RELATIVIDADE GERAL 2011-2012

Exame, 11 de Junho de 2012, 9h30 - 12h30

1. Considere um espaço bidimensional (coordenadas $r \in [0, \infty)$, $\varphi \in [0, 2\pi)$), cuja métrica é dada pelo elemento de linha

$$ds^2 = dr^2 + r^2 d\varphi^2 \quad .$$

a Mostre que o elemento de linha toma a forma

$$ds^2 = dx^2 + dy^2$$

sob a transformação de coordenadas dada por

$$x = r\cos(\varphi) \quad \text{e} \quad y = r\sin(\varphi) \quad .$$

b Determine a constante (o escalar) da curvatura e mostre que as geodésicas são dadas por (A, B, C e D constantes)

$$x(s) = As + B \quad \text{e} \quad y(s) = Cs + D \quad ,$$

ou, eliminando s, ainda por

$$y(x) = \tan(\alpha)x - R \quad .$$

Determine as relações entre $\tan(\alpha)$ e R e as constantes A, B, C e D e demonstre

$$r^2 = \frac{R^2\left(1 + \tan^2(\varphi)\right)}{\left(\tan(\alpha) - \tan(\varphi)\right)^2} \quad .$$

Indique o significado das constantes R e α.

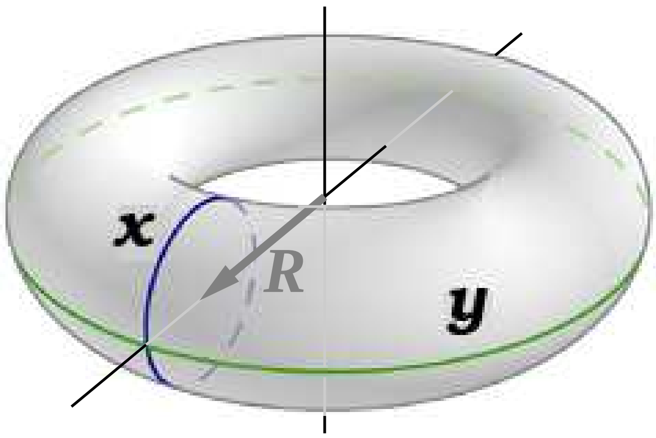
2. Considere um espaço bidimensional (coordenadas $x, y \in [0, 2\pi)$), cuja métrica é dada por

$$ds^2 \;=\; dx^2 \;+\; (2+\cos(x))^2\, dy^2 \;\; .$$

a Mostre que as geodésicas são dadas por (J constante)

$$\frac{dy}{ds} = \frac{J}{(2+\cos(x))^2} \quad \text{e} \quad \frac{d^2x}{ds^2} = \frac{-J^2\sin(x)}{(2+\cos(x))^3} \quad .$$

b Determine o valor do escalar da curvatura para $x = 0$, $\pi/2$ e π e interprete os resultados usando, nos sítios correspondentes, os principais raios da curvatura de um toro com raios $R = 2$ e $r = 1$ (ver figura).



3. Em geometria analítica descreve-se um hipérbole no plano (x, y) pela equação

$$\frac{(x+c)^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{com} \quad b^2 = c^2 - a^2 \quad ,$$

onde $x = -c$ representa a posição do centro do hipérbole relativamente ao centro do sistema das coordenadas (x, y) situado no foco do hipérbole.
A excentricidade $e > 1$ do hipérbole é definida por $e = c/a$.

a Considerando a curva hiperbólica com $x > -c$, mostre que em coordenadas cilindricas (r, φ), definidas por $x = r\cos(\varphi)$ e $y = r\sin(\varphi)$, a equação do hipérbole pode ser dada por

$$r(\varphi) = \frac{a(e^2-1)}{1 - e\cos(\varphi)} \quad .$$

b Determine a relação entre as constantes a e c e a aproximação máxima do centro do sistema das coordenadas, r_0, da curva hiperbólica.

4. Considere um campo gravítico no espaço-tempo de uma distribuição esférica simétrica de massa M, dada por (*Schwarzschild, 1916*)

$$ds^2 \;=\; A(r)\,dt^2 \;-\; \frac{dr^2}{A(r)} \;-\; r^2\,\left\{d\vartheta^2 \;+\; \sin^2(\vartheta)\,d\varphi^2\right\} \quad ,$$

com $A(r) = 1 - \frac{2MG}{r}$.

a Mostre que para luz e no plano $\vartheta = \pi/2$ a métrica de Schwarzschild se reduz a

$$dt^2 \;=\; \frac{dr^2}{A^2(r)} \;+\; \frac{r^2\,d\varphi^2}{A(r)} \quad .$$

b Mostre que as geodésicas são dadas por (J constante)

$$\frac{d\varphi}{dt} = \frac{JA(r)}{r^2} \qquad \text{e} \qquad \frac{dr}{dt} = A(r)\sqrt{1-\left(\frac{d\varphi}{dt}\right)^2\frac{r^2}{A(r)}} = A(r)\sqrt{1-\frac{J^2A(r)}{r^2}} \quad .$$

c Para um raio de luz cuja aproximação máxima do centro do sistema das coordenadas é dada por r_0, mostre

$$\frac{dr}{dt} = A(r)\sqrt{1-\frac{A(r)}{A\left(r_0\right)}\left(\frac{r_0}{r}\right)^2} \quad .$$

5. Considere um Universo isotrópico que consiste de poeira localmente em repouso. O raio instantâneo e a densidade deste Universo são respectivamente designados por $a(t)$ e $\rho(t)$; o seu tensor da energia-momento é dado por

$$T^{00} \;=\; \rho(t)\, g^{00} \qquad \text{e} \qquad T^{0i} \;=\; T^{i0} \;=\; T^{ij} \;=\; 0 \quad (i,j=1,2,3),$$

onde a métrica obedece

$$ds^2 \;=\; dt^2 \;-\; \frac{a^2(t)}{\left(1+r^2\right)^2}\, d\vec{r}^{\,2} \quad .$$

$r^2 = \delta_{k\ell} x^k x^\ell$ e $d\vec{r}^{\,2} = \delta_{ij} dx^i dx^j$.
O tensor de Ricci é dado por ($\dot{a}(t) = da(t)/dt$ e $\ddot{a}(t) = d^2a(t)/dt^2$)

$$R_{00} = \frac{3\ddot{a}(t)}{a(t)} \quad , \qquad R_{0i} = R_{i0} = 0 \quad (i = 1,\ 2,\ 3) \quad \text{e}$$

$$R_{ij} = -\frac{\delta_{ij}}{\left(1+r^2\right)^2} \left\{ 8 + 2\dot{a}^2(t) + \ddot{a}(t) a(t) \right\} \quad (i,j = 1,\ 2,\ 3) \quad .$$

Mostre que

$$\dot{a}^2(t) \;+\; 4 \;=\; \frac{8\pi G}{3}\, \rho(t)\, a^2(t) \quad .$$

Os exercícios do trabalho suplementar:

6.

a Para um espaço bidimensional (coordenadas $r \in [0, \infty)$, $\varphi \in [0, 2\pi)$), cuja métrica é dada por

$$ds^2 \;=\; dr^2 \;+\; r^2 d\varphi^2 \quad ,$$

determine a constante da curvatura e mostre que as geodésicas são dadas por

$$r^2(\varphi) \;=\; \frac{R^2\left(1+\tan^2(\varphi)\right)}{\left(\tan(\alpha)-\tan(\varphi)\right)^2} \quad .$$

Indique o significado das constantes R e α.

b Para um espaço bidimensional (coordenadas $x, y \in [0, 2\pi)$), cuja métrica é dada por

$$ds^2 \;=\; dx^2 \;+\; \left(2+\cos(x)\right)^2 dy^2 \quad ,$$

determine o escalar da curvatura $R(x, y)$ e mostre que as geodésicas são dadas por

$$\frac{d^2x}{dy^2} \;+\; \frac{2\sin(x)}{2+\cos(x)} \left(\frac{dx}{dy}\right)^2 \;+\; \sin(x)\left(2+\cos(x)\right) \;=\; 0 \quad .$$

Interprete os casos (i) $y = \text{constant}$, e (ii) $x = \text{constant}$.

7. Considere um campo gravítico no espaço-tempo de uma distribuição esférica simétrica de massa M, dada por (*Schwarzschild, 1916*)

$$ds^2 \;=\; A(r)\,dt^2 \;-\; \frac{dr^2}{A(r)} \;-\; r^2\,\left\{d\vartheta^2 \;+\; \sin^2(\vartheta)\,d\varphi^2\right\} \quad , \qquad (1)$$

com $A(r) = 1 - \frac{2MG}{r}$.

a Mostre que os elementos da conexão afim que não se anulam são os seguintes

$$\Gamma^{t}_{\;tr} = \Gamma^{t}_{\;rt} = -\Gamma^{r}_{\;rr} = \frac{A'(r)}{2A(r)} \quad , \quad \Gamma^{r}_{\;tt} \;=\; \tfrac{1}{2}A(r)A'(r) \quad , \quad \Gamma^{r}_{\;\vartheta\vartheta} \;=\; -rA(r) \;\;,$$

$$\Gamma^{r}_{\;\varphi\varphi} \;=\; -r\sin^2(\vartheta)A(r) \quad , \quad \Gamma^{\vartheta}_{\;r\vartheta} \;=\; \Gamma^{\vartheta}_{\;\vartheta r} \;=\; \Gamma^{\varphi}_{\;r\varphi} \;=\; \Gamma^{\varphi}_{\;\varphi r} \;=\; \frac{1}{r} \quad ,$$

$$\Gamma^{\vartheta}_{\;\varphi\varphi} \;=\; -\sin(\vartheta)\cos(\vartheta) \;\;\text{e}\;\; \Gamma^{\varphi}_{\;\vartheta\varphi} \;=\; \Gamma^{\varphi}_{\;\varphi\vartheta} \;=\; \text{cotg}(\vartheta) \quad ; \quad A'(r) = \frac{dA(r)}{dr} \;.$$

b. Mostre que no plano $\vartheta = \pi/2$ as três seguintes expressões são constantes do movimento:

$$\tau = A(r)\frac{dt}{ds} \quad , \quad \ell = r^2\frac{d\varphi}{ds} \quad \text{e} \quad \epsilon = \frac{\tau^2}{A(r)} - \frac{\ell^2}{r^2} - \frac{1}{A(r)}\left(\frac{dr}{ds}\right)^2 \quad .$$

c. Mostre que, se r é parametrizado por φ, obtém-se da equação geodésica no plano $\vartheta = \pi/2$ para $r(\varphi)$ a expressão:

$$\frac{d}{d\varphi}\left\{\frac{1}{A(r)}\left(\frac{\ell}{r^2}\right)^2\left(\frac{dr}{d\varphi}\right)^2 + r^2\left(\frac{\ell}{r^2}\right)^2 - \frac{\tau^2}{A(r)}\right\} = 0 \quad .$$

d. Mostre que, para $r \gg 2MG$, a equação geodésica no plano $\vartheta = \pi/2$ se reduz a

$$\frac{1}{r}\frac{d^2r}{d\varphi^2} - \frac{2}{r^2}\left(\frac{dr}{d\varphi}\right)^2 \approx 1 - \frac{MG}{\ell^2}r \quad ,$$

e ainda que as soluções desta equação são dadas por

$$r(\varphi) = \frac{\ell^2/MG}{1 - e\cos(\varphi)} \quad . \qquad (2)$$

Interprete os casos (*i*) $e = 0$, (*ii*) $0 < e < 1$, (*iii*) $e = 1$ e (*iv*) $e > 1$.

e. Determine a velocidade radial $v(t) = dr/dt$ de um fotão radialmente em queda livre no plano $\vartheta = \pi/2$ a distâncias $r > 2MG$.

