RELATIVIDADE GERAL 2011-2012

Exame, 2 de Julho de 2012, 9h30 - 12h30

1. Considere três referenciais A, B e C, cujas coordenadas são dadas por respectivamente

$$\begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix} \quad , \quad \begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} \quad \text{e} \quad \begin{pmatrix} t^{(C)} \\ x_1^{(C)} \\ x_2^{(C)} \\ x_3^{(C)} \end{pmatrix} \quad .$$

O referencial A está em movimento relativamente ao referencial B com velocidade constante $\vec{\beta}^{(AB)} = (v\,,\,0\,,\,0)$, enquanto o referencial B está em movimento relativamente ao referencial C com velocidade constante $\vec{\beta}^{(BC)} = (0\,,\,u\,,\,0)$.
A transformação entre as coordenadas dos referenciais A e B é dado pelo "*boost*" $\Lambda(AB)$:

$$\begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} = \Lambda(AB) \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda(AB)^0{}_0 & \Lambda(AB)^0{}_1 & \Lambda(AB)^0{}_2 & \Lambda(AB)^0{}_3 \\ \Lambda(AB)^1{}_0 & \Lambda(AB)^1{}_1 & \Lambda(AB)^1{}_2 & \Lambda(AB)^1{}_3 \\ \Lambda(AB)^2{}_0 & \Lambda(AB)^2{}_1 & \Lambda(AB)^2{}_2 & \Lambda(AB)^2{}_3 \\ \Lambda(AB)^3{}_0 & \Lambda(AB)^3{}_1 & \Lambda(AB)^3{}_2 & \Lambda(AB)^3{}_3 \end{pmatrix} \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix} \tag{1}$$

onde

$$\Lambda(AB)^0{}_0 = \gamma_{AB} = \frac{1}{\sqrt{1 - \vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)}}} \quad , \quad \Lambda(AB)^0{}_i = \Lambda(AB)^i{}_0 = \gamma_{AB}\beta_i^{(AB)}$$

$$\text{e} \quad \Lambda(AB)^i{}_j = \delta_{ij} + \frac{\beta_i^{(AB)}\beta_j^{(AB)}}{\vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)}}(\gamma_{AB} - 1) \quad ,$$

para $i, j = 1, 2, 3$.
A transformação $\Lambda(BC)$ entre as coordenadas dos referenciais B e C obtem-se pela substituição de A por B e B por C na Equação (1).

a. Mostre que os "*boosts*" $\Lambda(AB)$ e $\Lambda(BC)$ são dados por respectivamente

$$\Lambda(AB) = \begin{pmatrix} \gamma_{AB} & v\gamma_{AB} & 0 & 0 \\ v\gamma_{AB} & \gamma_{AB} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{e} \quad \Lambda(BC) = \begin{pmatrix} \gamma_{BC} & 0 & u\gamma_{BC} & 0 \\ 0 & 1 & 0 & 0 \\ u\gamma_{BC} & 0 & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad ,$$

com $\gamma_{AB} = 1/\sqrt{1-v^2}$ e $\gamma_{BC} = 1/\sqrt{1-u^2}$.

b. Demonstre que a velocidade $\vec{\beta}^{(AC)}$ do referencial A relativamente ao referencial C é tal que

$$\gamma_{AC} = \frac{1}{\sqrt{1-\vec{\beta}^{(AC)}\cdot\vec{\beta}^{(AC)}}} = \gamma_{BC}\gamma_{AB} \quad .$$

c. Demonstre que um "*boost*" com velocidade $\vec{\beta}^{(AC)}$ é dado por

$$\Lambda(AC) = \begin{pmatrix} \gamma_{AC} & v\gamma_{AB} & u\gamma_{AC} & 0 \\ v\gamma_{AB} & 1+\dfrac{v^2\gamma_{AB}^2}{(\gamma_{AC}+1)} & \dfrac{uv\gamma_{AB}\gamma_{AC}}{(\gamma_{AC}+1)} & 0 \\ u\gamma_{AC} & \dfrac{uv\gamma_{AB}\gamma_{AC}}{(\gamma_{AC}+1)} & 1+\dfrac{u^2\gamma_{AC}^2}{(\gamma_{AC}+1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

d. Demonstre que a transformação entre as coordenadas dos referenciais A e C é dada pela "*transformação de Lorentz*" $L(AC)$, representada por

$$\begin{pmatrix} t^{(C)} \\ x_1^{(C)} \\ x_2^{(C)} \\ x_3^{(C)} \end{pmatrix} = \Lambda(BC) \begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} = \Lambda(BC)\Lambda(AB) \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix} = L(AC) \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix} \quad ,$$

com

$$L(AC) = \begin{pmatrix} \gamma_{AC} & v\gamma_{AC} & u\gamma_{BC} & 0 \\ v\gamma_{AB} & \gamma_{AB} & 0 & 0 \\ u\gamma_{AC} & uv\gamma_{AC} & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

e. Demonstre que $L(AC) = \Lambda(BC)\Lambda(AB) = \Lambda(AC)R$, onde R represente uma rotação espacial, dada por

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} & -\dfrac{uv\gamma_{AB}\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} & 0 \\ 0 & \dfrac{uv\gamma_{AB}\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} & \dfrac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

f. Determine

$$\left(\frac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}}\right)^2 + \left(\frac{uv\gamma_{AB}\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}}\right)^2$$

e o eixo e o ângulo α da rotação.

g. Para o caso $v = 0.6c$ e $u = 0.8c$ determine α.

2. Para um espaço bidimensional (coordenadas $\vartheta \in [0, \pi)$, $\varphi \in [0, 2\pi)$), cuja métrica é dada por

$$ds^2 = R^2 d\vartheta^2 + R^2 \sin^2(\vartheta) d\varphi^2 \quad ,$$

mostre que geodésicas são dadas por (a, b e c constantes arbitrárias)

$$a \sin(\vartheta)\cos(\varphi) + b \sin(\vartheta)\sin(\varphi) + c\cos(\vartheta) = 0 \quad .$$

3. Considere um campo gravítico no espaço-tempo de uma distribuição esférica simétrica de massa M, dada por (*Schwarzschild, 1916*)

$$ds^2 = A(r)\, dt^2 - \frac{dr^2}{A(r)} - r^2 \left\{ d\vartheta^2 + \sin^2(\vartheta)\, d\varphi^2 \right\} \quad , \qquad (2)$$

com $A(r) = 1 - \frac{2MG}{r}$.

Se r é parametrizado por φ, obtém-se da equação geodésica no plano $\vartheta = \pi/2$ para $r(\varphi)$ a expressão:

$$\frac{d}{d\varphi}\left\{ \frac{1}{A(r)}\left(\frac{\ell}{r^2}\right)^2 \left(\frac{dr}{d\varphi}\right)^2 + r^2\left(\frac{\ell}{r^2}\right)^2 - \frac{1}{A(r)} \right\} = 0$$

e, portanto, uma constante de movimento, designada por E e dada por

$$E = -\frac{1}{A(r)}\left(\frac{\ell}{r^2}\right)^2 \left(\frac{dr}{d\varphi}\right)^2 - r^2\left(\frac{\ell}{r^2}\right)^2 + \frac{1}{A(r)} \quad .$$

a. Mostre que φ pode ser dado por

$$\varphi = \int \frac{dr}{r^2} \sqrt{\frac{1}{A(r)\left\{\frac{1}{\ell^2 A(r)} - \frac{E}{\ell^2} - \frac{1}{r^2}\right\}}} \ .$$

b. Demonstre que para uma órbita de um planeta ligado ao sol nos periélios e nos afélios, onde r alcança as distâncias do sol respectivamente minimas r_- e máximas r_+, o valor do constante de movimento E é dado por

