MECÂNICA QUÂNTICA II

Exame de recurso, Terça-feira, 2 de Julho de 2013

1. Considere um mesão (bosão com momento angular total $J_{a\bar{a}} = 0$) que consiste de um quark a (fermião com spin $s_a = \frac{1}{2}$) e um antiquark $\bar{a}$ (fermião com spin $s_{\bar{a}} = \frac{1}{2}$) cujo momento angular relativo entre o quark a e o antiquark $\bar{a}$ é dado por $\ell_{a\bar{a}} = 0$. A relação entre o momento angular total $J_{a\bar{a}}$, o momento angular relativo $\ell_{a\bar{a}}$ e o spin total dos fermiões, $s_{a\bar{a}} = s_a \otimes s_{\bar{a}} = 0$ ou 1, é dado por

$$J_{a\bar{a}} = \ell_{a\bar{a}} \otimes s_{a\bar{a}} \quad .$$

a Mostre que $s_{a\bar{a}} = 0$.

O mesão pode desintegrar em dois mesões sob a criação interna de um novo par quark-antiquark $b\bar{b}$. Os números quânticos do par $b\bar{b}$ são dados por $J_{b\bar{b}} = 0$, $\ell_{b\bar{b}} = 1$ e $s_{b\bar{b}} = 1$.

b Mostre que $J_{b\bar{b}} = \ell_{b\bar{b}} \otimes s_{b\bar{b}}$ pode ser igual a 0.

O momento angular relativo entre o sistemas $a\bar{a}$ e $b\bar{b}$ é igual a $L_{a\bar{a}b\bar{b}} = 0$. O spin total do sistema $a\bar{a}b\bar{b}$ é dado por $s_{a\bar{a}b\bar{b}} = J_{a\bar{a}} \otimes J_{b\bar{b}} = 0$ e o momento angular total por $J = L_{a\bar{a}b\bar{b}} \otimes s_{a\bar{a}b\bar{b}} = 0$. Portanto, $J = J_{a\bar{a}}$, o momento angular total é conservado neste processo de desintegração.

Considere a desintegração do sistema $a\bar{a}b\bar{b}$ em dois mesões $a\bar{b}$ e $b\bar{a}$ com os seguintes números quânticos: os internos do sistema $a\bar{b}$: $J_{a\bar{b}}$, $s_{a\bar{b}}$ e $\ell_{a\bar{b}}$; os internos do sistema $b\bar{a}$: $J_{b\bar{a}}$, $s_{b\bar{a}}$ e $\ell_{b\bar{a}}$; o spin total $S = J_{a\bar{b}} \otimes J_{b\bar{a}}$ dos sistemas $a\bar{b}$ e $b\bar{a}$; o momento angular relativo L entre os sistemas $a\bar{b}$ e $b\bar{a}$.

O momento angular total J é conservado nestes processos de desintegração. Portanto, $J = L \otimes S$.

c Mostre que os seguintes combinações conservam $J = 0$.

$J_{a\bar{b}}$	$s_{a\bar{b}}$	$\ell_{a\bar{b}}$	$J_{b\bar{a}}$	$s_{b\bar{a}}$	$\ell_{b\bar{a}}$	S	L
0	0	0	1	1	0	1	1
0	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1
1	1	0	1	1	1	0	0
1	1	0	1	0	1	0	0

2. Considere uma partícula de massa m que se desloca dentro de um poço potencial unidimensional e cuja dinâmica é dada pelo seguinte Hamiltoniano

$$H = \frac{p^2}{2m} + U(x) + \lambda\delta(x) \quad \text{com} \quad U(x) = \begin{cases} 0 & \text{para} \quad x < |a| \\ \infty & \text{para} \quad x \geq |a| \end{cases} \quad .$$

Caso $\lambda \ll 1$ determine o espectro deste sistema até o primeiro ordem em λ.