8. Considere um objecto em movimento livre no espaço definido por Eq. (1) do exercício 7 ao longo duma trajectória dada por fórmula (2) para um valor muito elevado do parámetro e (*i.e.* $e \gg 1$).

a. Define o ângulo de desvio $\Delta\varphi$ e determine como este está relacionado com os ângulos $\varphi(t\downarrow -\infty)$ e $\varphi(t\uparrow +\infty)$, dados por

$$-\varphi(t\downarrow -\infty) = \varphi(t\uparrow +\infty) = \frac{\pi}{2} - \frac{1}{e} \quad .$$

b. Determine a relação entre ℓ^2/MG, e e a distância mínima, r_0, entre o objecto e a fonte do campo gravítico.

c. Determine a relação entre r_0 e ℓ e a velocidade v_0 do objecto no ponto r_0 e, utilizando o facto de que no ponto r_0 se verifica $dr/d\varphi = 0$, mostre que no caso em que $2MG/r_0 \ll 1$, a energia total E do objecto é dada por $E = m(1-\epsilon)/2$.

d. Determine o valor "semi-clássico" de $\Delta\varphi$ para um fotão no campo gravítico do sol.

e. Mostre que para fotões se verifica:

$$\varphi(r) - \varphi(\infty) = \int_r^\infty dr \, \frac{1}{r\sqrt{A(r)}\sqrt{\frac{A(r_0)}{A(r)}\left(\frac{r}{r_0}\right)^2 - 1}} \quad ,$$

e mostre que $\Delta\varphi = 2\left|\varphi(r_0) - \varphi(\infty)\right| - \pi$.

f. No caso em que $2MG/r_0 \ll 1$, mostre que

$$\varphi(r_0) - \varphi(\infty) = \int_{r_0}^\infty dr \, \frac{1 + \frac{MG}{r} + \frac{MGr}{r_0(r+r_0)}}{r\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} \quad .$$

Determine $\Delta\varphi$ em primeira ordem em MG/r_0.
Se fôr preciso, utilize as seguintes expressões:

$$\frac{d}{dr}\arcsin\left(\frac{r_0}{r}\right) = -\frac{1}{r\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} \ , \quad \frac{d}{dr}\sqrt{1 - \left(\frac{r_0}{r}\right)^2} = \frac{r_0}{r^2\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}}$$

$$\text{e} \quad \frac{d}{dr}\sqrt{\frac{r-r_0}{r+r_0}} = r_0(r-r_0)^{-1/2}(r+r_0)^{-3/2} \quad .$$

Determine o valor deste desvio para um fotão no campo gravítico do sol no caso em que r_0 é igual ao raio do sol.

9. Mostre que a métrica de Schwarzschild (1, exercício 7) toma as seguintes formas sob transformações de coordenadas.

a. $ds^2 = \left(1 - \frac{2MG}{r\,(r^*)}\right) \left\{dt^2 \; - \; dr^{*2}\right\} \; - \; r^2\,(r^*)\;\; d\Omega^2,$

nas coordenadas (*Tortoise*), dadas por

$$t\;,\; r^* \;=\; r \;+\; 2MG\,\log\left|\frac{r}{2MG} \;-\; 1\right|\;,\; \vartheta\;,\; \varphi \quad .$$

b. $ds^2 = \left(1 - \frac{2MG}{r}\right)\; d\tilde{v}^2 \;-\; (d\tilde{v}dr \;+\; drd\tilde{v}) \;\; - \; r^2\, d\Omega^2,$

nas coordenadas ($\tilde{v} \;=\; t \;+\; r^*\;,\; r\;,\; \vartheta\;,\; \varphi$).

c $ds^2 = \left(1 - \frac{2MG}{r}\right)\; d\tilde{u}^2 \;+\; (d\tilde{u}dr \;+\; drd\tilde{u}) \;\; - \; r^2\, d\Omega^2,$

nas coordenadas ($\tilde{u} \;=\; t \;-\; r^*\;,\; r\;,\; \vartheta\;,\; \varphi$).

d $ds^2 = \left(1 - \frac{2MG}{r\,(\tilde{u}, \tilde{v})}\right)\; d\tilde{u}d\tilde{v} \;-\; r^2\,(\tilde{u}, \tilde{v})\;\; d\Omega^2,$

nas coordenadas (*Eddington e Finkelstein 1924*) ($\tilde{u}\;,\; \tilde{v}\;,\; \vartheta\;,\; \varphi$).

e $ds^2 = \frac{32M^3G^3}{r\,(u', v')}\; e^{-r\,(u', v')\,/2MG}\; dv'du' \;-\; r^2\,(u', v')\;\; d\Omega^2,$

nas coordenadas, dadas por

$$v' \;=\; e^{\tilde{v}/4MG}\;,\; u' \;=\; e^{-\tilde{u}/4MG}\;,\; \vartheta\;,\; \varphi \quad .$$

f $ds^2 = \frac{32M^3G^3}{r\,(u, v)}\; e^{-r\,(u, v)\,/2MG}\;\; (dv^2 \;-\; du^2) \;-\; r^2\,(u, v)\;\; d\Omega^2,$

nas coordenadas (*Kruskal 1960*) dadas por

$$u \;=\; \frac{1}{2}\,(v' \;-\; u')\;,\; v \;=\; \frac{1}{2}\,(v' \;+\; u')\;,\; \vartheta\;,\; \varphi \quad .$$

g Mostra que em ambos os casos b) e c) se tem $\det(g) = -r^4\sin^2(\vartheta)$.

10. Considere um Universo isotrópico que consiste de poeira localmente em repouso. O raio instantâneo e a densidade deste Universo são respectivamente designados por $a(t)$ e $\rho(t)$; o seu tensor da energia-momento é dado por

$$T^{00} = \rho(t)\, g^{00} \quad \text{e} \quad T^{0i} = T^{i0} = T^{ij} = 0 \quad (i, j = 1, 2, 3),$$

onde a métrica obedece

$$ds^2 = dt^2 - \frac{a^2(t)}{\left(1 + r^2\right)^2}\, d\vec{r}^{\,2} \quad .$$

$r^2 = \delta_{k\ell} x^k x^\ell$ e $d\vec{r}^{\,2} = \delta_{ij} dx^i dx^j$.

a Demonstre as seguintes relações ($\dot{a}(t) = da(t)/dt$ e $\ddot{a}(t) = d^2a(t)/dt^2$):

$$\Gamma^0_{ij} = \frac{a(t)\dot{a}(t)}{\left(1 + r^2\right)^2}\, \delta_{ij} \quad , \quad \Gamma^i_{0j} = \frac{\dot{a}(t)}{a(t)}\, \delta^i_j \quad ,$$

$$\Gamma^i_{jk} = -\frac{2}{1 + r^2} \left\{ \left(\delta^i_j \delta_{k\ell} + \delta^i_k \delta_{j\ell} \right) x^\ell - x^i\, \delta_{jk} \right\} \quad ,$$

$$R_{00} = \frac{3\ddot{a}(t)}{a(t)} \quad , \quad R_{ij} = -\frac{\delta_{ij}}{\left(1 + r^2\right)^2} \left\{ 8 + 2\dot{a}^2(t) + \ddot{a}(t)a(t) \right\} \quad .$$

b Mostre também que

$$\dot{a}^2(t) + 4 = \frac{8\pi G}{3}\, \rho(t)\, a^2(t) \quad .$$

Solutions

Exercício 1

a. By the choice of coordinates

$$x = r\cos(\varphi) \quad \text{and} \quad y = r\sin(\varphi) \quad ,$$

we obtain the following relations between dx, dy, dr and $d\varphi$.

$$dx = dr\cos(\varphi) - rd\varphi\sin(\varphi) \quad \text{and} \quad dy = dr\sin(\varphi) + rd\varphi\cos(\varphi) \quad ,$$

hence, for the infinitesimal line element, we find

$$dx^2 + dy^2 = \left(dr\cos(\varphi) - rd\varphi\sin(\varphi)\right)^2 + \left(dr\sin(\varphi) + rd\varphi\cos(\varphi)\right)^2 = dr^2 + r^2 d\varphi^2 \quad .$$

b. We find that in those coordinates the metric

$$\begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is constant and, consequently, the afine connections and the curvature scalar vanish. Hence, this two-dimensional space has no curvature.

Furthermore, the geodesic equations that follow are given by

$$\frac{d^2x}{ds^2} = 0 \quad \text{and} \quad \frac{d^2y}{ds^2} = 0 \quad ,$$

which has solutions (A, B, C and D constants)

$$x(s) = As + B \quad \text{and} \quad y(s) = Cs + D \quad .$$

By elimination of s, one finds

$$y(x) = \frac{C}{A}(x - B) + D \quad ,$$

which represents a straight line in the plane and which, by the choices $\tan(\alpha) = C/A$ and $R = CB/A - D$, can be casted in the form

$$y(x) = \tan(\alpha)x - R \quad .$$

It follows

$$R = \tan(\alpha)x - y = x\left(\tan(\alpha) - \tan(\varphi)\right) \quad .$$

The remaining calculus is straightforward.

$$r^2 = x^2 + y^2 = x^2\left(1 + \tan^2(\varphi)\right) = \frac{R^2\left(1 + \tan^2(\varphi)\right)}{\left(\tan(\alpha) - \tan(\varphi)\right)^2} \quad .$$

$\tan(\alpha)$ indicates the gradient (or slope) of the straight line with respect to the x axis, whereby α represents the angle of inclination of the straight line. $y = -R$ represents the position where the straight line cuts the y axis.

Exercício 2

a. In solving exercise (6b), we found

$$\frac{d}{ds}\left[(2+\cos(x))^2\left(\frac{dy}{ds}\right)\right] = 0 \quad ,$$

which implies that the expression within the brackets is constant, say J, along a geodesic curve:

$$(2+\cos(x))^2\left(\frac{dy}{ds}\right) = J \quad \Longrightarrow \quad \frac{dy}{ds} = \frac{J}{(2+\cos(x))^2} \quad .$$

When this result is inserted in the geodesic equation

$$\frac{d^2x}{ds^2} = -\left(\frac{dy}{ds}\right)^2 \sin(x)\,(2+\cos(x)) \quad ,$$

one finds for $x(s)$ the equation

$$\frac{d^2x}{ds^2} = \frac{-J^2\sin(x)}{\{2+\cos(x)\}^3} \quad .$$

b. In exercise (6b) we also found for the curvature scalar

$$eqnarray{*}R(x,y) = \frac{-2\cos(x)}{2+\cos(x)} \quad .$$

In the lectures (see formula 193 of the lecture notes), we found that the curvature scalar R is related to the curvature constant K (see formula 175 of the lecture notes) by $R = -2K$. Furthermore, the curvature constant K is related to the two extreme values R_1 and R_2 of the curvature radius (see discussion of section 22 of the lecture notes) by $K = 1/R_1R_2$. Hence

$$R = -2K = -\frac{2}{R_1R_2} \quad .$$

Now, the torus of the present problem could be represented by radii $R = 2$ and $r = 1$. Then, at the outside, $x = 0$, for which $R = -2/3$, one has the extreme radii $R_1 = r = 1$ and $R_2 = R + r = 3$, hence $K = 1/3$, whereas, at the inside, $x = \pi$, for which $R = 2$, one has the extreme radii $R_1 = -r = -1$ (minus, since curved away from the center) and $R_2 = R - r = 1$, hence $K = -1$. At the "top" of the torus, where $x = \pi/2$ and $R = 0$ and where the torus locally has the aspect of a cylinder, one has $R_1 = r = 1$ and $R_2 = \infty$, hence $K = 0$.