$$E = \frac{1}{A\left(r_\pm\right)} - \frac{\ell^2}{r_\pm^2}$$

e mostre que

$$\varphi = \int \frac{dr}{r^2} \frac{1}{\sqrt{A(r)\left\{\frac{r_+^2\left(A^{-1}(r) - A^{-1}\left(r_+\right)\right) - r_-^2\left(A^{-1}(r) - A^{-1}\left(r_-\right)\right)}{r_+^2 r_-^2\left(A^{-1}\left(r_-\right) - A^{-1}\left(r_+\right)\right)} - \frac{1}{r^2}\right\}}} \ .$$

c. Demonstre que

$$A(r)\left\{\frac{r_+^2\left(A^{-1}(r) - A^{-1}\left(r_+\right)\right) - r_-^2\left(A^{-1}(r) - A^{-1}\left(r_-\right)\right)}{r_+^2 r_-^2\left(A^{-1}\left(r_-\right) - A^{-1}\left(r_+\right)\right)} - \frac{1}{r^2}\right\} =$$

$$= \frac{\left(r_+ - r\right)\left(r - r_-\right)}{r^2 r_+ r_-}\left\{1 - 2MG\left(\frac{r_+ + r_-}{r_+ r_-} + \frac{1}{r}\right)\right\} \ .$$

d. Demonstre que para

$$\left|2MG\frac{r_+ + r_-}{r_+ r_-}\right| \ll 1 \quad \text{and} \quad \left|\frac{2MG}{r}\right| \ll 1$$

podemos utilizar a seguinte aproximação:

$$\frac{1}{\sqrt{A(r)\left\{\frac{r_+^2\left(A^{-1}(r) - A^{-1}\left(r_+\right)\right) - r_-^2\left(A^{-1}(r) - A^{-1}\left(r_-\right)\right)}{r_+^2 r_-^2\left(A^{-1}\left(r_-\right) - A^{-1}\left(r_+\right)\right)} - \frac{1}{r^2}\right\}}} \approx$$

$$\approx \frac{\left\{1 + MG\frac{r_+ + r_-}{r_+ r_-}\right\}\left(1 + \frac{MG}{r}\right)}{\sqrt{\frac{\left(r - r_-\right)\left(r_+ - r\right)}{r^2 r_- r_+}}} \ .$$

e. Demonstre que, com a definição

$$\sin(\alpha) = \frac{r\,(r_+ + r_-) - 2r_+r_-}{r\,(r_+ - r_-)} \quad\Longleftrightarrow\quad \frac{2}{r} = \frac{r_+ + r_-}{r_+r_-} + \frac{r_- - r_+}{r_+r_-}\sin(\alpha) \quad ,$$

obtem-se

$$\varphi \approx \left\{1 + MG\frac{r_+ + r_-}{r_+r_-}\right\}\int d\alpha\;\frac{2r_+r_- + MG\,(r_+ + r_- + (r_- - r_+)\sin(\alpha))}{2r_+r_-} \quad .$$

f. Demonstre que

$$\varphi\,(r_+) - \varphi\,(r_-) \approx \pi\left\{1 + MG\frac{r_+ + r_-}{r_+r_-}\right\}\left\{1 + MG\frac{(r_+ + r_-)}{2r_+r_-}\right\} \quad .$$

g. Que pode concluir sobre a órbita do planeta em causa?

4. Considere um campo gravítico fraco e estático no espaço-tempo dada por

$$ds^2 \;=\; [1 + 2\Phi\,(\vec{x}\,)]\;dt^2 \;-\; [1 - 2\Phi\,(\vec{x}\,)]\;d\vec{x}^{\,2} \quad .$$

Para objectos com velocidades muitas inferiores à velocidade de luz mostre que

$$\frac{d^2\,\vec{x}}{dt^2} \;=\; -\bigtriangledown\;\Phi\,(\vec{x}\,) \quad .$$

Solutions

Exercício 1

a.

$$\Lambda(AB)^0{}_0 = \gamma_{AB} = \frac{1}{\sqrt{1 - \vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)}}} = \frac{1}{\sqrt{1 - v^2}} \quad ,$$

$$\Lambda(AB)^0{}_i = \Lambda(AB)^i{}_0 = \gamma_{AB}\beta_i^{(AB)} = \begin{cases} \gamma_{AB} v & \text{for } i = 1 \\ 0 & \text{for } i = 2\,,\,3 \end{cases}$$

$$\text{and} \quad \Lambda(AB)^i{}_j = \delta_{ij} + \frac{\beta_i^{(AB)}\beta_j^{(AB)}}{\vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)}}(\gamma_{AB} - 1) \quad .$$

Now, $\vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)} = v^2$ and

$$\beta_i^{(AB)}\beta_j^{(AB)} = \begin{cases} v^2 & \text{for } i = j = 1 \\ 0 & \text{for } i \neq 1\,, \text{ or } j \neq 1\,. \end{cases}$$

Hence, $\dfrac{\beta_i^{(AB)}\beta_j^{(AB)}}{\vec{\beta}^{(AB)} \cdot \vec{\beta}^{(AB)}}(\gamma_{AB} - 1) = \delta_{ij} + \delta_{i1}\delta_{j1}(\gamma_{AB} - 1)$.
Consequently,

$$\Lambda(AB)^i{}_j = \delta_{ij} + \delta_{i1}\delta_{j1}(\gamma_{AB} - 1) = \begin{cases} 1 + (\gamma_{AB} - 1) = \gamma_{AB} & \text{for } i = j = 1 \\ 1 & \text{for } i = j = 2\,, \text{ or } i = j = 3 \\ 0 & \text{for } i \neq j\,. \end{cases}$$

Conclusion:

$$\Lambda(AB)^i{}_j = \begin{pmatrix} \gamma_{AB} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

Hence,

$$\Lambda(AB) = \begin{pmatrix} \gamma_{AB} & v\gamma_{AB} & 0 & 0 \\ v\gamma_{AB} & \gamma_{AB} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

Similarly,

$$\Lambda(BC)^0{}_0 = \gamma_{BC} = \frac{1}{\sqrt{1 - \vec{\beta}^{(BC)} \cdot \vec{\beta}^{(BC)}}} = \frac{1}{\sqrt{1 - u^2}} \quad ,$$

$$\Lambda(BC)^0{}_i = \Lambda(BC)^i{}_0 = \gamma_{BC}\beta_i^{(BC)} = \begin{cases} \gamma_{BC} u & \text{for } i = 2 \\ 0 & \text{for } i = 1 \text{ and } 3 \end{cases}$$

$$\text{and} \quad \Lambda(BC)^i{}_j = \delta_{ij} + \frac{\beta_i^{(BC)}\beta_j^{(BC)}}{\vec{\beta}^{(BC)} \cdot \vec{\beta}^{(BC)}}(\gamma_{BC} - 1) = \delta_{ij} + \delta_{i2}\delta_{j2}(\gamma_{BC} - 1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_{BC} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

Hence,

$$\Lambda(BC) = \begin{pmatrix} \gamma_{BC} & 0 & u\gamma_{BC} & 0 \\ 0 & 1 & 0 & 0 \\ u\gamma_{BC} & 0 & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

b. First, we need to know $\vec{\beta}^{(AC)}$. From the coordinate transformation

$$\begin{pmatrix} t^{(C)} \\ x_1^{(C)} \\ x_2^{(C)} \\ x_3^{(C)} \end{pmatrix} = \Lambda(BC) \begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} = \begin{pmatrix} \gamma_{BC} & 0 & u\gamma_{BC} & 0 \\ 0 & 1 & 0 & 0 \\ u\gamma_{BC} & 0 & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix}$$

we deduce

$$t^{(C)} = \gamma_{BC} t^{(B)} + u\gamma_{BC} x_2^{(B)}$$

and

$$\begin{pmatrix} x_1^{(C)} \\ x_2^{(C)} \\ x_3^{(C)} \end{pmatrix} = \begin{pmatrix} x_1^{(B)} \\ u\gamma_{BC} t^{(B)} + \gamma_{BC} x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} \quad .$$