3. Considere no espaço de funções de três componentes a seguinte base de nove estados $|\ell=1,\ell_z;s=1,s_z=+1\rangle$, $|\ell=1,\ell_z;s=1,s_z=0\rangle$ e $|\ell=1,\ell_z;s=1,s_z=-1\rangle$ respectivamente dados por

$$\begin{pmatrix} Y_{\ell,\,\ell_z}(\hat{r}) \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ Y_{\ell,\,\ell_z}(\hat{r}) \\ 0 \end{pmatrix} \quad \text{e} \quad \begin{pmatrix} 0 \\ 0 \\ Y_{\ell,\,\ell_z}(\hat{r}) \end{pmatrix} \quad , \quad \text{com} \;\; \ell_z = 0, \pm 1 \quad , \tag{1}$$

onde as funções $Y_{\ell,\,\ell_z}(\hat{r})$ representam as funções esféricas harmónicas.
Seja $\vec{J}=\vec{L}+\vec{S}$, onde L_x, L_y e L_z representam as geradores de rotações e onde S_x, S_y e S_z são respectivamente dados pelas seguintes matrizes

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{e} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2}$$

a Determina o estado próprio de J^2 designado por $|j=1,j_z=0\rangle$, combinação linear das funções dadas na equação (1), para que J^2 tem valor próprio 2.

b Determina o coeficiente de Clebsch-Gordan, designado por:

$$\begin{pmatrix} \ell=1 & s=1 & j=1 \\ \ell_z=0 & s_z=0 & j_z=0 \end{pmatrix} \quad .$$

4. Considere funções de onda de duas componentes $\psi(\vec{r},t)$, cuja dinâmica é dada pela seguinte equação de Schrödinger

$$i\frac{\partial}{\partial t}\,\psi(\vec{r},t) \;=\; \begin{pmatrix} H_0 & i\omega \\ -i\omega & H_0 \end{pmatrix}\,\psi(\vec{r},t) \quad .$$

No instante $t=0$ a função de onda é dada por

$$\psi(\vec{r},t=0) \;=\; \begin{pmatrix} \varphi(\vec{r}\,) \\ 0 \end{pmatrix} \quad ,$$

onde $\varphi(\vec{r}\,)$ é solução de $H_0\varphi(\vec{r}\,)=E\varphi(\vec{r}\,)$. Determine a função de onda $\psi(\vec{r},t)$ no instante genérico t.
Notar que $\exp(A+B)=\exp(A)\exp(B)$ quando $[A\ ,\ B]=0$.

Solutions

Exercício 1

a. One can only combine $\ell_{a\bar{a}} = 0$ with spin $s_{a\bar{a}} = 0$ into $J_{a\bar{a}} = 0$, any other spin would imply more components than one.

b. We must combine $\ell_{b\bar{b}} = 1$ which means three components with $s_{b\bar{b}} = 1$. Hence the combination contains $3 \times 3 = 9$ components. Those can be grouped into one set of five components ($J_{b\bar{b}} = 2$), one set of three components ($J_{b\bar{b}} = 1$) and one set of one component ($J_{b\bar{b}} = 0$). Hence, $J_{b\bar{b}} = 0$ is possible.

c. The first three columns contain all $\ell_{a\bar{b}} = 0$. Hence, one has $J_{a\bar{b}} = \{\ell_{a\bar{b}} = 0\} \otimes s_{a\bar{b}} = s_{a\bar{b}}$, which is indeed the case.
In the next three columns we have similarly $\ell_{b\bar{a}} = 0$ in the first and the third rows, which implies $J_{b\bar{a}} = s_{b\bar{a}}$. Furthermore, in the last row we have $s_{b\bar{a}} = 0$, hence $J_{b\bar{a}} = \ell_{b\bar{a}}$.
The remaining two rows for the $b\bar{a}$ system we find $s_{b\bar{a}} = 1$ and $\ell_{b\bar{a}} = 1$. This is similar to the situation of alinea (**b**) where we have seen that $J_{b\bar{a}}$ may be 0, 1 or 2. Hence, $J_{b\bar{a}} = 0$ is possible (second row) and $J_{b\bar{a}} = 1$ is possible (fourth row).
Next, we must combine $S = J_{a\bar{b}} \otimes J_{b\bar{a}}$.
In the first two rows we have $J_{a\bar{b}} = 0$, hence $S = J_{b\bar{a}}$. The remaining rows have $J_{a\bar{b}} = J_{b\bar{a}} = 1$, similar to the situation of alinea (**b**). Hence, $S = 1$ and $S = 0$ are possible.
Finally, we must combine $J = L \otimes S$ for $J = 0$. This implies that L and S must be equal.