Exercício 3

a. From the equation for the hyperbola, we find

$$\frac{(x+c)^2}{a^2}-\frac{y^2}{b^2}=1 \iff \frac{(r\cos(\varphi)+c)^2}{a^2}-\frac{r^2\sin^2(\varphi)}{b^2}=1 \iff$$

$$\iff r^2\left\{\frac{\cos^2(\varphi)}{a^2}-\frac{\sin^2(\varphi)}{b^2}\right\}+r\frac{2c\cos(\varphi)}{a^2}+\frac{c^2}{a^2}-1=0 \iff$$

$$\iff r^2\left\{\frac{\cos^2(\varphi)}{a^2}-\frac{\sin^2(\varphi)}{b^2}\right\}+r\frac{2c\cos(\varphi)}{a^2}+\frac{b^2}{a^2}=0 \quad .$$

Solutions to that equation are given by

$$r_\pm =$$

$$= \frac{1}{2}\left(\frac{\cos^2(\varphi)}{a^2}-\frac{\sin^2(\varphi)}{b^2}\right)^{-1}\left\{-\frac{2c\cos(\varphi)}{a^2}\pm\sqrt{\frac{4c^2\cos^2(\varphi)}{a^4}-4\left(\frac{\cos^2(\varphi)}{a^2}-\frac{\sin^2(\varphi)}{b^2}\right)\frac{b^2}{a^2}}\right\}$$

$$= \frac{1}{a^2}\left(\frac{\cos^2(\varphi)}{a^2}-\frac{\sin^2(\varphi)}{b^2}\right)^{-1}\left\{-c\cos(\varphi)\pm\sqrt{(c^2-b^2)\cos^2(\varphi)+a^2\sin^2(\varphi)}\right\}$$

$$= b^2\left(b^2\cos^2(\varphi)-a^2\sin^2(\varphi)\right)^{-1}\left\{-c\cos(\varphi)\pm\sqrt{a^2\cos^2(\varphi)+a^2\sin^2(\varphi)}\right\}$$

$$= b^2\left(\left(c^2-a^2\right)\cos^2(\varphi)-a^2\sin^2(\varphi)\right)^{-1}\left\{-c\cos(\varphi)\pm a\right\}$$

$$= b^2\frac{-c\cos(\varphi)\pm a}{c^2\cos^2(\varphi)-a^2} = b^2\frac{-c\cos(\varphi)\pm a}{(c\cos(\varphi)+a)\,(c\cos(\varphi)-a)}$$

$$= \begin{cases} \dfrac{-b^2}{a+c\cos(\varphi)} = \dfrac{-a\,(e^2-1)}{1+e\cos(\varphi)} \\[2ex] \dfrac{b^2}{a-c\cos(\varphi)} = \dfrac{a\,(e^2-1)}{1-e\cos(\varphi)} \end{cases} .$$

The second solution gives positive r for φ in the domain of interest.

c. The nearest approximation occurs for $\varphi=\pi$, hence

$$r_0 = r(\varphi=\pi) = \frac{a(e^2-1)}{1+e} = a(e-1) = c-a \quad .$$

Exercício 4

The solutions of **a**, **b** and **c** are discussed for exercise (8).

Exercício 5

The solutions of **a** and **b** are discussed for exercise (10).

Exercício 6

a. The metric and its inverse are given by

$$g_{rr} = 1 \quad , \quad g_{\varphi\varphi} = r^2 \quad , \quad g^{rr} = 1 \quad \text{and} \quad g^{\varphi\varphi} = r^{-2} \quad .$$

The only non-vanishing derivative of the metric is given by

$$g_{\varphi\varphi,\, r} = 2r \quad .$$

The Christoffel symbols are given by

$$\Gamma_{\mu\alpha\beta} = \tfrac{1}{2}\left\{ g_{\mu\alpha,\, \beta} + g_{\mu\beta,\, \alpha} - g_{\alpha\beta,\, \mu} \right\} \quad . \tag{3}$$

Hence, the non-vanishing Christoffel symbols are

$$-\Gamma_{r\varphi\varphi} = \Gamma_{\varphi r\varphi} = \Gamma_{\varphi\varphi r} = \tfrac{1}{2} g_{\varphi\varphi,\, r} = r \quad ,$$

and the non-vanishing affine connections are given by

$$\begin{aligned} \Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} &= g^{\varphi\varphi}\Gamma_{\varphi\varphi r} = r^{-2} r = \frac{1}{r} \\ \text{and} \quad \Gamma^{r}_{\varphi\varphi} &= g^{rr}\Gamma_{r\varphi\varphi} = -r \quad . \end{aligned}$$

Curvature is defined by

$$R_{\mu\nu\rho\sigma} = g_{\mu\alpha} R^{\alpha}_{\nu\rho\sigma} \quad , \tag{4}$$

where

$$R^{\alpha}_{\nu\rho\sigma} = \Gamma^{\alpha}_{\nu\rho,\, \sigma} - \Gamma^{\alpha}_{\nu\sigma,\, \rho} + \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\beta}_{\nu\rho} - \Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\nu\sigma} \quad .$$

The curvature tensor (4) has the following symmetry properties:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho} \\ R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\sigma\rho} \\ R_{\mu\nu\rho\sigma} &= 0 \quad , \quad \text{for } \rho = \sigma \\ R_{\mu\nu\rho\sigma} &= 0 \quad , \quad \text{for } \mu = \nu \quad . \end{aligned}$$

With those symmetry relations, we have for the curvature tensor at a two-dimensional surface only one independent nonzero element, *i.e.*

$$R_{r\varphi r\varphi} = -R_{r\varphi\varphi r} = R_{\varphi r\varphi r} = -R_{\varphi r r\varphi} \quad .$$

We determine

$$R_{r\varphi r\varphi} = g_{r\alpha} R^{\alpha}_{\ \varphi r\varphi} = g_{rr} R^{r}_{\ \varphi r\varphi} + g_{r\varphi} R^{\varphi}_{\ \varphi r\varphi} = g_{rr} R^{r}_{\ \varphi r\varphi} + 0 =$$

$$= g_{rr} \left\{ \Gamma^{r}_{\varphi r,\varphi} - \Gamma^{r}_{\varphi\varphi,r} + \Gamma^{r}_{\varphi\beta}\Gamma^{\beta}_{\varphi r} - \Gamma^{r}_{r\beta}\Gamma^{\beta}_{\varphi\varphi} \right\}$$

$$= g_{rr} \left\{ \Gamma^{r}_{\varphi r,\varphi} - \Gamma^{r}_{\varphi\varphi,r} + \Gamma^{r}_{\varphi r}\Gamma^{r}_{\varphi r} + \Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi r} - \Gamma^{r}_{rr}\Gamma^{r}_{\varphi\varphi} - \Gamma^{r}_{r\varphi}\Gamma^{\varphi}_{\varphi\varphi} \right\}$$

$$= g_{rr} \left\{ 0 - \Gamma^{r}_{\varphi\varphi,r} + 0 + \Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi r} - 0 - 0 \right\} = -\Gamma^{r}_{\varphi\varphi,r} + \Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi r}$$

$$= 1 + (-r)\frac{1}{r} = 0$$

The local *curvature scalar* is in general defined by

$$R \ = \ g^{\mu\rho} \, g^{\nu\sigma} \, R_{\mu\nu\rho\sigma} \quad . \tag{5}$$

Here, since $R = 0$, we find that the space has no curvature!

The geodesic equations are given by

$$0 \ = \ \frac{d^2u^{\mu}}{ds^2} \ + \ \frac{du^{\alpha}}{ds}\,\frac{du^{\beta}}{ds}\,\Gamma^{\mu}_{\alpha\beta} \quad , \tag{6}$$

which in the present case lead to

$$\frac{d^2r}{ds^2} = -\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{r}_{\alpha\beta} = -\left(\frac{d\varphi}{ds}\right)^2 \Gamma^{r}_{\varphi\varphi} = r\left(\frac{d\varphi}{ds}\right)^2$$

and

$$\frac{d^2\varphi}{ds^2} = -\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{\varphi}_{\alpha\beta} = -2\frac{dr}{ds}\frac{d\varphi}{ds}\Gamma^{\varphi}_{r\varphi} = -\frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} \quad .$$

The latter equation can be casted in the form

$$\frac{1}{r^2}\frac{d}{ds}\left(r^2\frac{d\varphi}{ds}\right) = 0 \quad ,$$

which leaves us, for $r \neq 0$, with a constant of motion, say J, given by

$$J = r^2\frac{d\varphi}{ds} \quad .$$

From the line element

$$ds^2 = dr^2 + r^2 d\varphi^2 \quad ,$$

we deduce

$$1 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\varphi}{ds}\right)^2 \quad ,$$

hence,

$$\frac{r^4}{J^2} = \frac{1}{\left(\frac{d\varphi}{ds}\right)^2} = \left(\frac{dr}{d\varphi}\right)^2 + r^2 = \left(\frac{1}{2r}\frac{dr^2}{d\varphi}\right)^2 + r^2 = \frac{1}{4r^2}\left(\frac{dr^2}{d\varphi}\right)^2 + r^2 \quad .$$

Now, for

$$r^2 = \frac{R^2\left(1+\tan^2(\varphi)\right)}{\left(\tan(\alpha)-\tan(\varphi)\right)^2} \quad ,$$

also observing that

$$\frac{d\tan(\varphi)}{d\varphi} = 1+\tan^2(\varphi) \quad ,$$

one has

$$\frac{dr^2}{d\varphi} = \frac{2R^2\tan(\varphi)\left(1+\tan^2(\varphi)\right)}{\left(\tan(\alpha)-\tan(\varphi)\right)^2} + \frac{2R^2\left(1+\tan^2(\varphi)\right)^2}{\left(\tan(\alpha)-\tan(\varphi)\right)^3} = 2r^2\,\frac{1+\tan(\alpha)\tan(\varphi)}{\tan(\alpha)-\tan(\varphi)} \quad .$$

Consequently,

$$\frac{1}{4r^2}\left(\frac{dr^2}{d\varphi}\right)^2 + r^2 = r^2\,\frac{\left(1+\tan(\alpha)\tan(\varphi)\right)^2}{\left(\tan(\alpha)-\tan(\varphi)\right)^2} + r^2 =$$

$$= r^2\,\frac{\left(1+\tan^2(\alpha)\right)\left(1+\tan^2(\varphi)\right)}{\left(\tan(\alpha)-\tan(\varphi)\right)^2} = r^4\,\frac{1+\tan^2(\alpha)}{R^2} = \frac{r^4}{R^2\cos^2(\alpha)} \quad .$$

We find

$$J = \pm R\cos(\alpha) \quad .$$

For a more detailed interpretation of R and α, let us consider $0<\alpha<\pi/2$. We have then

$$r^2(\varphi=0) = \frac{R^2}{\tan^2(\alpha)} \quad \Longrightarrow \quad r(\varphi=0) = \pm\frac{R}{\tan(\alpha)} \quad .$$

This is the place where the geodesic cuts the x axis.

Furthermore, by taking the appropriate limit $\varphi\uparrow\pi/2$, we obtain

$$r^2(\varphi=\pi/2) = R^2 \quad \Longrightarrow \quad r(\varphi=\pi/2) = \pm R \quad .$$

This is the place where the geodesic cuts the y axis.

Now, geodesics in flat space are just straight lines. Its distance to the (x,y) origin is given by

$$J = \pm R\cos(\alpha) \quad .$$

Hence, a particle with unit mass and which moves with unit velocity (in s) along the line, has angular momentum J.

b. The Christoffel symbols are given by Eq. (3), whereas the only non-vanishing derivative of the metric is given by

$$g_{yy,\,x} = -2\sin(x)\left(2+\cos(x)\right) \quad .$$

Hence, the non-vanishing Christoffel symbols are given by

$$\Gamma_{xyy} = -\Gamma_{yxy} = -\Gamma_{yyx} = -\tfrac{1}{2}g_{yy,\,x} = \sin(x)\left(2+\cos(x)\right) \quad .$$

Furthermore

$$g^{xx} = 1 \quad \text{and} \quad g^{yy} = (2+\cos(x))^{-2} \quad .$$

Hence, the non-vanishing afine connections are given by

$$\Gamma^{y}_{xy} = \Gamma^{y}_{yx} = g^{yy}\Gamma_{yyx} = \tfrac{1}{2}g^{yy}g_{yy,\,x} = \frac{-\sin(x)}{2+\cos(x)}$$

$$\text{and} \quad \Gamma^{x}_{yy} = g^{xx}\Gamma_{xyy} = -\tfrac{1}{2}g^{xx}g_{yy,\,x} = \sin(x)\left\{2+\cos(x)\right\} \quad .$$