Hence,

$$\frac{dt^{(C)}}{dt^{(B)}} = \gamma_{BC} + u\gamma_{BC} \frac{dx_2^{(B)}}{dt^{(B)}}$$

and

$$\begin{pmatrix} \frac{dx_1^{(C)}}{dt^{(B)}} \\ \frac{dx_2^{(C)}}{dt^{(B)}} \\ \frac{dx_3^{(C)}}{dt^{(B)}} \end{pmatrix} = \begin{pmatrix} \frac{dx_1^{(B)}}{dt^{(B)}} \\ u\gamma_{BC} + \gamma_{BC} \frac{dx_2^{(B)}}{dt^{(B)}} \\ \frac{dx_3^{(B)}}{dt^{(B)}} \end{pmatrix} \quad .$$

We then deduce for the transformation of velocities from reference system B to reference system C, the following.

$$\begin{pmatrix} \frac{dx_1^{(C)}}{dt^{(C)}} \\ \frac{dx_2^{(C)}}{dt^{(C)}} \\ \frac{dx_3^{(C)}}{dt^{(C)}} \end{pmatrix} = \begin{pmatrix} \frac{dx_1^{(C)}}{dt^{(B)}} \frac{dt^{(B)}}{dt^{(C)}} \\ \frac{dx_2^{(C)}}{dt^{(B)}} \frac{dt^{(B)}}{dt^{(C)}} \\ \frac{dx_3^{(C)}}{dt^{(B)}} \frac{dt^{(B)}}{dt^{(C)}} \end{pmatrix} = \begin{pmatrix} \frac{dx_1^{(C)}}{dt^{(B)}} \\ \frac{dx_2^{(C)}}{dt^{(B)}} \\ \frac{dx_3^{(C)}}{dt^{(B)}} \end{pmatrix} \frac{1}{\frac{dt^{(C)}}{dt^{(B)}}} = \begin{pmatrix} \frac{dx_1^{(B)}}{dt^{(B)}} \\ u\gamma_{BC} + \gamma_{BC} \frac{dx_2^{(B)}}{dt^{(B)}} \\ \frac{dx_3^{(B)}}{dt^{(B)}} \end{pmatrix} \frac{1}{\gamma_{BC} + u\gamma_{BC} \frac{dx_2^{(B)}}{dt^{(B)}}} \quad .$$

Now, the moving reference system A has in B the velocity given by

$$\begin{pmatrix} \dfrac{dx_1^{(A\,\mathrm{in}\,B)}}{dt^{(B)}} \\ \dfrac{dx_2^{(A\,\mathrm{in}\,B)}}{dt^{(B)}} \\ \dfrac{dx_3^{(A\,\mathrm{in}\,B)}}{dt^{(B)}} \end{pmatrix} = \vec{\beta}^{(AB)} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \quad .$$

Hence,

$$\vec{\beta}^{(AC)} = \begin{pmatrix} \dfrac{dx_1^{(A\,\mathrm{in}\,C)}}{dt^{(B)}} \\ \dfrac{dx_2^{(A\,\mathrm{in}\,C)}}{dt^{(B)}} \\ \dfrac{dx_3^{(A\,\mathrm{in}\,C)}}{dt^{(B)}} \end{pmatrix} = \begin{pmatrix} \dfrac{v}{\gamma_{BC}} \\ u \\ 0 \end{pmatrix} = \begin{pmatrix} v\sqrt{1-u^2} \\ u \\ 0 \end{pmatrix} \quad .$$

From this result, we may also conclude

$$\gamma_{AC} = \frac{1}{\sqrt{1-\vec{\beta}^{(AC)}\cdot\vec{\beta}^{(AC)}}} = \frac{1}{\sqrt{1-v^2\left(1-u^2\right)-u^2}} = \frac{1}{\sqrt{1-u^2}}\frac{1}{\sqrt{1-v^2}} = \gamma_{BC}\gamma_{AB} \quad .$$

c. The boost $\Lambda(AC)$ from reference system A to reference system C is fully determined by $\vec{\beta}^{(AC)}$ and γ_{AC}, as follows:

$$\Lambda(AC)^0{}_0 = \gamma_{AC} = \gamma_{BC}\gamma_{AB} \quad ,$$

$$\Lambda(AC)^0{}_i = \Lambda(AC)^i{}_0 = \gamma_{AC}\beta_i^{(AC)} = \begin{cases} \gamma_{AC}\dfrac{v}{\gamma_{BC}} = \gamma_{AB}v & \text{for } i=1 \\ \gamma_{AC}u & \text{for } i=2 \\ 0 & \text{for } i=3 \end{cases}$$

$$\text{and} \quad \Lambda(AC)^i{}_j = \delta_{ij} + \frac{\beta_i^{(AC)}\beta_j^{(AC)}}{\vec{\beta}^{(AC)}\cdot\vec{\beta}^{(AC)}}(\gamma_{AC}-1) = \delta_{ij} + \gamma_{AC}^2\frac{\beta_i^{(AC)}\beta_j^{(AC)}}{(\gamma_{AC}+1)} =$$

$$= \delta_{ij} + \frac{\gamma_{AC}^2}{(\gamma_{AC}+1)}\left\{\delta_{i1}\delta_{j1}\frac{v^2}{\gamma_{BC}^2} + \delta_{i1}\delta_{j2}\frac{uv}{\gamma_{BC}} + \delta_{i2}\delta_{j1}\frac{uv}{\gamma_{BC}} + \delta_{i2}\delta_{j2}u^2\right\}$$

$$= \begin{pmatrix} 1+\dfrac{\gamma_{AC}^2v^2}{\gamma_{BC}^2(\gamma_{AC}+1)} & \dfrac{\gamma_{AC}^2uv}{\gamma_{BC}(\gamma_{AC}+1)} & 0 \\ \dfrac{\gamma_{AC}^2uv}{\gamma_{BC}(\gamma_{AC}+1)} & 1+\dfrac{\gamma_{AC}^2u^2}{(\gamma_{AC}+1)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

Hence, also using $\gamma_{AC} = \gamma_{BC}\gamma_{AB}$,

$$\Lambda(AC) = \begin{pmatrix} \gamma_{BC}\gamma_{AB} & \gamma_{AB}v & \gamma_{AC}u & 0 \\ \gamma_{AB}v & 1 + \dfrac{\gamma_{AB}^2 v^2}{(\gamma_{AC}+1)} & \dfrac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)} & 0 \\ \gamma_{AC}u & \dfrac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)} & 1 + \dfrac{\gamma_{AC}^2 u^2}{(\gamma_{AC}+1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

d. The coordinate transformation from A to C is given by

$$\begin{pmatrix} t^{(C)} \\ x_1^{(C)} \\ x_2^{(C)} \\ x_3^{(C)} \end{pmatrix} = \Lambda(BC) \begin{pmatrix} t^{(B)} \\ x_1^{(B)} \\ x_2^{(B)} \\ x_3^{(B)} \end{pmatrix} = \Lambda(BC)\Lambda(AB) \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma_{BC} & 0 & u\gamma_{BC} & 0 \\ 0 & 1 & 0 & 0 \\ u\gamma_{BC} & 0 & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{AB} & v\gamma_{AB} & 0 & 0 \\ v\gamma_{AB} & \gamma_{AB} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t^{(A)} \\ x_1^{(A)} \\ x_2^{(A)} \\ x_3^{(A)} \end{pmatrix}.$$

