sinh^2(\kappa_1 a)} = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(\kappa_1 a)} \quad .$$

Exercício 4

a.

$$d\left(J_x\right)d\left(J_y\right)=\frac{1}{2}\begin{pmatrix} i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \end{pmatrix} \quad \text{and} \quad d\left(J_y\right)d\left(J_x\right)=\frac{1}{2}\begin{pmatrix} -i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & i \end{pmatrix}$$

$$d\left(J_x\right)d\left(J_y\right)-d\left(J_y\right)d\left(J_x\right)=\frac{1}{2}\begin{pmatrix} 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}=i\,d\left(J_z\right)$$

$$d\left(J_y\right)d\left(J_z\right)=\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & i \\ i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d\left(J_z\right)d\left(J_y\right)=\frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i & -i & 0 \end{pmatrix}$$

$$d\left(J_y\right)d\left(J_z\right)-d\left(J_z\right)d\left(J_y\right)=\frac{1}{2}\begin{pmatrix} 0 & i & i & 0 \\ i & 0 & 0 & i \\ i & 0 & 0 & i \\ 0 & i & i & 0 \end{pmatrix}=i\,d\left(J_x\right)$$

$$d\left(J_z\right)d\left(J_x\right)=\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad d\left(J_x\right)d\left(J_z\right)=\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$d\left(J_z\right)d\left(J_x\right)-d\left(J_x\right)d\left(J_z\right)=\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}=i\,d\left(J_y\right)$$

b.

$$d\left(J_x\right)^2=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad d\left(J_y\right)^2=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad d\left(J_z\right)^2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence

$$J^2=\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

c.

$$\det\left(J^2-\lambda\mathbf{1}\right)=\det\begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix}=$$

$$= (2-\lambda)^2 \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (2-\lambda)^2 \left\{ (1-\lambda)^2 - 1 \right\} = (\lambda - 2)^3 \lambda$$

Hence, J^2 has three eigenvalues equal to 2 and one equal to 0. The three eigenstates with eigenvalues equal to 2 are given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The one eigenstate with eigenvalue equal to 0 is given by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} .$$

d.

$$d\left(J_z\right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$d\left(J_z\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0 = 0 \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$d\left(J_z\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = -1 \times \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, we found three states for $j = 1$, $j(j+1) = 2$, with eigenvalues $j_z = 1, 0, -1$ for $d\left(J_z\right)$. Furthermore,

$$d\left(J_z\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0 = 0 \times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

we found one state for $j = 0$, $j(j+1) = 0$ with eigenvalue $j_z = 0$ for $d\left(J_z\right)$.