Using the expression (4) and its symmetry properties for the definition of curvature, we determine

$$R_{xyxy} = g_{x\alpha}R^{\alpha}_{yxy} = g_{xx}R^{x}_{yxy} + g_{xy}R^{y}_{yxy} = g_{xx}R^{x}_{yxy} + 0 =$$

$$= g_{xx}\left\{\Gamma^{x}_{yx,\,y} - \Gamma^{x}_{yy,\,x} + \Gamma^{x}_{y\beta}\Gamma^{\beta}_{yx} - \Gamma^{x}_{x\beta}\Gamma^{\beta}_{yy}\right\}$$

$$= g_{xx}\left\{\Gamma^{x}_{yx,\,y} - \Gamma^{x}_{yy,\,x} + \Gamma^{x}_{yx}\Gamma^{x}_{yx} + \Gamma^{x}_{yy}\Gamma^{y}_{yx} - \Gamma^{x}_{xx}\Gamma^{x}_{yy} - \Gamma^{x}_{xy}\Gamma^{y}_{yy}\right\}$$

$$= g_{xx}\left\{0 - \Gamma^{x}_{yy,\,x} + 0 + \Gamma^{x}_{yy}\Gamma^{y}_{yx} - 0 - 0\right\} = -\Gamma^{x}_{yy,\,x} + \Gamma^{x}_{yy}\Gamma^{y}_{yx}$$

$$= -\cos(x)\left(2+\cos(x)\right) + \sin^2(x) - \sin^2(x) = -\cos(x)\left(2+\cos(x)\right)$$

For a two-dimensional surface the only non-vanishing contributions to the curvature scalar (5) come from the one independent nonzero element, which results for the curvature scalar then in

$$R(x,y) = \left\{g^{xx}g^{yy} - g^{xy}g^{yx} + g^{yy}g^{xx} - g^{yx}g^{xy}\right\} R_{xyxy}$$

$$= 2g^{yy}R_{xyxy} = \frac{-2\cos(x)}{2+\cos(x)} \quad .$$

In the lectures (see formula 193 of the lecture notes), we found that the curvature scalar R is related to the curvature scalar K (see formula 175 of the lecture notes) by $R = -2K$. Furthermore, the curvature scalar K is related to the two extreme values R_1 and R_2 of the curvature radius (see discussion of section 22 of the lecture notes) by $K = 1/R_1R_2$. Hence

$$R = -2K = -\frac{2}{R_1R_2} \quad .$$

Now, the torus of the present problem could be represented by radii $R = 2$ and $r = 1$. Then, at the outside, $x = 0$, one has the extreme radii $R_1 = 1$ and $R_2 = 3$, hence $R = -2/3$, whereas, at the inside, $x = \pi$, one has the extreme radii $R_1 = -1$ (minus, since curved away from the center) and $R_2 = 1$, hence $R = 2$.

See also http://www.rdrop.com/∼half/math/torus/index.xhtml

The geodesic equations (6) lead in the present case to

$$\frac{d^2x}{ds^2} = -\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{x}_{\alpha\beta}(x,y) = -\left(\frac{dy}{ds}\right)^2\Gamma^{x}_{yy}(x,y) = -\left(\frac{dy}{ds}\right)^2\sin(x)\left(2+\cos(x)\right)$$

and

$$\frac{d^2y}{ds^2} = -\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\Gamma^y_{\alpha\beta}(x,y) = -2\frac{dx}{ds}\frac{dy}{ds}\Gamma^y_{xy}(x,y) = 2\frac{dx}{ds}\frac{dy}{ds}\frac{\sin(x)}{2+\cos(x)}$$

Two types of solutions can be discovered without any difficulty. Assuming again that the torus of the present problem could be represented by radii $R = 2$ and $r = 1$, one has one type of solutions for which

$$y = \text{constant} \quad \Rightarrow \quad \frac{dy}{ds} = 0 \quad \Rightarrow \quad \frac{d^2x}{ds^2} = 0 \quad \Rightarrow x = s \quad .$$

Those are the meridian circles of the torus.

The second type of solutions follows from

$$x = \text{constant} \quad \Rightarrow \quad \frac{d^2y}{ds^2} = 0 \ \Rightarrow \ y = s \quad .$$

Those are the circles centered in the symmetry axis of the torus.

When neither x, nor y, is constant, one may notice then that the second of the geodesic equations can also be written in the form

$$\frac{d}{ds}\left[\{2+\cos(x)\}^2\left(\frac{dy}{ds}\right)\right] = 0 \quad ,$$

which implies that the expression within the brackets is constant, say J, along a geodesic curve. When this result is inserted in the first of the geodesic equations, one finds for $x(s)$ the equation

$$\frac{d^2x}{ds^2} = \frac{-J^2\sin(x)}{\{2+\cos(x)\}^3} \quad .$$

We may simplify the first of equations, by considering x directly a function of y. In that case one has

$$\frac{dx}{ds} = \frac{dy}{ds}\frac{dx}{dy} \quad , \quad \frac{d^2x}{ds^2} = \frac{d^2y}{ds^2}\frac{dx}{dy} + \left(\frac{dy}{ds}\right)^2\frac{d^2x}{dy^2}$$

$$\text{and} \quad \frac{d^2y}{ds^2} = 2\left(\frac{dx}{dy}\right)\left(\frac{dy}{ds}\right)^2\frac{\sin(x)}{2+\cos(x)} = \frac{2J^2\sin(x)}{\{2+\cos(x)\}^5}\left(\frac{dx}{dy}\right) \quad ,$$

which formulas can be put together, resulting in a second order differential equation for x as a function of y.

From the first line of the previous equations we deduce

$$\frac{d^2x}{dy^2} = \frac{\frac{d^2x}{ds^2} - \frac{d^2y}{ds^2}\frac{dx}{dy}}{\left(\frac{dy}{ds}\right)^2} \quad .$$

Here, we may substitute d^2x/ds^2, d^2y/ds^2 and dy/ds.

$$\frac{d^2x}{dy^2} = \frac{\frac{-J^2\sin(x)}{(2+\cos(x))^3} - \frac{2J^2\sin(x)}{(2+\cos(x))^5}\left(\frac{dx}{dy}\right)\frac{dx}{dy}}{\left(\frac{J}{(2+\cos(x))^2}\right)^2} \quad .$$

After some reshufling, we obtain

$$\frac{d^2x}{dy^2} \;+\; \frac{2\sin(x)}{2+\cos(x)}\,\left(\frac{dx}{dy}\right)^2 \;+\; \sin(x)\,\{2+\cos(x)\} \;=\; 0 \quad .$$

Exercício 7

a The metric is fully given by

$$g_{tt} = A(r), \quad g_{rr} = -\frac{1}{A(r)}, \quad g_{\vartheta\vartheta} = -r^2, \quad g_{\varphi\varphi} = -r^2 \sin^2(\vartheta) \quad .$$

So, we have the derivatives

$$g_{tt,\,r} = A', \quad g_{rr,\,r} = \frac{A'}{A^2}, \quad g_{\vartheta\vartheta,\,r} = -2r, \quad g_{\varphi\varphi,\,r} = -2r\sin^2(\vartheta), \quad g_{\varphi\varphi,\,\vartheta} = -2r^2 \sin(\vartheta)\cos(\vartheta).$$

Using formula (3) we obtain for the Christoffel symbols

$$\Gamma_{ttr} = \Gamma_{trt} = -\Gamma_{rtt} = \tfrac{1}{2} g_{tt,\,r} = \frac{A'}{2} \quad ,$$

$$\Gamma_{rrr} = \tfrac{1}{2} g_{rr,\,r} = \frac{A'}{2A^2} \quad ,$$

$$\Gamma_{\vartheta\vartheta r} = \Gamma_{\vartheta r\vartheta} = -\Gamma_{r\vartheta\vartheta} = \tfrac{1}{2} g_{\vartheta\vartheta,\,r} = -r \quad ,$$

$$\Gamma_{\varphi\varphi r} = \Gamma_{\varphi r\varphi} = -\Gamma_{r\varphi\varphi} = \tfrac{1}{2} g_{\varphi\varphi,\,r} = -r\sin^2(\vartheta) \quad \text{and}$$

$$\Gamma_{\varphi\varphi\vartheta} = \Gamma_{\varphi\vartheta\varphi} = -\Gamma_{\vartheta\varphi\varphi} = \tfrac{1}{2} g_{\varphi\varphi,\,\vartheta} = -r^2 \sin(\vartheta)\cos(\vartheta) \quad .$$

The inverse metric is given by

$$g^{tt} = \frac{1}{A(r)}, \quad g^{rr} = -A(r), \quad g^{\vartheta\vartheta} = -r^{-2}, \quad g^{\varphi\varphi} = -r^{-2}\sin^{-2}(\vartheta) \quad .$$

Hence for the affine connections we find

$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = g^{tt}\Gamma_{ttr} = \frac{A'}{2A} \quad ,$$

$$\Gamma^{r}_{tt} = g^{rr}\Gamma_{rtt} = \frac{AA'}{2}$$

$$\Gamma^{r}_{rr} = g^{rr}\Gamma_{rrr} = -\frac{A'}{2A} \quad ,$$

$$\Gamma^{r}_{\vartheta\vartheta} = g^{rr}\Gamma_{r\vartheta\vartheta} = -rA \quad ,$$

$$\Gamma^{\vartheta}_{r\vartheta} = \Gamma^{\vartheta}_{\vartheta r} = g^{\vartheta\vartheta}\Gamma_{\vartheta r\vartheta} = \frac{1}{r} \quad ,$$

$$\Gamma^{r}_{\varphi\varphi} = g^{rr}\Gamma_{r\varphi\varphi} = -rA\sin^2(\vartheta) \quad ,$$

$$\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = g^{\varphi\varphi}\Gamma_{\varphi r\varphi} = \frac{1}{r} \quad ,$$

$$\Gamma^{\vartheta}_{\varphi\varphi} = g^{\vartheta\vartheta}\Gamma_{\vartheta\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta) \quad \text{and}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \Gamma^{\varphi}_{\varphi\vartheta} = g^{\varphi\varphi}\Gamma_{\varphi\vartheta\varphi} = \frac{\cos(\vartheta)}{\sin(\vartheta)} \quad .$$

b. Using expression (6) and the above expressions for the affine connections, we construct the geodesic equations for the mass distribution.

$$\begin{aligned}
0 &= \frac{d^2t}{ds^2} + \frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{t}_{\alpha\beta}(u) \\
&= \frac{d^2t}{ds^2} + 2\frac{dt}{ds}\frac{dr}{ds}\Gamma^{t}_{tr} = \frac{d^2t}{ds^2} + \frac{dt}{ds}\frac{dr}{ds}\frac{A'}{A} = \frac{1}{A(r)}\frac{d}{ds}\left\{\frac{dt}{ds}A(r)\right\} \quad , \\
0 &= \frac{d^2r}{ds^2} + \frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{r}_{\alpha\beta}(u) \\
&= \frac{d^2r}{ds^2} + \left(\frac{dt}{ds}\right)^2\Gamma^{r}_{tt} + \left(\frac{dr}{ds}\right)^2\Gamma^{r}_{rr} + \left(\frac{d\vartheta}{ds}\right)^2\Gamma^{r}_{\vartheta\vartheta} + \left(\frac{d\varphi}{ds}\right)^2\Gamma^{r}_{\varphi\varphi} \\
&= \frac{d^2r}{ds^2} + \left(\frac{dt}{ds}\right)^2\frac{AA'}{2} - \left(\frac{dr}{ds}\right)^2\frac{A'}{2A} - \left(\frac{d\vartheta}{ds}\right)^2 rA - \left(\frac{d\varphi}{ds}\right)^2 rA\sin^2(\vartheta) \\
0 &= \frac{d^2\vartheta}{ds^2} + \frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{\vartheta}_{\alpha\beta}(u) \\
&= \frac{d^2\vartheta}{ds^2} + 2\frac{dr}{ds}\frac{d\vartheta}{ds}\Gamma^{\vartheta}_{r\vartheta} + \left(\frac{d\varphi}{ds}\right)^2\Gamma^{\vartheta}_{\varphi\varphi} = \frac{d^2\vartheta}{ds^2} + 2\frac{dr}{ds}\frac{d\vartheta}{ds}\frac{1}{r} - \left(\frac{d\varphi}{ds}\right)^2\sin(\vartheta)\cos(\vartheta) \\
0 &= \frac{d^2\varphi}{ds^2} + \frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{\varphi}_{\alpha\beta}(u) \\
&= \frac{d^2\varphi}{ds^2} + 2\frac{dr}{ds}\frac{d\varphi}{ds}\Gamma^{\varphi}_{r\varphi} + 2\frac{d\vartheta}{ds}\frac{d\varphi}{ds}\Gamma^{\varphi}_{\vartheta\varphi} = \frac{d^2\varphi}{ds^2} + 2\frac{dr}{ds}\frac{d\varphi}{ds}\frac{1}{r} + 2\frac{d\vartheta}{ds}\frac{d\varphi}{ds}\frac{\cos(\vartheta)}{\sin(\vartheta)} \quad .
\end{aligned}$$