Hence,

$$L(AC) = \Lambda(BC)\Lambda(AB) = \begin{pmatrix} \gamma_{BC} & 0 & u\gamma_{BC} & 0 \\ 0 & 1 & 0 & 0 \\ u\gamma_{BC} & 0 & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{AB} & v\gamma_{AB} & 0 & 0 \\ v\gamma_{AB} & \gamma_{AB} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{BC}\gamma_{AB} & v\gamma_{BC}\gamma_{AB} & u\gamma_{BC} & 0 \\ v\gamma_{AB} & \gamma_{AB} & 0 & 0 \\ u\gamma_{BC}\gamma_{AB} & uv\gamma_{BC}\gamma_{AB} & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_{AC} & v\gamma_{AC} & u\gamma_{BC} & 0 \\ v\gamma_{AB} & \gamma_{AB} & 0 & 0 \\ u\gamma_{AC} & uv\gamma_{AC} & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

e. We must show that $L(AC) = \Lambda(AC)R$.

$\Lambda(AC)R =$

$$= \begin{pmatrix} \gamma_{AC} & \gamma_{AB}v & \gamma_{AC}u & 0 \\ \gamma_{AB}v & 1 + \dfrac{\gamma_{AB}^2 v^2}{(\gamma_{AC}+1)} & \dfrac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)} & 0 \\ \gamma_{AC}u & \dfrac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)} & 1 + \dfrac{\gamma_{AC}^2 u^2}{(\gamma_{AC}+1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AC}} & -\dfrac{uv\gamma_{AC}}{1+\gamma_{AC}} & 0 \\ 0 & \dfrac{uv\gamma_{AC}}{1+\gamma_{AC}} & \dfrac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AC}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Part of the matrix product can easily be performed:

$$\Lambda(AC)R =$$

$$= \begin{pmatrix} \gamma_{AC} & X & X & 0 \\ \gamma_{AB}v & X & X & 0 \\ \gamma_{AC}u & X & X & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & X & X & 0 \\ 0 & X & X & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_{AC} & X & X & 0 \\ \gamma_{AB}v & X & X & 0 \\ \gamma_{AC}u & X & X & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

The nontrivial part of the matrix product is then given by

$$\Lambda(AC)R =$$

$$= \begin{pmatrix} \gamma_{AC} & \gamma_{AB}v & \gamma_{AC}u & 0 \\ \gamma_{AB}v & 1+\frac{\gamma_{AB}^2 v^2}{(\gamma_{AC}+1)} & \frac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)} & 0 \\ \gamma_{AC}u & \frac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)} & 1+\frac{\gamma_{AC}^2 u^2}{(\gamma_{AC}+1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} & -\frac{uv\gamma_{AC}}{(1+\gamma_{AC})} & 0 \\ 0 & \frac{uv\gamma_{AC}}{(1+\gamma_{AC})} & \frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \left(\begin{matrix} \gamma_{AC} & \gamma_{AB}v\frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} + \gamma_{AC}u\frac{uv\gamma_{AC}}{(1+\gamma_{AC})} \\ \gamma_{AB}v & \left(1+\frac{\gamma_{AB}^2 v^2}{(\gamma_{AC}+1)}\right)\frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} + \frac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)}\frac{uv\gamma_{AC}}{(1+\gamma_{AC})} \\ \gamma_{AC}u & \frac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)}\frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} + \left(1+\frac{\gamma_{AC}^2 u^2}{(\gamma_{AC}+1)}\right)\frac{uv\gamma_{AC}}{(1+\gamma_{AC})} \\ 0 & 0 \end{matrix} \right.$$

$$\left. \begin{matrix} -\gamma_{AB}v\frac{uv\gamma_{AC}}{(1+\gamma_{AC})} + \gamma_{AC}u\frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} & 0 \\ -\left(1+\frac{\gamma_{AB}^2 v^2}{(\gamma_{AC}+1)}\right)\frac{uv\gamma_{AC}}{(1+\gamma_{AC})} + \frac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)}\frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} & 0 \\ -\frac{\gamma_{AB}\gamma_{AC}uv}{(\gamma_{AC}+1)}\frac{uv\gamma_{AC}}{(1+\gamma_{AC})} + \left(1+\frac{\gamma_{AC}^2 u^2}{(\gamma_{AC}+1)}\right)\frac{\gamma_{AB}+\gamma_{BC}}{(1+\gamma_{AC})} & 0 \\ 0 & 1 \end{matrix} \right) .$$

For the remaining arithmetic we use the following relations:

$$\gamma^2 = \frac{1}{\sqrt{1-\beta^2}} \quad \Longleftrightarrow \quad \beta^2 = \frac{\gamma^2 - 1}{\gamma^2} \quad \text{and} \quad \beta^2\gamma^2 = \gamma^2 - 1$$

and, furthermore, $\gamma_{AB}\gamma_{BC} = \gamma_{AC}$, $v = \beta_{AB}$ and $u = \beta_{BC}$.

This leads to the following identities:

$$\frac{\gamma_{AB}+\gamma_{BC}+u^2\gamma_{AB}\gamma_{BC}^2}{1+\gamma_{AC}}=\frac{\gamma_{AB}+\gamma_{BC}+\gamma_{AB}\left(\gamma_{BC}^2-1\right)}{1+\gamma_{AC}}=\gamma_{BC}\frac{1+\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}}=\gamma_{BC}\quad,$$

$$\frac{\gamma_{AB}+\gamma_{BC}+v^2\gamma_{AB}^2\gamma_{BC}}{1+\gamma_{AC}}=\frac{\gamma_{AB}+\gamma_{BC}+\gamma_{BC}\left(\gamma_{AB}^2-1\right)}{1+\gamma_{AC}}=\gamma_{AB}\frac{1+\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}}=\gamma_{AB}$$

and

$$\frac{-\gamma_{AB}^2v^2+\gamma_{AB}^2+\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}}=\frac{1+\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}}=1\quad.$$

$$\Lambda(AC)R=$$

$$=\begin{pmatrix}\gamma_{AC} & \gamma_{AB}v\dfrac{\gamma_{AB}+\gamma_{BC}+u^2\gamma_{AB}\gamma_{BC}^2}{1+\gamma_{AC}} \\ \gamma_{AB}v & \dfrac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AC}}+\dfrac{\gamma_{AB}^2v^2}{(\gamma_{AC}+1)^2}\left(\gamma_{AB}+\gamma_{BC}+u^2\gamma_{AB}\gamma_{BC}^2\right) \\ \gamma_{AC}u & \dfrac{uv\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}}+\dfrac{uv\gamma_{AB}^2\gamma_{BC}}{(\gamma_{AC}+1)^2}\left(\gamma_{AB}+\gamma_{BC}+u^2\gamma_{AB}\gamma_{BC}^2\right) \\ 0 & 0\end{pmatrix.$$

$$\begin{matrix}\gamma_{BC}u\dfrac{-\gamma_{AB}^2v^2+\gamma_{AB}^2+\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}} & 0 \\ -\dfrac{uv\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}}+\dfrac{\gamma_{AC}uv}{(\gamma_{AC}+1)^2}\left(-v^2\gamma_{AB}^2+\gamma_{AB}^2+\gamma_{AB}\gamma_{BC}\right) & 0 \\ \dfrac{\gamma_{AB}+\gamma_{BC}+u^2\gamma_{AB}\gamma_{BC}^2}{1+\gamma_{AC}} & 0 \\ 0 & 1\end{matrix}\Bigg)$$

$$=\begin{pmatrix}\gamma_{AC} & v\gamma_{AB}\gamma_{BC} & u\gamma_{BC} & 0 \\ \gamma_{AB}v & \dfrac{\gamma_{AB}+\gamma_{BC}+v^2\gamma_{AB}^2\gamma_{BC}}{1+\gamma_{AC}} & \dfrac{uv\gamma_{AB}\gamma_{BC}-uv\gamma_{AB}\gamma_{BC}}{1+\gamma_{AC}} & 0 \\ \gamma_{AC}u & \dfrac{uv\gamma_{AB}\gamma_{BC}+uv\gamma_{AB}^2\gamma_{BC}^2}{1+\gamma_{AC}} & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}\gamma_{AC} & v\gamma_{AB}\gamma_{BC} & u\gamma_{BC} & 0 \\ \gamma_{AB}v & \gamma_{AB} & 0 & 0 \\ \gamma_{AC}u & uv\gamma_{AB}\gamma_{BC} & \gamma_{BC} & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}=L(AC)\quad.$$