Exercício 2

We start by writing

$$H = H_0 + V \quad \text{where} \quad H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)$$

and where $V = \lambda\delta(x)$ is considered a perturbation for $\lambda \ll 1$.
We first determine the eigenvalues and eigenfunctions for H_0. The Schrödinger equation for $x < |a|$ is given by

$$E\psi^{(0)}(x) = H_0\psi^{(0)}(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi^{(0)}(x) \quad ,$$

with boundary conditions

$$\psi^{(0)}(x = -a) = 0 \quad \text{and} \quad \psi^{(0)}(x = a) = 0 \quad .$$

We define $\hbar^2 k^2 = 2mE$, to obtain for the Schrödinger equation

$$\frac{\partial^2}{\partial x^2}\psi^{(0)}(x) = -k^2\psi^{(0)}(x) \quad ,$$

for which solutions are given by (A and B are normalization constants)

$$\psi^{(0)}(x) = A\sin(kx) \quad \text{and} \quad \psi^{(0)}(x) = B\cos(kx) \quad .$$

From the boundary conditions we obtain

$$0 = \sin(ka) \quad \text{and} \quad 0 = \cos(ka) \quad .$$

For the formar condition we find $ka = \pi,\ 2\pi,\ 3\pi,\ \ldots$, whereas, for the latter condition we obtain $ka = \frac{1}{2}\pi,\ \frac{3}{2}\pi,\ \frac{5}{2}\pi,\ \ldots$. So, we found for k the allowed values

$$k_n = \frac{\pi}{2a}n \quad \text{for} \quad n = 1,\ 2,\ 3,\ 4,\ \ldots \quad .$$

So, we obtain the eigenvalues and eigenfunctions

$$E_n^{(0)} = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2}{8ma^2}n^2 \quad \text{and} \quad \begin{cases} B_n\cos(k_n x) & \text{for} \quad n = 1,\ 3,\ 5,\ \ldots \\ A_n\sin(k_n x) & \text{for} \quad n = 2,\ 4,\ 6,\ \ldots \end{cases} \quad .$$

The first order corrections to the spectrum are given by

$$\begin{aligned} E_n^{(1)} &= \int_{-\infty}^{+\infty} dx\, \psi_n^{0*}(x)\, V(x)\, \psi_n^0(x) = \lambda\int_{-a}^{+a} dx\, \psi_n^{0*}(x)\, \delta(x)\, \psi_n^0(x) = \lambda\left|\psi_n^0(x=0)\right|^2 \\ &= \lambda\begin{cases} |B_n|^2 & \text{for} \quad n = 1,\ 3,\ 5,\ \ldots \\ 0 & \text{for} \quad n = 2,\ 4,\ 6,\ \ldots \end{cases} \quad . \end{aligned}$$

So, it is left to determine the normalization constants B_n.

$$\begin{aligned} 1 &= |B_n|^2\int_{-a}^{+a} dx\, \cos^2(k_n x) = |B_n|^2\int_{-a}^{+a} dx\, \tfrac{1}{2}\left(1 + \cos(2k_n x)\right) \\ &= |B_n|^2\, \tfrac{1}{2}\left(x + \frac{\sin(2k_n x)}{2k_n}\right)\Bigg|_{-a}^{+a} = |B_n|^2\, a \quad . \end{aligned}$$

Notice that $2k_n a = \pi n$ for $n = 1, 3, 5, \ldots$, for which $\sin(\pm 2k_n a) = 0$.
Hence $|B_n|^2 = 1/a$ and thus

$$E_n^{(1)} = \frac{\lambda}{a}\begin{cases} 1 & \text{for} \quad n = 1,\ 3,\ 5,\ \ldots \\ 0 & \text{for} \quad n = 2,\ 4,\ 6,\ \ldots \end{cases} .$$

To first order in λ we obtain for the spectrum

$$E_n = E_n^{(0)} + E_n^{(1)} = \frac{\hbar^2\pi^2}{8ma^2}n^2 + \frac{\lambda}{a}\begin{cases} 1 & \text{for} \quad n = 1,\ 3,\ 5,\ \ldots \\ 0 & \text{for} \quad n = 2,\ 4,\ 6,\ \ldots \end{cases} .$$