In the plane $\vartheta = \pi/2$ those differential equations reduce to ($d\vartheta/ds = 0$)

$$0 = \frac{1}{A(r)}\frac{d}{ds}\left\{\frac{dt}{ds}A(r)\right\} \quad ,$$

$$
\begin{aligned}
0 &= \frac{d^2r}{ds^2} + \left(\frac{dt}{ds}\right)^2 \frac{AA'}{2} - \left(\frac{dr}{ds}\right)^2 \frac{A'}{2A} - \left(\frac{d\varphi}{ds}\right)^2 rA \\
0 &= \frac{d^2\varphi}{ds^2} + 2\frac{dr}{ds}\frac{d\varphi}{ds}\frac{1}{r} = \frac{1}{r}\frac{d}{ds}\left\{r^2\frac{d\varphi}{ds}\right\} \quad .
\end{aligned}
$$

We find two constants of motion in the plane $\vartheta = \pi/2$ for $A(r) \neq 0$ and $r \neq 0$, namely

$$
\tau = \frac{dt}{ds}A(r) \quad \text{and} \quad \ell = r^2\frac{d\varphi}{ds} \quad .
$$

When we substitute those equations of motion in the second of the above differential equations, we find

$$
\begin{aligned}
0 &= \frac{d^2r}{ds^2} + \tau^2\frac{A'}{2A} - \left(\frac{dr}{ds}\right)^2 \frac{A'}{2A} - \ell^2\frac{A}{r^3} \\
&= \left(\frac{2}{A}\frac{dr}{ds}\right)^{-1} \frac{d}{ds}\left\{\frac{1}{A}\left(\frac{dr}{ds}\right)^2 - \frac{\tau^2}{A} + \frac{\ell^2}{r^2}\right\} \quad ,
\end{aligned}
$$

which establishes a third constant of motion, defined by

$$
\epsilon = \frac{\tau^2}{A(r)} - \frac{\ell^2}{r^2} - \frac{1}{A(r)}\left(\frac{dr}{ds}\right)^2 \quad .
$$

c. For $\varphi(s)$ one has

$$
\frac{d}{ds} = \frac{d\varphi}{ds}\frac{d}{d\varphi} = \frac{\ell}{r^2}\frac{d}{d\varphi} \quad .
$$

Hence,

$$
\frac{d}{ds}\left\{\frac{1}{A}\left(\frac{dr}{ds}\right)^2 - \frac{\tau^2}{A} + \frac{\ell^2}{r^2}\right\} = 0 \quad \Longrightarrow \quad \frac{\ell}{r^2}\frac{d}{d\varphi}\left\{\frac{1}{A}\left(\frac{\ell}{r^2}\frac{dr}{d\varphi}\right)^2 - \frac{\tau^2}{A} + \frac{\ell^2}{r^2}\right\} = 0 \quad .
$$

d. For $r \gg 2MG$ one has

$$
\frac{A'}{A} = \frac{\frac{2MG}{r^2}}{1 - \frac{2MG}{r}} = \frac{1}{r}\frac{2MG}{r - 2MG} \approx \frac{2MG}{r^2} \quad \text{and} \quad A = 1 - \frac{2MG}{r} \approx 1 \quad .
$$

Hence, we obtain for the constant of motion τ of paragraph **c** the relation

$$
\tau = \frac{dt}{ds}A(r) \approx \frac{dt}{ds} \quad \Longrightarrow \quad t = \tau s + \text{constant} \quad .
$$

This allows us to choose $\tau = 1$ and set the constant to zero for $r \gg 2MG$, which leaves us with the identification

$$
t = s \quad .
$$

Next, we start from the final equation of paragraph **c**, which for $A \neq 0$ and $r \neq 0$ gives

$$\begin{aligned} 0 &= \frac{d}{d\varphi}\left\{\frac{1}{A}\left(\frac{\ell}{r^2}\frac{dr}{d\varphi}\right)^2 - \frac{\tau^2}{A} + \frac{\ell^2}{r^2}\right\} \\ &= -\frac{dr}{d\varphi}\frac{A'}{A^2}\left(\frac{\ell}{r^2}\frac{dr}{d\varphi}\right)^2 - \frac{1}{A}\frac{dr}{d\varphi}\frac{4\ell^2}{r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{2}{A}\left(\frac{\ell}{r^2}\right)^2\frac{dr}{d\varphi}\frac{d^2r}{d\varphi^2} + \frac{dr}{d\varphi}\frac{A'}{A^2} - 2\frac{dr}{d\varphi}\frac{\ell^2}{r^3} \\ &= \frac{2\ell^2}{Ar^3}\frac{dr}{d\varphi}\left\{-\frac{A'}{2A}\frac{1}{r}\left(\frac{dr}{d\varphi}\right)^2 - \frac{2}{r^2}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r}\frac{d^2r}{d\varphi^2} + \frac{A'r^3}{2A\ell^2} - A\right\} \quad , \end{aligned}$$

which, for $A \neq 0$, $r \neq 0$ and $dr/d\varphi \neq 0$, leaves us with the geodesic equation

$$-\frac{A'}{2A}\frac{1}{r}\left(\frac{dr}{d\varphi}\right)^2 - \frac{2}{r^2}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r}\frac{d^2r}{d\varphi^2} + \frac{A'r^3}{2A\ell^2} - A = 0 \quad .$$

We then end up with the equation

$$\begin{aligned} -\frac{MG}{r^2}\frac{1}{r}\left(\frac{dr}{d\varphi}\right)^2 - \frac{2}{r^2}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r}\frac{d^2r}{d\varphi^2} + \frac{MGr}{\ell^2} - 1 &= \\ -\frac{2}{r^2}\left\{1 + \frac{MG}{2r}\right\}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r}\frac{d^2r}{d\varphi^2} + \frac{MGr}{\ell^2} - 1 &= 0 \quad , \end{aligned}$$

which for $r \gg 2MG$ gives

$$-\frac{2}{r^2}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r}\frac{d^2r}{d\varphi^2} + \frac{MGr}{\ell^2} - 1 = 0 \quad .$$

Let us verify that $r(\varphi) = \ell^2/MG\,(1 - e\cos(\varphi))$ is a solution of the above geodesic equation.

$$\frac{dr}{d\varphi} = -\frac{\ell^2}{MG}\frac{e\sin(\varphi)}{(1-e\cos(\varphi))^2} = -\frac{MGe\sin(\varphi)}{\ell^2}r^2 \quad .$$

Hence,

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{(MGe)^2\sin^2(\varphi)}{\ell^4}r^4 \quad .$$

Furthermore

$$\begin{aligned} \frac{d^2r}{d\varphi^2} &= -\frac{\ell^2}{MG}\frac{e\cos(\varphi)}{(1-e\cos(\varphi))^2} + \frac{\ell^2}{MG}\frac{2e^2\sin^2(\varphi)}{(1-e\cos(\varphi))^3} \\ &= -\frac{MGe\cos(\varphi)}{\ell^2}r^2 + \frac{2(MGe)^2\sin^2(\varphi)}{\ell^4}r^3 \quad . \end{aligned}$$

Substitution of the above results into the differential equation gives

$$\begin{aligned} &-\frac{2(MGe)^2\sin^2(\varphi)}{\ell^4}r^2 - \frac{MGe\cos(\varphi)}{\ell^2}r + \frac{2(MGe)^2\sin^2(\varphi)}{\ell^4}r^2 + \frac{MGr}{\ell^2} - 1 \\ &= \frac{MG\,(1-e\cos(\varphi))}{\ell^2}r - 1 = +1 - 1 = 0 \quad . \end{aligned}$$

In Fig. 1 we show the particle's orbit for the case that $e > 1$. That orbit has the shape of a hyperbola.

In analytical geometry, with orthogonal coordinates X and Y, the hyperbola can be given by the expression

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad \text{with} \quad b^2 = c^2 - a^2 \quad .$$

Here a is the shortest distance of the hyperbola to the center of the (X, Y) coordinate system, whereas c is the position of its focal point.

In Fig. 1 we show the (x, y) coordinate system of the Schwarzschild metric given by Eq. (1) and which has its center in the focal point $X = c$ of the hyperbola. Hence, in the coordinates x and y the hyperbola of Fig. 1 is given by

$$\frac{(x+c)^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{with} \quad b^2 = c^2 - a^2 \quad . \tag{7}$$

The center of the hyperbola, which is the center of the (X, Y) coordinate system, comes at $x = -c$. In that case, the hyperbola may also be expressed as

$$r = \frac{a(e^2 - 1)}{1 - e\cos(\varphi)} \quad , \quad e = \frac{c}{a} \quad \text{and} \quad a^2 + b^2 = c^2 \quad , \tag{8}$$

where we take negative values for the angle φ when $y < 0$, *i.e.*

$$\varphi\,(t \downarrow -\infty) = -\arccos\left(\frac{1}{e}\right) \quad \text{and} \quad \varphi\,(t \uparrow 0) = -\pi \quad ,$$

and positive values for the angle φ when $y > 0$, *i.e.*

$$\varphi\,(t \downarrow 0) = +\pi \quad \text{and} \quad \varphi\,(t \uparrow +\infty) = +\arccos\left(\frac{1}{e}\right) \quad .$$

In the following, we will show that Eq. (7) is equivalent to Eq. (8). For that, we start with

$$e^2 - 1 = \frac{c^2}{a^2} - 1 = \frac{c^2 - a^2}{a^2} = \frac{b^2}{a^2} \quad \Longrightarrow \quad 1 = \frac{a^2(e^2-1)}{b^2} \quad \text{and} \quad a^2 = \frac{b^2}{e^2 - 1} \quad .$$

Furthermore $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$. Hence,

$$1 = \frac{a^2(e^2-1)}{b^2} = \frac{r^2}{b^2(e^2-1)}\left(1 - e\cos(\varphi)\right)^2 = \frac{r^2}{b^2(e^2-1)}\left(1 - 2e\cos(\varphi) + e^2\cos^2(\varphi)\right) =$$

$$= \frac{r^2}{b^2(e^2-1)}\left\{\cos^2(\varphi) + \sin^2(\varphi) - 2e\cos(\varphi) + e^2\left(1 - \sin^2(\varphi)\right)\right\}$$

$$= \frac{r^2}{b^2(e^2-1)}\left\{\cos^2(\varphi) - 2e\cos(\varphi) + e^2 - e^2\sin^2(\varphi) + \sin^2(\varphi)\right\}$$

$$= \frac{r^2}{a^2b^2(e^2-1)}\left\{a^2\left(\cos^2(\varphi) - 2e\cos(\varphi) + e^2\right) - a^2(e^2-1)\sin^2(\varphi)\right\}$$

$$= \frac{r^2}{a^2b^2(e^2-1)}\left\{\frac{b^2}{(e^2-1)}\left(e - \cos(\varphi)\right)^2 - a^2(e^2-1)\sin^2(\varphi)\right\}$$