This completes the proof that $L(AC) = \Lambda(AC)R$.
f. Hence,

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} & -\dfrac{\gamma_{AB}\gamma_{BC}uv}{1+\gamma_{AB}\gamma_{BC}} & 0 \\ 0 & \dfrac{\gamma_{AB}\gamma_{BC}uv}{1+\gamma_{AB}\gamma_{BC}} & \dfrac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

This represents a pure rotation around the z-axis. The rotation angle α is given by

$$\cos(\alpha) = \frac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} \quad \text{and} \quad \sin(\alpha) = -\frac{\gamma_{AB}\gamma_{BC}uv}{1+\gamma_{AB}\gamma_{BC}} \quad .$$

We verify

$$\cos^2(\alpha) + \sin^2(\alpha) = \frac{1}{(1+\gamma_{AB}\gamma_{BC})^2}\left\{(\gamma_{AB}+\gamma_{BC})^2 + \gamma_{AB}^2\gamma_{BC}^2u^2v^2\right\} =$$

$$= \frac{1}{(1+\gamma_{AB}\gamma_{BC})^2}\left\{\gamma_{AB}^2 + \gamma_{BC}^2 + 2\gamma_{AB}\gamma_{BC} + \gamma_{AB}^2\gamma_{BC}^2u^2v^2\right\}$$

$$= \frac{1}{(1+\gamma_{AB}\gamma_{BC})^2}\left\{\gamma_{AB}^2 + \gamma_{BC}^2 + 2\gamma_{AB}\gamma_{BC} + \left(\gamma_{AB}^2 - 1\right)\left(\gamma_{BC}^2 - 1\right)\right\}$$

$$= \frac{1}{(1+\gamma_{AB}\gamma_{BC})^2}\left\{2\gamma_{AB}\gamma_{BC} + \gamma_{AB}^2\gamma_{BC}^2 + 1\right\} = 1 \quad .$$

g.

$$\gamma_{AB} = \frac{1}{\sqrt{1-v^2}} = 1.25 \quad \text{and} \quad \gamma_{BC} = \frac{1}{\sqrt{1-u^2}} = 1.67 \quad .$$

Hence,

$$\cos(\alpha) = \frac{\gamma_{AB}+\gamma_{BC}}{1+\gamma_{AB}\gamma_{BC}} = 0.945 \quad \text{and} \quad \sin(\alpha) = -\frac{\gamma_{AB}\gamma_{BC}uv}{1+\gamma_{AB}\gamma_{BC}} = -0.324 \quad .$$

$\alpha = -18.9°$.

Exercício 2

The non-zero elements of the affine connection are given by

$$\Gamma^{\vartheta}_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta) \quad \text{and} \quad \Gamma^{\varphi}_{\vartheta\varphi} = \Gamma^{\varphi}_{\varphi\vartheta} = \frac{\cos(\vartheta)}{\sin(\vartheta)} \quad .$$

Hence, the geodesic equations are

$$\frac{d^2\vartheta}{ds^2} \; - \; \sin(\vartheta)\cos(\vartheta) \left(\frac{d\varphi}{ds}\right)^2 \; = \; 0 \quad \text{and} \quad \frac{d^2\varphi}{ds^2} \; + \; 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\frac{d\vartheta}{ds}\,\frac{d\varphi}{ds} \; = \; 0 \quad .$$

One possible strategy of solving the geodesic equations is to express one of the coordinates as a function of the other along the geodesic curve, for example let

$$\varphi(s) \; = \; \varphi(\vartheta(s)) \quad .$$

In order to simplify the formulas to come, we define

$$\varphi' \; = \; \frac{d\varphi}{d\vartheta} \quad \text{and} \quad \varphi'' \; = \; \frac{d^2\varphi}{d\vartheta^2} \quad ,$$

in which notation we obtain for the derivatives of φ with respect to the proper length parameter s, the expressions:

$$\frac{d\varphi}{ds} \; = \; \frac{d\vartheta}{ds}\,\varphi' \quad \text{and} \quad \frac{d^2\varphi}{ds^2} \; = \; \frac{d^2\vartheta}{ds^2}\,\varphi' \; + \; \left(\frac{d\vartheta}{ds}\right)^2 \varphi'' \quad ,$$

and hence for the geodesic equations

$$\frac{d^2\vartheta}{ds^2} \; - \; \sin(\vartheta)\cos(\vartheta) \left(\frac{d\vartheta}{ds}\,\varphi'\right)^2 \; = \; 0 \quad \text{and}$$

$$\frac{d^2\vartheta}{ds^2}\,\varphi' \; + \; \left(\frac{d\vartheta}{ds}\right)^2 \varphi'' \; + \; 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)} \left(\frac{d\vartheta}{ds}\right)^2 \varphi' \; = \; 0 \quad .$$

When we substitue moreover the first of the geodesic equations into the second, then we find the differential equation

$$\left(\frac{d\vartheta}{ds}\right)^2 \left\{\varphi'' \; + \; \sin(\vartheta)\cos(\vartheta)\;(\varphi')^3 \; + \; 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\varphi'\right\} \; = \; 0 \quad .$$

General solutions of the geodesic equations are found from the second piece of this equation, *i.e.*

$$\varphi'' \; + \; \sin(\vartheta)\cos(\vartheta)\;(\varphi')^3 \; + \; 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\varphi' \; = \; 0 \quad .$$

In order to show that

$$a\cos(\varphi) + b\sin(\varphi) = -c\,\frac{\cos(\vartheta)}{\sin(\vartheta)}$$

is solution to this equation, we determine the first and second order derivatives of the above equation with respect to ϑ:

$$\varphi' \ \{-a\ \sin(\varphi)\ +\ b\ \cos(\varphi)\}\ =\ \frac{c}{\sin^2(\vartheta)}$$

and

$$\varphi'' \ \{-a\ \sin(\varphi)\ +\ b\ \cos(\varphi)\}\ -\ (\varphi')^2\ \{a\ \cos(\varphi)\ +\ b\ \sin(\varphi)\}\ =\ -2c\,\frac{\cos(\vartheta)}{\sin^3(\vartheta)}$$

By multiplying the second equation with φ' and by substitution of relation $a\sin(\vartheta)\cos(\varphi) + b\sin(\vartheta)\sin(\varphi) + c\cos(\vartheta) = 0$ and the first equation in the resulting expression, we obtain

$$\varphi''\,\frac{c}{\sin^2(\vartheta)}\ +\ (\varphi')^3\ c\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\ +\ 2\varphi'\ c\,\frac{\cos(\vartheta)}{\sin^3(\vartheta)}\ =\ 0\quad ,$$

which, for $\sin(\vartheta) \neq 0$, is completely equivalent to the geodesic equation

$$\varphi''\ +\ \sin(\vartheta)\cos(\vartheta)\ (\varphi')^3\ +\ 2\,\frac{\cos(\vartheta)}{\sin(\vartheta)}\,\varphi'\ =\ 0\quad .$$