Exercício 3

a. We want a linear combination of the functions which are given in Eq. 1 such that J^2 has eigenvalue 2 ($j = 1$) and $j_z = 0$. Now, $j_z = \ell_z + s_z$, whereas $\ell_z = 0, \pm 1$ and $s_z = 0, \pm 1$. So, we can take the combinations ($s_z = +1$, $\ell_z = -1$), ($s_z = 0$, $\ell_z = 0$) and ($s_z = -1$, $\ell_z = +1$), *i.e.*

$$|j = 1, j_z = 0\rangle \;=\; \begin{pmatrix} aY_{1,-1}(\hat{r}) \\ bY_{1,0}(\hat{r}) \\ cY_{1,+1}(\hat{r}) \end{pmatrix} \quad ,$$

where the constants a, b and c must be determined.
Now,

$$J^2 = \left(\vec{L} + \vec{S}\right)^2 = L^2 + S^2 + \vec{L}\cdot\vec{S} + \vec{S}\cdot\vec{L} = L^2 + S^2 + 2\vec{L}\cdot\vec{S} \quad .$$

The last step is allowed since $\vec{L}$ and $\vec{S}$ commute. For $\vec{L}\cdot\vec{S}$ one has

$$\vec{L}\cdot\vec{S} = L_xS_x + L_yS_y + L_zS_z =$$

$$= \begin{pmatrix} L_z & \frac{1}{\sqrt{2}}\left(L_x - iL_y\right) & 0 \\ \frac{1}{\sqrt{2}}\left(L_x + iL_y\right) & 0 & \frac{1}{\sqrt{2}}\left(L_x - iL_y\right) \\ 0 & \frac{1}{\sqrt{2}}\left(L_x + iL_y\right) & -L_z \end{pmatrix} = \begin{pmatrix} L_z & \frac{1}{\sqrt{2}}L_- & 0 \\ \frac{1}{\sqrt{2}}L_+ & 0 & \frac{1}{\sqrt{2}}L_- \\ 0 & \frac{1}{\sqrt{2}}L_+ & -L_z \end{pmatrix} \quad .$$

We know that $\ell = 1$ and $s = 1$, hence for $L^2 + S^2$ we will obtain $2 + 2 = 4$ when it operates on our linear combination. Furtermore,

$$L_zY_{\ell,\,\ell_z} = \ell_zY_{\ell,\,\ell_z} \quad \text{and} \quad L_\pm Y_{\ell,\,\ell_z} = \sqrt{\ell(\ell+1) - \ell_z(\ell_z \pm 1)}\,Y_{\ell,\,\ell_z} \quad .$$

For $\ell = 1$ the latter expression simplifies to

$$\begin{array}{ccc} L_+Y_{1,+1} = 0 & L_+Y_{1,0} = \sqrt{2}\,Y_{1,+1} & L_+Y_{1,-1} = \sqrt{2}\,Y_{1,0} \\ L_-Y_{1,+1} = \sqrt{2}\,Y_{1,0} & L_-Y_{1,0} = \sqrt{2}\,Y_{1,-1} & L_-Y_{1,-1} = 0 \end{array}$$

.

With all this information we may determine

$$2\begin{pmatrix} aY_{1,-1}(\hat{r}) \\ bY_{1,0}(\hat{r}) \\ cY_{1,+1}(\hat{r}) \end{pmatrix} = J^2|j = 1, j_z = 0\rangle =$$

$$= 4|j = 1, j_z = 0\rangle + 2\begin{pmatrix} L_z & \frac{1}{\sqrt{2}}L_- & 0 \\ \frac{1}{\sqrt{2}}L_+ & 0 & \frac{1}{\sqrt{2}}L_- \\ 0 & \frac{1}{\sqrt{2}}L_+ & -L_z \end{pmatrix}\begin{pmatrix} aY_{1,-1}(\hat{r}) \\ bY_{1,0}(\hat{r}) \\ cY_{1,+1}(\hat{r}) \end{pmatrix}$$

$$= \begin{pmatrix} (4a - 2a + 2b) Y_{1,-1}(\hat{r}) \\ \\ (4b + 2a + 2c) Y_{1,0}(\hat{r}) \\ \\ (4c + 2b - 2c) Y_{1,+1}(\hat{r}) \end{pmatrix} \quad .$$