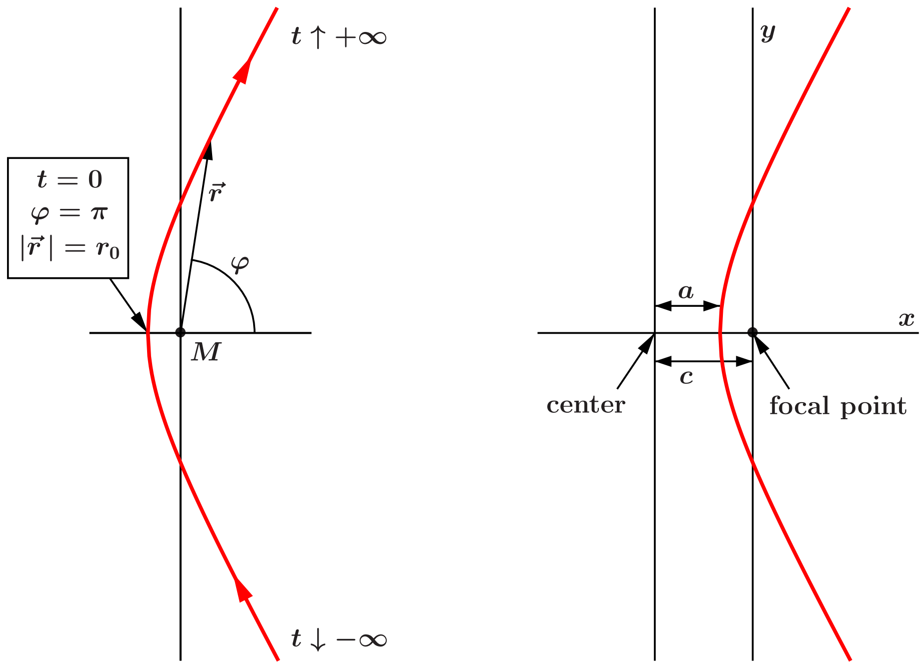


Figure 1: The orbit for $e > 1$ has the shape of a hyperbola.

$$= \frac{r^2}{a^2}\left(\frac{e-\cos(\varphi)}{e^2-1}\right)^2 - \frac{r^2\sin^2(\varphi)}{b^2} = \frac{r^2}{a^2}\left(\frac{e(1-e\cos(\varphi))+(e^2-1)\cos(\varphi)}{e^2-1}\right)^2 - \frac{y^2}{b^2}$$

$$= \frac{r^2}{a^2}\left(\cos(\varphi)+\frac{c(1-e\cos(\varphi))}{a(e^2-1)}\right)^2 - \frac{y^2}{b^2} = \frac{r^2}{a^2}\left(\frac{x}{r}+\frac{c}{r}\right)^2 - \frac{y^2}{b^2} = \frac{(x+c)^2}{a^2} - \frac{y^2}{b^2} \quad ,$$

which is, indeed, the geometric relation (7) for the hyperbola centered at $x = -c$.

d(i). For $e = 0$ one has $c = 0$, hence focal point and center of the orbit are in the same place. Furthermore, $a = b$, which describes a circular motion at distance $r = a = \ell^2/MG$ from the origin of the coordinate system.

d(ii). For $0 < e < 1$ the expression $(1 - e\cos(\varphi))$ takes values between $1 - e > 0$, for $\varphi = 0$, in which case $r(\varphi) = \ell^2/MG\,(1 - e\cos(\varphi))$ has a maximum, and $1 + e > 0$, for $\varphi = \pi$, in which case $r(\varphi) = \ell^2/MG\,(1 - e\cos(\varphi))$ has a minimum. The orbit has the shape of an ellipse with one focus at the origin of the coordinate system.
In the case of an ellipse, one may use the above derivation by substituting b^2 by $-b^2$ and use $r = \dfrac{a(1-e^2)}{1-e\cos(\varphi)}$.

d(iii). For $e = 1$ one has $c = a$, hence $b = 0$. One ends up with $y^2 = 4ax$ in a derivation which is similar to the one above. In that case the orbit has the shape of a parabola with its focal point at the origin of the coordinate system.

d(iv). The case $e > 1$ has been studied in the above. In that case the orbit has the shape of a hyperbola with its focal point at the origin of the coordinate system.

e. When also φ does not vary (radially infalling photon), then the Schwarzschild line element (1) becomes

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} \quad .$$

Furthermore, for light one has $ds^2 = 0$. Hence, for the radial velocity of light we find

$$v(t) = \frac{dr}{dt} = A(r) = 1 - \frac{2MG}{r} \quad .$$

Exercício 8

a. For $e \gg 1$ one has

$$\cos\left(\frac{\pi}{2} - \frac{1}{e}\right) = \sin\left(\frac{1}{e}\right) \approx \frac{1}{e} \quad .$$

Consequently

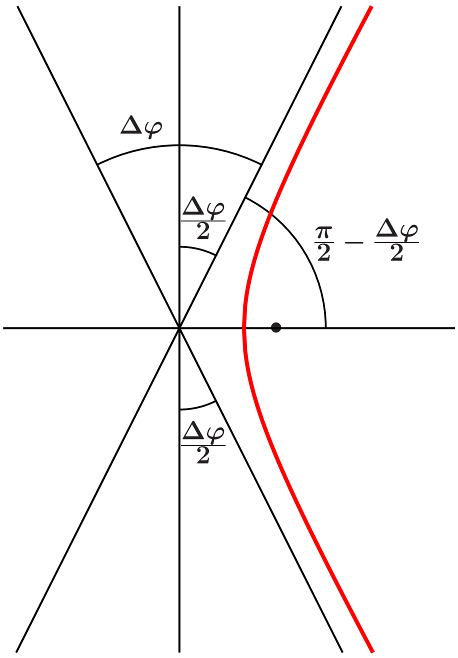
$$1 - e\cos\left(\frac{\pi}{2} - \frac{1}{e}\right) \approx 0 \quad .$$

The angles

$$-\varphi(t \downarrow -\infty) = \varphi(t \uparrow +\infty) \approx \frac{\pi}{2} - \frac{1}{e}$$

correspond to $r \to \pm\infty$.
For an orbit which has the shape of a straight line through the origin, we assume

$$\varphi(t \uparrow +\infty) - \varphi(t \downarrow -\infty) = \pi \quad .$$

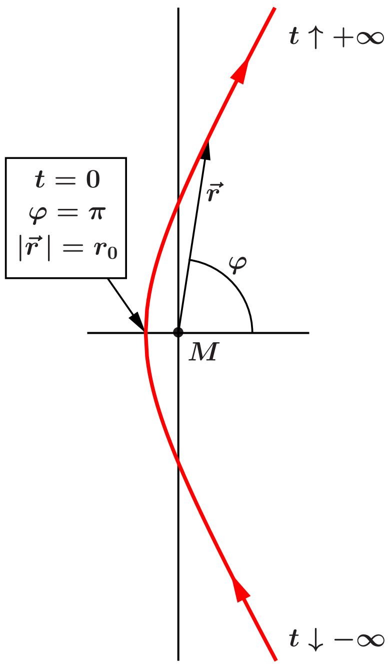


Hence, for an orbit which has the shape of a hyperbola, we obtain

$$\Delta\varphi = |\varphi(t \uparrow +\infty) - \varphi(t \downarrow -\infty) - \pi| = \frac{2}{e} \quad .$$

b. The minimum distance happens for $\varphi = \pi$, which leads for $e \gg 1$ to

$$r_0 = \frac{\ell^2}{MG(1+e)} \approx \frac{\ell^2}{MGe} \quad .$$



For $r_0 \gg MG$ the constant of motion ϵ takes the form (remember $\tau = 1$, whereas, furthermore, $dr/d\varphi = 0$ implies $dr/ds = 0$)

$$\epsilon = \frac{\tau^2}{A(r_0)} - \frac{\ell^2}{r_0^2} - \frac{1}{A(r_0)}\left(\frac{dr}{ds}\right)^2 = \frac{1}{A(r_0)} - \frac{\ell^2}{r_0^2} \approx 1 + \frac{2MG}{r_0} - \frac{\ell^2}{r_0^2} \quad .$$

In terms of classical quantities one has that the potential energy of a mass m in the gravitational field of the sun (mass M) and at a distance r_0 from the sun, is given by

$$U_{\text{pot}} = -\frac{mMG}{r_0} \quad .$$

Furthermore, its kinetic energy is given by

$$U_{\text{kin}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\frac{\ell^2}{r_0^2} \quad .$$

Hence,

$$U_{\text{pot}} + U_{\text{kin}} = \tfrac{1}{2}m\left\{-\frac{2MG}{r_0} + \frac{\ell^2}{r_0^2}\right\} \approx \tfrac{1}{2}m\left\{1 - \epsilon\right\} \quad .$$

Here, we used $\ell = vr_0$, hence for light $\ell = r_0$. We find then for the deflection of light

$$\Delta\varphi = \frac{2}{e} \approx \frac{2MGr_0}{\ell^2} = \frac{2MG}{r_0} \quad .$$

e. We return to the Schwarzschild solution for the metric around the sun. For light one has $ds^2 = 0$. Furthermore, we consider again the plane $\vartheta = \pi/2$. Eq. (1) turns then into

$$dt^2 = \frac{dr^2}{A^2(r)} + \frac{r^2 d\varphi^2}{A(r)} \quad ,$$

with $A(r) = 1 - 2MG/r$.

The above line-element can be regarded describing a two-dimensional surface. Light rays are given by the geodesics on that surface by Fermat's shortest-time principle.

So, we have then a metric and its inverse given by

$$g_{rr} = A^{-2}(r) \quad , \quad g_{\varphi\varphi} = \frac{r^2}{A(r)} \quad ; \quad g^{rr} = A^2(r) \quad \text{and} \quad g^{\varphi\varphi} = \frac{A(r)}{r^2} \quad .$$

The only non-vanishing derivatives of the metric follow from

$$g_{rr,\,r} = -\frac{2A'}{A^3} \quad \text{and} \quad g_{\varphi\varphi,\,r} = \frac{2r}{A} - \frac{r^2A'}{A^2} \quad .$$

Hence, the non-vanishing Christoffel symbols follow from

$$\Gamma_{rrr} = \tfrac{1}{2} g_{rr,\,r} = -\frac{A'}{A^3} \quad \text{and}$$

$$-\Gamma_{r\varphi\varphi} = \Gamma_{\varphi r\varphi} = \Gamma_{\varphi\varphi r} = \tfrac{1}{2} g_{\varphi\varphi,\,r} = \frac{r}{A} - \frac{r^2A'}{2A^2} \quad .$$

Next, we determine the non-vanishing affine connections

$$\Gamma^r_{rr} = g^{rr}\Gamma_{rrr} = A^2\left(-\frac{A'}{A^3}\right) = -\frac{A'}{A} \quad ,$$

$$\Gamma^r_{\varphi\varphi} = g^{rr}\Gamma_{r\varphi\varphi} = A^2\left(-\frac{r}{A} + \frac{r^2A'}{2A^2}\right) = -rA + \tfrac{1}{2}r^2A' \quad \text{and}$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = g^{\varphi\varphi}\Gamma_{\varphi r\varphi} = \frac{A}{r^2}\left(\frac{r}{A} - \frac{r^2A'}{2A^2}\right) = \frac{1}{r} - \frac{A'}{2A} \quad .$$

We obtain the geodesic equations (6)

$$\frac{d^2r}{dt^2} = -\frac{du^\alpha}{dt}\frac{du^\beta}{dt}\Gamma^r_{\alpha\beta} = -\left(\frac{dr}{dt}\right)^2\Gamma^r_{rr} - \left(\frac{d\varphi}{dt}\right)^2\Gamma^r_{\varphi\varphi}$$

$$= \left(\frac{dr}{dt}\right)^2\frac{A'}{A} + \left(\frac{d\varphi}{dt}\right)^2\left(rA - \tfrac{1}{2}r^2A'\right)$$

and

$$\frac{d^2\varphi}{dt^2} = -\frac{du^\alpha}{dt}\frac{du^\beta}{dt}\Gamma^\varphi_{\alpha\beta} = -2\frac{dr}{dt}\frac{d\varphi}{dt}\Gamma^\varphi_{r\varphi}$$

$$= -2\frac{dr}{dt}\frac{d\varphi}{dt}\left(\frac{1}{r} - \frac{A'}{2A}\right) = \frac{dr}{dt}\frac{d\varphi}{dt}\left(\frac{A'}{A} - \frac{2}{r}\right) \quad .$$

The latter equation is equivalent to

$$\frac{d}{dt}\left(\frac{d\varphi}{dt}\frac{r^2}{A}\right) = 0 \quad .$$

Hence,

$$J=\frac{d\varphi}{dt}\frac{r^2}{A}$$

is a constant of motion. Notice that $\frac{d\varphi}{dt}$ is negative, since φ decreases for increasing t. Consequently, $J<0$.