Exercício 3

a. It is given that

$$E=-\frac{1}{A(r)}\left(\frac{\ell}{r^2}\right)^2\left(\frac{dr}{d\varphi}\right)^2-r^2\left(\frac{\ell}{r^2}\right)^2+\frac{1}{A(r)} \ .$$

Hence,

$$\left(\frac{dr}{d\varphi}\right)^2=\frac{1}{\dfrac{1}{A(r)}\left(\dfrac{\ell}{r^2}\right)^2}\left(-E-r^2\left(\frac{\ell}{r^2}\right)^2+\frac{1}{A(r)}\right) \ ,$$

or

$$\left(\frac{d\varphi}{dr}\right)^2=\frac{\dfrac{1}{A(r)}\left(\dfrac{\ell}{r^2}\right)^2}{-E-r^2\left(\dfrac{\ell}{r^2}\right)^2+\dfrac{1}{A(r)}}=\frac{1}{r^4}\ \frac{1}{A(r)\left\{-\dfrac{E}{\ell^2}-\dfrac{1}{r^2}+\dfrac{1}{\ell^2A(r)}\right\}} \ .$$

Taking the square root on both sides gives the desired result.

b. When $r(\varphi)$ is maximum (*aphelion* r_+) or minimum (*perihelion* r_-), then the first order derivative $dr/d\varphi$ vanishes. As a consequence

$$\frac{1}{A\left(r_-\right)}-\frac{\ell^2}{r_-^2}=E=\frac{1}{A\left(r_+\right)}-\frac{\ell^2}{r_+^2}\quad\Longleftrightarrow\quad \ell^2\left\{\frac{1}{r_-^2}-\frac{1}{r_+^2}\right\}=\frac{1}{A\left(r_-\right)}-\frac{1}{A\left(r_+\right)}$$

$$\Longleftrightarrow\quad \ell^2=\frac{\dfrac{1}{A\left(r_-\right)}-\dfrac{1}{A\left(r_+\right)}}{\dfrac{1}{r_-^2}-\dfrac{1}{r_+^2}}$$

$$\frac{1}{\ell^2}=\frac{\dfrac{1}{r_-^2}-\dfrac{1}{r_+^2}}{\dfrac{1}{A\left(r_-\right)}-\dfrac{1}{A\left(r_+\right)}}=\frac{r_+^2-r_-^2}{\dfrac{r_-^2r_+^2}{A\left(r_-\right)}-\dfrac{r_-^2r_+^2}{A\left(r_+\right)}}$$

and

$$E=\frac{1}{A\left(r_-\right)}-\frac{\ell^2}{r_-^2}=\frac{1}{A\left(r_-\right)}-\frac{1}{r_-^2}\frac{\dfrac{1}{A\left(r_-\right)}-\dfrac{1}{A\left(r_+\right)}}{\dfrac{1}{r_-^2}-\dfrac{1}{r_+^2}}=$$

$$=\frac{\dfrac{1}{A\left(r_-\right)}\left(1-\dfrac{r_-^2}{r_+^2}\right)-\dfrac{1}{A\left(r_-\right)}+\dfrac{1}{A\left(r_+\right)}}{1-\dfrac{r_-^2}{r_+^2}}=\frac{\dfrac{1}{A\left(r_-\right)}\left(r_+^2-r_-^2\right)-\dfrac{r_+^2}{A\left(r_-\right)}+\dfrac{r_+^2}{A\left(r_+\right)}}{r_+^2-r_-^2}$$

$$=\frac{\dfrac{r_+^2}{A\left(r_+\right)}-\dfrac{r_-^2}{A\left(r_-\right)}}{r_+^2-r_-^2} \ .$$

Hence,

$$\frac{1}{\ell^2 A(r)} - \frac{E}{\ell^2} = \frac{1}{A(r)} \frac{r_+^2 - r_-^2}{\dfrac{r_-^2 r_+^2}{A(r_-)} - \dfrac{r_-^2 r_+^2}{A(r_+)}} - \frac{\dfrac{r_+^2}{A(r_+)} - \dfrac{r_-^2}{A(r_-)}}{r_+^2 - r_-^2} \frac{r_+^2 - r_-^2}{\dfrac{r_-^2 r_+^2}{A(r_-)} - \dfrac{r_-^2 r_+^2}{A(r_+)}}$$

$$= \frac{\dfrac{1}{A(r)}\left(r_+^2 - r_-^2\right) - \dfrac{r_+^2}{A(r_+)} + \dfrac{r_-^2}{A(r_-)}}{\dfrac{r_-^2 r_+^2}{A(r_-)} - \dfrac{r_-^2 r_+^2}{A(r_+)}}$$

$$= \frac{r_+^2\left(A^{-1}(r) - A^{-1}(r_+)\right) - r_-^2\left(A^{-1}(r) - A^{-1}(r_-)\right)}{r_+^2 r_-^2\left(A^{-1}(r_-) - A^{-1}(r_+)\right)}$$

c. We develop further the full expression under the square root:

$$A(r)\left\{\frac{r_+^2\left(A^{-1}(r) - A^{-1}(r_+)\right) - r_-^2\left(A^{-1}(r) - A^{-1}(r_-)\right)}{r_+^2 r_-^2\left(A^{-1}(r_-) - A^{-1}(r_+)\right)} - \frac{1}{r^2}\right\} =$$

$$= \frac{r_+^2\left(1 - A(r)A^{-1}(r_+)\right) - r_-^2\left(1 - A(r)A^{-1}(r_-)\right)}{r_+^2 r_-^2\left(A^{-1}(r_-) - A^{-1}(r_+)\right)} - \frac{A(r)}{r^2}$$

$$= \frac{r_+^2\left(A(r_+)A(r_-) - A(r)A(r_-)\right) - r_-^2\left(A(r_+)A(r_-) - A(r)A(r_+)\right)}{r_+^2 r_-^2\left(A(r_+) - A(r_-)\right)} - \frac{A(r)}{r^2}$$

$$= \frac{A(r_+)A(r_-)\left(r_+^2 - r_-^2\right) - A(r)\left(r_+^2 A(r_-) - r_-^2 A(r_+)\right)}{2MGr_+r_-\left(r_+ - r_-\right)} - \frac{A(r)}{r^2}$$

$$= \frac{A(r_+)A(r_-)\left(r_+^2 - r_-^2\right) - A(r)\left(r_+^2 - r_-^2\right) + 2MGA(r)\left(\dfrac{r_+^2}{r_-} - \dfrac{r_-^2}{r_+}\right)}{2MGr_+r_-\left(r_+ - r_-\right)} - \frac{A(r)}{r^2}$$

$$= \frac{\left\{-2MG\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \frac{4M^2G^2}{r_+r_-} + \frac{2MG}{r}\right\}\left(r_+^2 - r_-^2\right) + 2MGA(r)\left(\dfrac{r_+^2}{r_-} - \dfrac{r_-^2}{r_+}\right)}{2MGr_+r_-\left(r_+ - r_-\right)} - \frac{A(r)}{r^2}$$

$$= \frac{\left\{-\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \frac{2MG}{r_+r_-} + \frac{1}{r}\right\}\left(r_+^2 - r_-^2\right) + A(r)\left(\dfrac{r_+^3 - r_-^3}{r_+r_-}\right)}{r_+r_-\left(r_+ - r_-\right)} - \frac{A(r)}{r^2}$$