Hence, we obtain equations

$$2a = 4a - 2a + 2b \qquad 2b = 4b + 2a + 2c \qquad 2c = 4c + 2b - 2c \quad ,$$

which is solved by $c = -a$ and $b = 0$. Consequently,

$$|j = 1, j_z = 0\rangle = a \begin{pmatrix} Y_{1,-1}(\hat{r}) \\ \\ 0 \\ \\ -Y_{1,+1}(\hat{r}) \end{pmatrix} \quad .$$

b. We found in alinea (**a**)

$$|j = 1, j_z = 0\rangle = a \begin{pmatrix} Y_{1,-1}(\hat{r}) \\ \\ 0 \\ \\ -Y_{1,+1}(\hat{r}) \end{pmatrix} = a|\ell = 1, \ell_z = -1; s = 1, s_z = +1\rangle +$$

$$+0|\ell = 1, \ell_z = 0; s = 1, s_z = 0\rangle - a|\ell = 1, \ell_z = +1; s = 1, s_z = -1\rangle$$

Hence, the coefficient for $|\ell = 1, \ell_z = 0; s = 1, s_z = 0\rangle$ in $|j = 1, j_z = 0\rangle$ equals 0! Hence,

$$\begin{pmatrix} \ell = 1 & s = 1 & j = 1 \\ \ell_z = 0 & s_z = 0 & j_z = 0 \end{pmatrix} = 0 \quad .$$

Exercício 4

A general solution is given by

$$\psi(\vec{r},t) \;=\; e^{-iHt}\psi(\vec{r},t=0) \quad ,$$

where

$$H \;=\; \begin{pmatrix} H_0 & i\omega \\ -i\omega & H_0 \end{pmatrix} \quad .$$

So, we must handle the exponent.

First, we observe that the following two matrices commute:

$$\begin{pmatrix} H_0 & 0 \\ 0 & H_0 \end{pmatrix}\begin{pmatrix} 0 & i\omega \\ -i\omega & 0 \end{pmatrix} \;=\; \begin{pmatrix} 0 & i\omega \\ -i\omega & 0 \end{pmatrix}\begin{pmatrix} H_0 & 0 \\ 0 & H_0 \end{pmatrix} \quad .$$

Hence

$$\begin{aligned} e^{-iHt} &= \exp\left\{-i\begin{pmatrix} H_0 & 0 \\ 0 & H_0 \end{pmatrix}t\right\}\,\exp\left\{-i\begin{pmatrix} 0 & i\omega \\ -i\omega & 0 \end{pmatrix}t\right\} \\ &= \exp\left\{-i\mathbf{1}H_0t\right\}\,\exp\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\omega t\right\} \\ &= \exp\left\{-i\mathbf{1}H_0t\right\}\,\exp\left\{A\omega t\right\} \quad , \end{aligned}$$

where

$$A \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Expansion of the second exponent gives

$$\begin{aligned} e^{A\omega t} &= \mathbf{1} + (A\omega t) + \tfrac{1}{2!}(A\omega t)^2 + \tfrac{1}{3!}(A\omega t)^3 + \ldots \\ &= \mathbf{1} + A\omega t - \tfrac{1}{2!}\mathbf{1}(\omega t)^2 - \tfrac{1}{3!}A(\omega t)^3 + \tfrac{1}{4!}\mathbf{1}(\omega t)^4 + \tfrac{1}{5!}A(\omega t)^5 + \ldots \\ &= \mathbf{1}\left\{1 - \tfrac{1}{2!}(\omega t)^2 + \tfrac{1}{4!}(\omega t)^4 + \ldots\right\} + A\left\{(\omega t) - \tfrac{1}{3!}(\omega t)^3 + \tfrac{1}{5!}(\omega t)^5 + \ldots\right\} \\ &= \mathbf{1}\cos(\omega t) + A\sin(\omega t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \quad . \end{aligned}$$