The constant of motion J can be determined at perihelion ($r=r_0$), where r is minimum, hence $dr/dt=0$. For that, we divide the line element for dt^2 by dt^2, to obtain

$$1=\frac{dr^2}{dt^2}\frac{1}{A^2}+\frac{d\varphi^2}{dt^2}\frac{r^2}{A}=\left(\frac{dr}{dt}\right)^2\frac{1}{A^2}+\left(\frac{d\varphi}{dt}\right)^2\frac{r^2}{A}\quad , \tag{9}$$

which at $r=r_0$ reads

$$1=\left(\frac{d\varphi}{dt}\right)^2\frac{r_0^2}{A\left(r_0\right)}\quad\Longrightarrow\quad\frac{d\varphi}{dt}=-\sqrt{\frac{A\left(r_0\right)}{r_0^2}}\quad .$$

Hence,

$$J=\frac{r_0^2}{A\left(r_0\right)}\frac{d\varphi}{dt}\left(r=r_0\right)=-\frac{r_0^2}{A\left(r_0\right)}\sqrt{\frac{A\left(r_0\right)}{r_0^2}}\quad .$$

When we substitute this result in the differential equation for $d\varphi/dt$, we obtain

$$\frac{d\varphi}{dt}=\frac{A(r)}{r^2}J=-\frac{A(r)}{A\left(r_0\right)}\left(\frac{r_0}{r}\right)^2\sqrt{\frac{A\left(r_0\right)}{r_0^2}}\quad .$$

From Eq. (9) we may also obtain

$$\frac{dr}{dt}=A(r)\sqrt{1-\left(\frac{d\varphi}{dt}\right)^2\frac{r^2}{A(r)}}=A(r)\sqrt{1-\frac{A(r)}{A\left(r_0\right)}\left(\frac{r_0}{r}\right)^2}\quad .$$

Notice that dr/dt is positive for $t>0$, since r increases for increasing t when $t>0$.

Next, we divide $d\varphi/dt$ by dr/dt, to find

$$\frac{d\varphi}{dr}=\frac{-1}{r\sqrt{A(r)}\sqrt{\frac{A(r_0)}{A(r)}\left(\frac{r}{r_0}\right)^2-1}}\quad .$$

f. For $r>r_0\gg 2MG$ we expand as follows.

$$\frac{A(r_0)}{A(r)}=\frac{1-\frac{2MG}{r_0}}{1-\frac{2MG}{r}}\approx\left[1-\frac{2MG}{r_0}\right]\left[1+\frac{2MG}{r}\right]\approx 1+2MG\left(\frac{1}{r}-\frac{1}{r_0}\right)\quad .$$

Next

$$2MG\left(\frac{1}{r}-\frac{1}{r_0}\right)\left(\frac{r}{r_0}\right)^2=2MG\frac{r}{r_0}\left(\frac{1}{r_0}-\frac{r}{r_0^2}\right)=$$

$$=\frac{2MGr}{r_0}\left(\frac{r_0-r}{r_0^2}\right)=\frac{2MGr}{r_0\left(r+r_0\right)}\left(\frac{r_0^2-r^2}{r_0^2}\right)=\frac{2MGr}{r_0\left(r+r_0\right)}\left(1-\frac{r^2}{r_0^2}\right)$$

$$=\frac{2MGr}{r_0\left(r+r_0\right)}-\frac{2MGr}{r_0\left(r+r_0\right)}\left(\frac{r}{r_0}\right)^2\quad .$$

and

$$\frac{A(r_0)}{A(r)}\left(\frac{r}{r_0}\right)^2 - 1 \approx \left(\frac{r}{r_0}\right)^2 + 2MG\left(\frac{1}{r} - \frac{1}{r_0}\right)\left(\frac{r}{r_0}\right)^2 - 1 =$$

$$= \left(\frac{r}{r_0}\right)^2 + \frac{2MGr}{r_0\,(r+r_0)} - \frac{2MGr}{r_0\,(r+r_0)}\left(\frac{r}{r_0}\right)^2 - 1 = \left\{\left(\frac{r}{r_0}\right)^2 - 1\right\}\left[1 - \frac{2MGr}{r_0\,(r+r_0)}\right] \quad .$$

Furthermore

$$\sqrt{1 - \frac{2MGr}{r_0\,(r+r_0)}} \approx 1 - \frac{MGr}{r_0\,(r+r_0)} \quad .$$

and

$$\frac{1}{\sqrt{1 - \frac{2MGr}{r_0\,(r+r_0)}}} \approx 1 + \frac{MGr}{r_0\,(r+r_0)} \quad .$$

Hence,

$$\frac{1}{\sqrt{\frac{A(r_0)}{A(r)}\left(\frac{r}{r_0}\right)^2 - 1}} \approx \frac{1}{\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}}\left\{1 + \frac{MGr}{r_0\,(r+r_0)}\right\} \quad .$$

Also

$$\frac{1}{\sqrt{A(r)}} = \frac{1}{\sqrt{1 - \frac{2MG}{r}}} \approx 1 + \frac{MG}{r} \quad .$$

So, for the product of the previous two terms we find

$$\frac{1}{\sqrt{A(r)}}\frac{1}{\sqrt{\frac{A(r_0)}{A(r)}\left(\frac{r}{r_0}\right)^2 - 1}} \approx \frac{1}{\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}}\left\{1 + \frac{MG}{r} + \frac{MGr}{r_0\,(r+r_0)}\right\} \quad .$$

We obtain for the differential equation of $d\varphi/dr$

$$\frac{d\varphi}{dr} \approx \frac{-1}{r\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}}\left\{1 + \frac{MG}{r} + \frac{MGr}{r_0\,(r+r_0)}\right\} =$$

$$= \frac{-1}{r\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} - \frac{MG}{r^2\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} - \frac{MG}{r_0\,(r+r_0)\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}}$$

$$= \frac{-1}{r\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} - \frac{MG}{r^2\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} - \frac{MG}{(r+r_0)\sqrt{r^2 - r_0^2}}$$

$$= \frac{-1}{r\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} - \frac{MG}{r^2\sqrt{\left(\frac{r}{r_0}\right)^2 - 1}} - \frac{MG}{(r+r_0)^{3/2}\sqrt{r - r_0}} \quad .$$

Hence, besides an integration constant C, we have

$$\varphi(r) = C + \arcsin\left(\frac{r_0}{r}\right) - \frac{MG}{r_0}\sqrt{1 - \left(\frac{r_0}{r}\right)^2} - \frac{MG}{r_0}\sqrt{\frac{r - r_0}{r + r_0}} \quad .$$

We want $\varphi(r_0) = \pi$. Hence, the integration constant is given by

$$\pi = \varphi(r_0) = C + \arcsin(1) = C + \frac{\pi}{2} \quad \Longrightarrow \quad C = \frac{\pi}{2} \quad .$$

For $r \to \infty$, which is equivalent to $t \to +\infty$ for $t > 0$, we find then

$$\varphi(t \to +\infty) = \frac{\pi}{2} + \arcsin(0) - \frac{MG}{r_0} - \frac{MG}{r_0} = \frac{\pi}{2} - \frac{2MG}{r_0} \quad .$$

By symmetry the angle at the other side is then given by $\varphi(t \to -\infty) = -\pi/2 + 2MG/r_0$. Consequently, the light deflection is given by

$$\Delta\varphi = \frac{4MG}{r_0} \quad .$$

When we calculate this value for a light ray which just grazes the surface of the sun, we find 1.75 seconds of arc. Using $M_{\text{Sun}} = 2 \times 10^{30}$ kg, for the mass of the sun, $G/c^2 = 7.425 \times 10^{-28}$ m/kg, for the gravitational constant, and $R_{\text{Sun}} = 7 \times 10^8$ m, for the solar radius, we calculate

$$\Delta\varphi = \frac{4M_{\text{Sun}}G}{R_{\text{Sun}}c^2} = 8.5 \times 10^{-6} \text{ rad} = 1.75 \text{ arcsecond} \quad .$$

Below, you find some observations during solar eclipses and which are taken from http://mathpages.com/rr/s6-03/6-03.htm .

date	location	deflection (arcseconds)
May, 29th, 1919	Sobral	1.98±0.16
	Principe	1.16±0.40
Sep, 21st, 1922	Australia	1.77±0.40
		1.42-2.16
		1.72±0.15
		1.82±0.20
May, 9th, 1929	Sumatra	2.24±0.10
June, 19th, 1936	USSR	2.73±0.31
	Japan	1.28-2.13
May, 20th, 1947	Brazil	2.01±0.27
Feb, 25th, 1952	Sudan	1.70±0.10
June, 30st, 1973	Mauritania	1.66±0.19

Exercício 9

a.

$$dr^* = \left\{1 + \frac{1}{\frac{r}{2MG} - 1}\right\} dr = \frac{1}{A(r)} dr \quad .$$

Hence

$$\frac{1}{A(r)} dr^2 = A(r) dr^{*2} \quad .$$

b.

$$d\tilde{v} = dt + dr^* = dt + \frac{1}{A(r)} dr \quad .$$

Hence

$$A(r) d\tilde{v}^2 - 2 d\tilde{v} dr = A dt^2 + 2 dt dr + \frac{1}{A} dr^2 - 2 dt dr - \frac{2}{A} dr^2 = A dt^2 - \frac{1}{A} dr^2 \quad .$$

c.

$$d\tilde{u} = dt - dr^* = dt - \frac{1}{A(r)} dr \quad .$$

Hence

$$A(r) d\tilde{u}^2 + 2 d\tilde{u} dr = A dt^2 - 2 dt dr + \frac{1}{A} dr^2 + 2 dt dr - \frac{2}{A} dr^2 = A dt^2 - \frac{1}{A} dr^2 \quad .$$

d.

$$A d\tilde{v} d\tilde{u} = A \left(dt + \frac{1}{A} dr\right)\left(dt - \frac{1}{A} dr\right) = A dt^2 - \frac{1}{A} dr^2 \quad .$$

e.

$$dv' = \frac{1}{4MG} v' d\tilde{v} \quad \text{and} du' = -\frac{1}{4MG} u' d\tilde{u} \quad .$$

Hence,

$$A d\tilde{v} d\tilde{u} = -\frac{16 M^2 G^2 A}{v' u'} dv' du' \quad .$$

Here

$$v' u' = e^{\tilde{v}/4MG} e^{-\tilde{u}/4MG} = e^{(\tilde{v} - \tilde{u})/4MG} = e^{r^*/2MG} \quad .$$

Furthermore, on substituting the definition of r^*, we obtain

$$v' u' = e^{\left(r + 2MG \log\left|\frac{r}{2MG} - 1\right|\right)/2MG} = e^{r/2MG}\left(\frac{r}{2MG} - 1\right) = e^{r/2MG} \frac{r}{2MG} A \quad .$$

Hence,

$$A d\tilde{v} d\tilde{u} = -\frac{32 M^3 G^3}{r} e^{-r/2MG} dv' du' \quad .$$

f.

$$du = \tfrac{1}{2}(dv' - du') \quad \text{and} dv = \tfrac{1}{2}(dv' + du') \quad .$$

Hence,

$$dv^2 - du^2 = dv'du' \quad ,$$

which gives

$$-\frac{32M^3G^3}{r}e^{-r/2MG}dv'du' = -\frac{32M^3G^3}{r}e^{-r/2MG}\left(dv^2 - du^2\right) \quad .$$

g. We write for the metric of **b**

$$g = \begin{pmatrix} g_{\tilde{v}\tilde{v}} & g_{\tilde{v}r} & g_{\tilde{v}\vartheta} & g_{\tilde{v}\varphi} \\ g_{r\tilde{v}} & g_{rr} & g_{r\vartheta} & g_{r\varphi} \\ g_{\vartheta\tilde{v}} & g_{\vartheta r} & g_{\vartheta\vartheta} & g_{\vartheta\varphi} \\ g_{\varphi\tilde{v}} & g_{\varphi r} & g_{\varphi\vartheta} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} A(r) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\vartheta) \end{pmatrix} \quad .$$

The determinant of the metric

$$\det(g) = \det\begin{pmatrix} A(r) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\vartheta) \end{pmatrix} = \det\begin{pmatrix} A(r) & -1 \\ -1 & 0 \end{pmatrix} \times \det\begin{pmatrix} -r^2 & 0 \\ 0 & -r^2\sin^2(\vartheta) \end{pmatrix}$$

obviously results in $-r^4\sin^2(\vartheta)$ for the above expression.