$$= \left\{ -\left(\frac{1}{r_+} + \frac{1}{r_-}\right) + \frac{2MG}{r_+ r_-} + \frac{1}{r} \right\} \frac{r_+ + r_-}{r_+ r_-} + A(r) \frac{r_+^2 + r_+ r_- + r_-^2}{r_+^2 r_-^2} - \frac{A(r)}{r^2}$$

$$= -\left(\frac{r_+ + r_-}{r_+ r_-}\right)^2 + \frac{r_+ + r_-}{r r_+ r_-} + \frac{r_+^2 + r_+ r_- + r_-^2}{r_+^2 r_-^2} - \frac{1}{r^2} +$$

$$+2MG \left\{ \frac{r_+ + r_-}{r_+^2 r_-^2} - \frac{r_+ + r_+ r_- + r_-}{r r_+^2 r_-^2} + \frac{1}{r^3} \right\}$$

$$= \frac{-r_+ r_-}{r_+^2 r_-^2} + \frac{r_+ + r_-}{r r_+ r_-} - \frac{1}{r^2} + 2MG \left\{ \frac{r^3 (r_+ + r_-) - r^2 \left(r_+^2 + r_+ r_- + r_-^2\right) + r_+^2 r_-^2}{r^3 r_+^2 r_-^2} \right\}$$

$$= \frac{-r^2 + r r_+ + r r_- - r_+ r_-}{r^2 r_+ r_-} - 2MG \left\{ \frac{(r (r_+ + r_-) + r_+ r_-) (r_+ - r) (r - r_-)}{r^3 r_+^2 r_-^2} \right\}$$

$$= \frac{(r_+ - r) (r - r_-)}{r^2 r_+ r_-} - 2MG \left\{ \frac{(r (r_+ + r_-) + r_+ r_-) (r_+ - r) (r - r_-)}{r^3 r_+^2 r_-^2} \right\}$$

$$= \frac{(r_+ - r) (r - r_-)}{r^2 r_+ r_-} \left\{ 1 - 2MG \frac{(r (r_+ + r_-) + r_+ r_-)}{r r_+ r_-} \right\}$$

$$= \frac{(r_+ - r) (r - r_-)}{r^2 r_+ r_-} \left\{ 1 - 2MG \left(\frac{r_+ + r_-}{r_+ r_-} + \frac{1}{r} \right) \right\} \quad .$$

d. For

$$\left| 2MG \frac{r_+ + r_-}{r_+ r_-} \right| \ll 1 \quad \text{and} \quad \left| \frac{2MG}{r} \right| \ll 1$$

we may approximate this result by

$$A(r) \left\{ \frac{r_+^2 \left(A^{-1}(r) - A^{-1}(r_+)\right) - r_-^2 \left(A^{-1}(r) - A^{-1}(r_-)\right)}{r_+^2 r_-^2 \left(A^{-1}(r_-) - A^{-1}(r_+)\right)} - \frac{1}{r^2} \right\} \approx$$

$$\approx \frac{(r - r_-)}{r r_-} \frac{(r_+ - r)}{r_+ r} \left\{ 1 - 2MG \frac{r_+ + r_-}{r_+ r_-} \right\} \left(1 - \frac{2MG}{r} \right) \quad ,$$

the square root by

$$\sqrt{A(r) \left\{ \frac{r_+^2 \left(A^{-1}(r) - A^{-1}(r_+)\right) - r_-^2 \left(A^{-1}(r) - A^{-1}(r_-)\right)}{r_+^2 r_-^2 \left(A^{-1}(r_-) - A^{-1}(r_+)\right)} - \frac{1}{r^2} \right\}}$$

$$\approx \sqrt{\frac{(r - r_-)}{r r_-} \frac{(r_+ - r)}{r_+ r}} \left\{ 1 - MG \frac{r_+ + r_-}{r_+ r_-} \right\} \left(1 - \frac{MG}{r} \right)$$

and the inverse of the square root by

$$\frac{1}{\sqrt{A(r)\left\{\frac{r_+^2\left(A^{-1}(r)-A^{-1}\left(r_+\right)\right)-r_-^2\left(A^{-1}(r)-A^{-1}\left(r_-\right)\right)}{r_+^2 r_-^2\left(A^{-1}\left(r_-\right)-A^{-1}\left(r_+\right)\right)}-\frac{1}{r^2}\right\}}}$$

$$\approx \frac{\left\{1+MG\frac{r_+ + r_-}{r_+ r_-}\right\}\left(1+\frac{MG}{r}\right)}{\sqrt{\frac{\left(r-r_-\right)}{r r_-}\frac{\left(r_+ - r\right)}{r_+ r}}} \quad .$$

e. So, we are left with the integral

$$\varphi \approx \left\{1+MG\frac{r_+ + r_-}{r_+ r_-}\right\}\int \frac{dr}{r^2}\,\frac{\left(1+\frac{MG}{r}\right)}{\sqrt{\frac{\left(r-r_-\right)}{r r_-}\frac{\left(r_+ - r\right)}{r_+ r}}} \quad .$$

With the substitution

$$\sin(\alpha)=\frac{r\left(r_+ + r_-\right)-2r_+ r_-}{r\left(r_+ - r_-\right)} \quad \Longleftrightarrow \quad \frac{2}{r}=\frac{r_+ + r_-}{r_+ r_-}+\frac{r_- - r_+}{r_+ r_-}\,\sin(\alpha)$$

we obtain for the integrand

$$\frac{\left(1+\frac{MG}{r}\right)}{\sqrt{\frac{\left(r-r_-\right)}{r r_-}\frac{\left(r_+ - r\right)}{r_+ r}}}=$$

$$=\frac{1+MG\left(\frac{r_+ + r_-}{2r_+ r_-}+\frac{r_- - r_+}{2r_+ r_-}\,\sin(\alpha)\right)}{\sqrt{\left(\frac{1}{r_-}-\frac{r_+ + r_-}{2r_+ r_-}-\frac{r_- - r_+}{2r_+ r_-}\,\sin(\alpha)\right)\left(\frac{r_+ + r_-}{2r_+ r_-}+\frac{r_- - r_+}{2r_+ r_-}\,\sin(\alpha)-\frac{1}{r_+}\right)}}$$

$$=\frac{2r_+ r_- + MG\left(r_+ + r_- + \left(r_- - r_+\right)\sin(\alpha)\right)}{\sqrt{\left(2r_+ - r_+ - r_- - \left(r_- - r_+\right)\sin(\alpha)\right)\left(r_+ + r_- + \left(r_- - r_+\right)\sin(\alpha)-2r_-\right)}}$$

$$=\frac{2r_+ r_- + MG\left(r_+ + r_- + \left(r_- - r_+\right)\sin(\alpha)\right)}{\sqrt{\left(r_+ - r_- - \left(r_- - r_+\right)\sin(\alpha)\right)\left(r_+ - r_- + \left(r_- - r_+\right)\sin(\alpha)\right)}}$$

$$=\frac{2r_+ r_- + MG\left(r_+ + r_- + \left(r_- - r_+\right)\sin(\alpha)\right)}{\left(r_+ - r_-\right)\sqrt{\left(1+\sin(\alpha)\right)\left(1-\sin(\alpha)\right)}}$$

$$=\frac{2r_+ r_- + MG\left(r_+ + r_- + \left(r_- - r_+\right)\sin(\alpha)\right)}{\left(r_+ - r_-\right)\cos(\alpha)} \quad .$$

and, furthermore,

$$-\frac{2}{r^2}dr = \frac{r_- - r_+}{r_+ r_-}\ \cos(\alpha)d\alpha$$

So, we are left with the integral

$$\varphi \approx \left\{1 + MG\frac{r_+ + r_-}{r_+ r_-}\right\} \int d\alpha\ \frac{2r_+ r_- + MG\,(r_+ + r_- + (r_- - r_+)\sin(\alpha))}{2r_+ r_-}\quad .$$

f.