Consequently

$$\psi(\vec{r},t) \;=\; e^{-iHt}\psi(\vec{r},t=0)$$

$$
\begin{aligned}
&= \exp\{-i\mathbf{1}H_0 t\}\ \exp\{A\omega t\}\begin{pmatrix}\varphi(\vec{r}\,)\\ 0\end{pmatrix}\\
&= \exp\{-i\mathbf{1}H_0 t\}\ \begin{pmatrix}\cos(\omega t) & \sin(\omega t)\\ -\sin(\omega t) & \cos(\omega t)\end{pmatrix}\begin{pmatrix}\varphi(\vec{r}\,)\\ 0\end{pmatrix}\\
&= \exp\{-i\mathbf{1}H_0 t\}\ \begin{pmatrix}\cos(\omega t)\varphi(\vec{r}\,)\\ -\sin(\omega t)\varphi(\vec{r}\,)\end{pmatrix} = \begin{pmatrix}\cos(\omega t)e^{-iH_0 t}\varphi(\vec{r}\,)\\ -\sin(\omega t)e^{-iH_0 t}\varphi(\vec{r}\,)\end{pmatrix}\\
&= \begin{pmatrix}\cos(\omega t)e^{-iEt}\varphi(\vec{r}\,)\\ -\sin(\omega t)e^{-iEt}\varphi(\vec{r}\,)\end{pmatrix} = e^{-iEt}\begin{pmatrix}\cos(\omega t)\varphi(\vec{r}\,)\\ -\sin(\omega t)\varphi(\vec{r}\,)\end{pmatrix} \quad .
\end{aligned}
$$

We may verify if it is the correct solution:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\,\psi(\vec{r},t) \ &= \ i\frac{\partial}{\partial t}\,e^{-iEt}\begin{pmatrix}\cos(\omega t)\varphi(\vec{r}\,)\\ -\sin(\omega t)\varphi(\vec{r}\,)\end{pmatrix} =\\
&= Ee^{-iEt}\begin{pmatrix}\cos(\omega t)\varphi(\vec{r}\,)\\ -\sin(\omega t)\varphi(\vec{r}\,)\end{pmatrix} + e^{-iEt}\begin{pmatrix}-i\omega\sin(\omega t)\varphi(\vec{r}\,)\\ -i\omega\cos(\omega t)\varphi(\vec{r}\,)\end{pmatrix}\\
&= e^{-iEt}\begin{pmatrix}\cos(\omega t)E\varphi(\vec{r}\,) - i\omega\sin(\omega t)\varphi(\vec{r}\,)\\ -i\omega\cos(\omega t)\varphi(\vec{r}\,) - \sin(\omega t)E\varphi(\vec{r}\,)\end{pmatrix}\\
&= e^{-iEt}\begin{pmatrix}\cos(\omega t)H_0\varphi(\vec{r}\,) - i\omega\sin(\omega t)\varphi(\vec{r}\,)\\ -i\omega\cos(\omega t)\varphi(\vec{r}\,) - \sin(\omega t)H_0\varphi(\vec{r}\,)\end{pmatrix}\\
&= e^{-iEt}\begin{pmatrix}H_0 & i\omega\\ -i\omega & H_0\end{pmatrix}\begin{pmatrix}\cos(\omega t)\varphi(\vec{r}\,)\\ -\sin(\omega t)\varphi(\vec{r}\,)\end{pmatrix}\\
&= \begin{pmatrix}H_0 & i\omega\\ -i\omega & H_0\end{pmatrix} e^{-iEt}\begin{pmatrix}\cos(\omega t)\varphi(\vec{r}\,)\\ -\sin(\omega t)\varphi(\vec{r}\,)\end{pmatrix} = \begin{pmatrix}H_0 & i\omega\\ -i\omega & H_0\end{pmatrix}\psi(\vec{r},t) \quad .
\end{aligned}
$$

Furthermore

$$
\psi(\vec{r},t=0) \ = \ e^{-iE0}\begin{pmatrix}\cos(\omega 0)\varphi(\vec{r}\,)\\ -\sin(\omega 0)\varphi(\vec{r}\,)\end{pmatrix} = \begin{pmatrix}\varphi(\vec{r}\,)\\ 0\end{pmatrix} \quad .
$$