Exercício 10

a. The metric (*Robertson-Walker* for $K=4$) is given by $(i, j = 1, 2, 3)$

$$g_{tt} = 1 \quad \text{and} \quad g_{xx} = g_{yy} = g_{zz} = -\frac{a^2(t)}{\left(1+r^2\right)^2} \quad \text{or} \quad g_{ij} = -\frac{a^2(t)}{\left(1+r^2\right)^2}\delta_{ij} \quad .$$

It's non-vanishing derivatives are given by

$$g_{ij,\,t} = -\frac{2\dot{a}a}{\left(1+r^2\right)^2}\delta_{ij} \quad \text{and} \quad g_{ij,\,k} = \frac{4a^2\delta_{k\ell}x^\ell}{\left(1+r^2\right)^3}\delta_{ij} \quad .$$

The non-vanishing Christoffel symbols (see Eq. 3) are given by

$$-\Gamma_{tij} = \Gamma_{ijt} = \Gamma_{itj} = \tfrac{1}{2}g_{ij,\,t} = -\frac{\dot{a}a}{\left(1+r^2\right)^2}\delta_{ij}$$

$$\begin{aligned}
\Gamma_{ijk} &= \tfrac{1}{2}\left\{g_{ij,\,k} + g_{ik,\,j} - g_{jk,\,i}\right\} \\
&= \tfrac{1}{2}\left\{\frac{4a^2\delta_{k\ell}x^\ell}{\left(1+r^2\right)^3}\delta_{ij} + \frac{4a^2\delta_{j\ell}x^\ell}{\left(1+r^2\right)^3}\delta_{ik} - \frac{4a^2\delta_{i\ell}x^\ell}{\left(1+r^2\right)^3}\delta_{jk}\right\} \\
&= \frac{2a^2}{\left(1+r^2\right)^3}\left\{\delta_{k\ell}x^\ell\delta_{ij} + \delta_{j\ell}x^\ell\delta_{ik} - \delta_{i\ell}x^\ell\delta_{jk}\right\} \quad .
\end{aligned}$$

The inverse of the metric is given by

$$g^{tt} = 1 \quad \text{and} \quad g^{xx} = g^{yy} = g^{zz} = -\frac{\left(1+r^2\right)^2}{a^2} \quad \text{or} \quad g^{ij} = -\frac{\left(1+r^2\right)^2}{a^2}\delta^{ij} \quad .$$

Hence, the non-vanishing afine connections are

$$\Gamma^t_{ij} = g^{tt}\Gamma_{tij} = \frac{\dot{a}a}{\left(1+r^2\right)^2}\delta_{ij} \quad ,$$

$$\Gamma^i_{tj} = \Gamma^i_{jt} = g^{ik}\Gamma_{ktj} = \frac{\left(1+r^2\right)^2}{a^2}\delta^{ik}\frac{\dot{a}a}{\left(1+r^2\right)^2}\delta_{kj} = \frac{\dot{a}}{a}\delta^i_j \quad ,$$

$$\begin{aligned}
\Gamma^i_{jk} = g^{im}\Gamma_{mjk} &= -\frac{\left(1+r^2\right)^2}{a^2}\delta^{im}\frac{2a^2}{\left(1+r^2\right)^3}\left\{\delta_{k\ell}x^\ell\delta_{mj} + \delta_{j\ell}x^\ell\delta_{mk} - \delta_{m\ell}x^\ell\delta_{jk}\right\} \\
&= -\frac{2}{1+r^2}\left\{\delta_{k\ell}x^\ell\delta^i_j + \delta_{j\ell}x^\ell\delta^i_k - x^i\delta_{jk}\right\} \quad .
\end{aligned}$$

The non-vanishing derivatives of the affine connections are

$$\Gamma^t_{ij,\,t} = \frac{\ddot{a}a + \dot{a}^2}{\left(1+r^2\right)^2}\delta_{ij} \quad , \quad \Gamma^t_{ij,\,k} = -\frac{4\dot{a}a\delta_{k\ell}x^\ell}{\left(1+r^2\right)^3}\delta_{ij} \quad ,$$

$$\Gamma^i_{tj,\,t} = \Gamma^i_{jt,\,t} = \left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)\delta^i_j \quad ,$$

$$\Gamma^i_{jk,\,\ell} = \frac{4\delta_{\ell m}x^m}{(1+r^2)^2}\left\{\delta_{kn}x^n\delta^i_j + \delta_{jn}x^n\delta^i_k - x^i\delta_{jk}\right\} - \frac{2}{1+r^2}\left\{\delta_{km}\delta^m_\ell\delta^i_j + \delta_{jm}\delta^m_\ell\delta^i_k - \delta^i_\ell\delta_{jk}\right\}$$

$$= \frac{4\delta_{\ell m}x^m}{(1+r^2)^2}\left\{\delta_{kn}x^n\delta^i_j + \delta_{jn}x^n\delta^i_k - x^i\delta_{jk}\right\} - \frac{2}{1+r^2}\left\{\delta_{k\ell}\delta^i_j + \delta_{j\ell}\delta^i_k - \delta^i_\ell\delta_{jk}\right\} \quad .$$

For the Ricci tensor we have, also using Eq. (4),

$$R_{tt} = R^\mu_{t\mu t} = R^t_{ttt} + R^i_{tit} = 0 + R^i_{tit} =$$

$$= \Gamma^i_{ti,\,t} - \Gamma^i_{tt,\,i} + \Gamma^i_{\beta t}\Gamma^\beta_{ti} - \Gamma^i_{\beta i}\Gamma^\beta_{tt}$$

$$= \Gamma^i_{ti,\,t} - \Gamma^i_{tt,\,i} + \Gamma^i_{tt}\Gamma^t_{ti} + \Gamma^i_{jt}\Gamma^j_{ti} - \Gamma^i_{ti}\Gamma^t_{tt} - \Gamma^i_{ji}\Gamma^j_{tt}$$

$$= \left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)\delta^i_i - 0 + 0 + \frac{\dot a}{a}\delta^i_j\frac{\dot a}{a}\delta^j_i - 0 - 0$$

$$= \left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)\delta^i_i + \frac{\dot a^2}{a^2}\delta^i_i = \frac{\ddot a}{a}\delta^i_i = 3\frac{\ddot a}{a} \quad .$$

and

$$R_{ij} = R^\mu_{i\mu j} = R^t_{itj} + R^k_{ikj} =$$

$$= \Gamma^t_{it,\,j} - \Gamma^t_{ij,\,t} + \Gamma^t_{\beta j}\Gamma^\beta_{it} - \Gamma^t_{\beta t}\Gamma^\beta_{ij} + \Gamma^k_{ik,\,j} - \Gamma^k_{ij,\,k} + \Gamma^k_{\beta j}\Gamma^\beta_{ik} - \Gamma^k_{\beta k}\Gamma^\beta_{ij}$$

$$= 0 - \Gamma^t_{ij,\,t} + \Gamma^t_{kj}\Gamma^k_{it} - 0 + \Gamma^k_{ik,\,j} - \Gamma^k_{ij,\,k} + \Gamma^k_{tj}\Gamma^t_{ik} + \Gamma^k_{\ell j}\Gamma^\ell_{ik} - \Gamma^k_{tk}\Gamma^t_{ij} - \Gamma^k_{\ell k}\Gamma^\ell_{ij}$$

$$= -\frac{\ddot a a + \dot a^2}{(1+r^2)^2}\delta_{ij} + \frac{\dot a a}{(1+r^2)^2}\delta_{kj}\frac{\dot a}{a}\delta^k_i +$$

$$+ \left(\frac{4\delta_{jm}x^m}{(1+r^2)^2}\left\{\delta_{kn}x^n\delta^k_i + \delta_{in}x^n\delta^k_k - x^k\delta_{ik}\right\} - \frac{2}{1+r^2}\left\{\delta_{kj}\delta^k_i + \delta_{ij}\delta^k_k - \delta^k_j\delta_{ik}\right\}\right) +$$

$$- \left(\frac{4\delta_{km}x^m}{(1+r^2)^2}\left\{\delta_{jn}x^n\delta^k_i + \delta_{in}x^n\delta^k_j - x^k\delta_{ij}\right\} - \frac{2}{1+r^2}\left\{\delta_{jk}\delta^k_i + \delta_{ik}\delta^k_j - \delta^k_k\delta_{ij}\right\}\right) +$$

$$+ \frac{\dot a}{a}\delta^k_j\frac{\dot a a}{(1+r^2)^2}\delta_{ik} +$$

$$+\frac{2}{1+r^2}\left\{\delta_{jm}x^m\delta^k_\ell+\delta_{\ell m}x^m\delta^k_j-x^k\delta_{\ell j}\right\}\frac{2}{1+r^2}\left\{\delta_{km}x^m\delta^\ell_i+\delta_{im}x^m\delta^\ell_k-x^\ell\delta_{ik}\right\}+$$

$$-\frac{\dot a}{a}\delta^k_k\frac{\dot a a}{(1+r^2)^2}\delta_{ij}+$$

$$-\frac{2}{1+r^2}\left\{\delta_{km}x^m\delta^k_\ell+\delta_{\ell m}x^m\delta^k_k-x^k\delta_{\ell k}\right\}\frac{2}{1+r^2}\left\{\delta_{jm}x^m\delta^\ell_i+\delta_{im}x^m\delta^\ell_j-x^\ell\delta_{ij}\right\}$$

$$=-\frac{\ddot a a+2\dot a^2}{(1+r^2)^2}\delta_{ij}+\frac{12\delta_{ik}x^k\delta_{j\ell}x^\ell}{(1+r^2)^2}-\frac{6\delta_{ij}}{1+r^2}-\frac{8\delta_{ik}x^k\delta_{j\ell}x^\ell-4r^2\delta_{ij}}{(1+r^2)^2}-\frac{2\delta_{ij}}{1+r^2}+$$

$$+\frac{20\delta_{ik}x^k\delta_{j\ell}x^\ell-8r^2\delta_{ij}}{(1+r^2)^2}-\frac{24\delta_{ik}x^k\delta_{j\ell}x^\ell-12r^2\delta_{ij}}{(1+r^2)^2}$$

$$=-\frac{\ddot a a+2\dot a^2+8}{(1+r^2)^2}\delta_{ij} \quad .$$

The scalar curvature equals

$$R=g^{\mu\nu}R_{\mu\nu}=g^{tt}R_{tt}+g^{ij}R_{ij}$$

$$=3\frac{\ddot a}{a}+\frac{(1+r^2)^2}{a^2}\delta^{ij}\frac{1}{(1+r^2)^2}\left(\ddot a a+2\dot a^2+8\right)\delta_{ij}$$

$$=3\frac{\ddot a}{a}+\frac{1}{a^2}3\left(\ddot a a+2\dot a^2+8\right)=6\frac{\ddot a}{a}+6\frac{\dot a^2}{a^2}+\frac{24}{a^2} \quad .$$

The Einstein tensor equals

$$R_{tt}-\tfrac{1}{2}g_{tt}R=3\frac{\ddot a}{a}-3\frac{\ddot a}{a}-3\frac{\dot a^2}{a^2}-\frac{12}{a^2}=-3\frac{\dot a^2}{a^2}-\frac{12}{a^2}$$

$$R_{ij}-\tfrac{1}{2}g_{ij}R=-\frac{\ddot a a+2\dot a^2+8}{(1+r^2)^2}\delta_{ij}+\frac{a^2(t)}{(1+r^2)^2}\delta_{ij}\left(3\frac{\ddot a}{a}+3\frac{\dot a^2}{a^2}+\frac{12}{a^2}\right)=\frac{2\ddot a a+\dot a^2+4}{(1+r^2)^2}\delta_{ij} \quad .$$

Hence, the Einstein equations read

$$-3\frac{\dot a^2}{a^2}-\frac{12}{a^2}=R_{tt}-\tfrac{1}{2}g_{tt}R=-8\pi GT_{00}=-8\pi G\rho(t)g_{00}=-8\pi G\rho(t)$$

$$\frac{2\ddot a a+\dot a^2+4}{(1+r^2)^2}\delta_{ij}=R_{ij}-\tfrac{1}{2}g_{ij}R=-8\pi GT_{ij}=0 \quad .$$