$$\varphi \approx \left\{1 + MG\frac{r_+ + r_-}{r_+ r_-}\right\} \left\{\frac{2r_+ r_- + MG\,(r_+ + r_-)}{2r_+ r_-}\ \alpha - MG\frac{r_- - r_+}{2r_+ r_-}\cos(\alpha)\right\}\quad .$$

For $r = r_+$ (aphelion) one has

$$\sin(\alpha(r = r_+)) = \frac{r_+\,(r_+ + r_-) - 2r_+ r_-}{r_+\,(r_+ - r_-)} = 1 \quad \Longleftrightarrow \quad \alpha = \tfrac{1}{2}\pi\ \ ,\ \ \cos(\alpha) = 0\ \ ,$$

whereas, for $r = r_-$ (perihelion) one has

$$\sin(\alpha(r = r_-)) = \frac{r_-\,(r_+ + r_-) - 2r_+ r_-}{r_-\,(r_+ - r_-)} = -1 \quad \Longleftrightarrow \quad \alpha = -\tfrac{1}{2}\pi\ \ ,\ \ \cos(\alpha) = 0\ \ .$$

Hence, φ sweeps from perihelion to aphelion an angle given by

$$\begin{aligned}
\varphi &\approx \left\{1 + MG\frac{r_+ + r_-}{r_+ r_-}\right\} \int_{r_-}^{r_+} \frac{dr}{r^2}\ \frac{\left(1 + \frac{MG}{r}\right)}{\sqrt{\frac{(r - r_-)}{r r_-}\frac{(r_+ - r)}{r_+ r}}} = \\
&\approx \left\{1 + MG\frac{r_+ + r_-}{r_+ r_-}\right\} \left\{\frac{2r_+ r_- + MG\,(r_+ + r_-)}{2r_+ r_-}\ \pi\right\} \\
&\approx \left\{1 + MG\frac{r_+ + r_-}{r_+ r_-}\right\} \left\{1 + MG\frac{(r_+ + r_-)}{2r_+ r_-}\right\}\ \pi \approx \pi + \tfrac{3}{2}\pi MG\frac{(r_+ + r_-)}{r_+ r_-}\quad .
\end{aligned}$$

g. We may conclude that the orbit is NOT a perfect ellipse, since the angle from perihelion to aphelion is different from π.

Exercício 4

The metric is given by

$$g_{tt} = 1 + 2\Phi \quad \text{and} \quad g_{ij} = \delta_{ij}\left(-1 + 2\Phi\right) \quad ,$$

for $i, j = 1$, 2 and 3.
There are no time derivatives for the elements of the metric.

$$g_{tt,\,i} = 2\Phi_{,\,i} \quad \text{and} \quad g_{ij,\,k} = 2\delta_{ij}\Phi_{,\,k} \quad ,$$

for $i, j, k = 1$, 2 and 3.
Hence the non-zero Christoffel symbols, defined by

$$\Gamma_{\mu\alpha\beta} = \tfrac{1}{2}\left\{ g_{\mu\alpha,\,\beta} + g_{\mu\beta,\,\alpha} - g_{\alpha\beta,\,\mu} \right\} \quad ,$$

are the following:

$$\Gamma_{tti} = \Gamma_{tit} = -\Gamma_{itt} = \Phi_{,\,i} \quad \text{and} \quad \Gamma_{ijk} = \delta_{ij}\Phi_{,\,k} + \delta_{ik}\Phi_{,\,j} - \delta_{jk}\Phi_{,\,i} \quad .$$

for $i, j = 1$, 2 and 3.
The inverse metric is given by

$$g^{tt} = \frac{1}{1 + 2\Phi} \quad \text{and} \quad g^{ij} = \frac{\delta^{ij}}{-1 + 2\Phi} \quad .$$

For the connection we find then

$$\Gamma^{t}_{it} = \Gamma^{t}_{ti} = g^{tt}\Gamma_{tti} = \frac{\Phi_{,\,i}}{1 + 2\Phi} \quad , \quad \Gamma^{i}_{tt} = g^{ij}\Gamma_{jtt} = \frac{\Phi_{,\,i}}{1 - 2\Phi} \quad ,$$

$$\Gamma^{i}_{jk} = g^{i\ell}\Gamma_{\ell jk} = \frac{\delta^{i\ell}}{-1 + 2\Phi}\left(\delta_{\ell j}\Phi_{,\,k} + \delta_{\ell k}\Phi_{,\,j} - \delta_{jk}\Phi_{,\,\ell}\right) =$$

$$= \frac{1}{-1 + 2\Phi}\left(\delta^{i}_{j}\Phi_{,\,k} + \delta^{i}_{k}\Phi_{,\,j} - \delta_{jk}\delta^{i\ell}\Phi_{,\,\ell}\right) \quad .$$

This gives for the geodesic equations, defined by

$$0 = \frac{d^2u^{\mu}}{ds^2} + \frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds}\Gamma^{\mu}_{\alpha\beta} \quad ,$$

the following results:

$$0 = \frac{d^2t}{ds^2} + 2\frac{dt}{ds}\frac{dx^i}{ds}\Gamma^{t}_{it} = \frac{d^2t}{ds^2} + 2\frac{dt}{ds}\frac{dx^i}{ds}\frac{\Phi_{,\,i}}{1 + 2\Phi}$$

$$0 = \frac{d^2x^i}{ds^2} + \frac{dt}{ds}\frac{dt}{ds}\Gamma^{i}_{tt} + \frac{dx^j}{ds}\frac{dx^k}{ds}\Gamma^{i}_{jk} =$$

$$= \frac{d^2x^i}{ds^2} + \frac{dt}{ds}\,\frac{dt}{ds}\frac{\Phi,_i}{1-2\Phi} + \frac{dx^j}{ds}\,\frac{dx^k}{ds}\frac{1}{-1+2\Phi}\left(\delta^i_j\Phi,_k + \delta^i_k\Phi,_j - \delta_{jk}\delta^{i\ell}\Phi,_\ell\right) =$$

$$= \frac{d^2x^i}{ds^2} + \frac{dt}{ds}\,\frac{dt}{ds}\frac{\Phi,_i}{1-2\Phi} + \frac{1}{-1+2\Phi}\left(2\frac{dx^i}{ds}\,\frac{dx^j}{ds}\Phi,_j - \frac{d\vec{x}}{ds}\cdot\frac{d\vec{x}}{ds}\Phi,_i\right) \quad .$$

Now, since the particle moves slowly, we may assume that

$$\left|\frac{d\vec{x}}{ds}\right| \ll \left|\frac{dt}{ds}\right| \quad ,$$

which gives us the following equations

$$0 \approx \frac{d^2t}{ds^2} \quad \text{and} \quad 0 \approx \frac{d^2x^i}{ds^2} + \frac{dt}{ds}\,\frac{dt}{ds}\frac{\Phi,_i}{1-2\Phi} \quad .$$

The first equation allows to choose $t = s$ and $dt/ds = 1$.
Since, moreover, the field is weak, we may assume

$$1 - 2\Phi \approx 1 \quad ,$$

which gives us the equation

$$0 \approx \frac{d^2x^i}{dt^2} + \Phi,_i \quad \text{or} \quad \frac{d^2\vec{x}}{dt^2} \approx -\bigtriangledown\Phi \quad .$$